# Time-Frequency Design and Processing of Signals Via Smoothed Wigner Distributions

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***Abstract*—The Wigner distribution (WD) and WD-based signal synthesis can be used for designing and processing signals in a joint time-frequency domain. Unfortunately, the performance of this method is adversely affected by the occurrence of interference terms (IT's) in the WD. This paper provides an analysis of IT effects in WD-based signal synthesis, and shows that IT effects can be substantially reduced by using a smoothed Wigner distribution (SWD) instead of the WD. An iterative algorithm for SWD-based signal synthesis is presented, and the improvement over WD-based signal synthesis is verified via computer simulation.**

## I. Introduction

THE Wigner distribution (WD) [1]

$$W_x(t,f) = \int_\tau x\left(t+\frac{\tau}{2}\right) x^*\left(t-\frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \tag{1.1}$$

is a quadratic time-frequency representation of a signal $x(t)$, which can be interpreted (with some restrictions due to the uncertainty principle) as a time-frequency distribution of the signal's energy. [In (1.1) and subsequent equations, $t$ and $f$ denote time and frequency, respectively, integrations are from $-\infty$ to $\infty$, and the superscript * denotes complex conjugation.]

In many fields, it is desirable to implement the design or processing of signals in a joint time-frequency domain. The WD can be applied to time-frequency signal design and signal processing by making use of WD-based signal synthesis techniques [2]. This method has been applied to the design of windows and filter impulse responses and to the separation of the components of seismic signals [2], [3]. Unfortunately, in many cases the method's performance is adversely affected by the occurrence of interference terms (IT's) [4]–[6] in the WD. The present paper analyzes IT effects in WD-based signal synthesis, and shows that these effects can be essentially avoided by using a smoothed Wigner distribution (SWD) instead of the WD.

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By way of introduction, we first review WD-based signal synthesis and its application to signal design and signal separation, concentrating on typical situations where IT effects are evident. A formal analysis of IT effects is then provided in Section II. Section III discusses signal synthesis based on the SWD, and presents an iterative signal synthesis algorithm for the SWD. Simulation results demonstrating the advantage of SWD-based signal synthesis over WD-based signal synthesis are finally given in Section IV.

### A. Review of WD-Based Signal Synthesis

WD-based signal synthesis [2], [3], [7], [8] is the generation of a signal from a real-valued square-integrable time-frequency "model function" $\tilde{W}(t,f)$, which, in general, is not a valid WD outcome. The signal is chosen such that its WD is closest to the model $\tilde{W}(t,f)$ in a least-square sense

$$x_{\text{opt}}(t) \triangleq \arg\min_x \epsilon_x \tag{1.2}$$

where the synthesis error $\epsilon_x$ to be minimized is defined as

$$\epsilon_x^2 \triangleq \|\tilde{W} - W_x\|^2 = \int_t \int_f [\tilde{W}(t,f) - W_x(t,f)]^2\, dt\, df. \tag{1.3}$$

The solution to the WD-based signal synthesis problem (1.2) is given by [2], [8]

$$x_{\text{opt}}(t) = \begin{cases} e^{j\varphi}\sqrt{\lambda_1}\, u_1(t), & \text{if } \lambda_1 > 0 \\ 0, & \text{if } \lambda_1 \le 0 \end{cases} \tag{1.4}$$

where $\varphi$ is an arbitrary constant phase and $\lambda_1$ and $u_1(t)$ are, respectively, the largest eigenvalue and the corresponding (normalized) eigenfunction of a kernel $\tilde{q}(t_1, t_2)$, which is derived from the model $\tilde{W}(t,f)$ as

$$\tilde{q}(t_1, t_2) = \tilde{c}\left(\frac{t_1+t_2}{2}, t_1 - t_2\right), \quad \text{with}$$

$$\tilde{c}(t,\tau) = \int_f \tilde{W}(t,f) e^{j2\pi\tau f}\, df. \tag{1.5}$$

If $\tilde{W}(t,f)$ is a valid WD of a signal $x(t)$, $\tilde{W}(t,f) = W_x(t,f)$, then $\tilde{q}(t_1, t_2) = x(t_1)x^*(t_2)$. We note, for later use, that the transformation (1.5) mapping $\tilde{W}(t,f)$ into

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$\tilde{q}(t_1, t_2)$ can be inverted as [cf. (1.1)]

$$\tilde{W}(t, f) = \int_\tau \tilde{q}\left(t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau. \quad (1.6)$$

The eigenvalues $\lambda_k$ and eigenfunctions $u_k(t)$ are the solutions to the eigenequation

$$\int_{t_2} \tilde{q}(t_1, t_2) u_k(t_2)\, dt_2 = \lambda_k u_k(t_1), \qquad k = 1, 2, \cdots.$$

Since $\tilde{q}^*(t_2, t_1) = \tilde{q}(t_1, t_2)$ for $\tilde{W}(t, f) \in \mathbb{F}$, the eigenvalues $\lambda_k$ are real-valued and the eigenfunctions $u_k(t)$ are orthonormal.

Using the eigendecomposition of the kernel $\tilde{q}(t_1, t_2)$

$$\tilde{q}(t_1, t_2) = \sum_{k=1}^{\infty} \lambda_k u_k(t_1) u_k^*(t_2)$$

and applying (1.6), we obtain an "eigendecomposition" of the model $\tilde{W}(t, f)$

$$\tilde{W}(t, f) = \sum_{k=1}^{\infty} \lambda_k W_{u_k}(t, f). \quad (1.7)$$

This shows that any real-valued square-integrable time-frequency function can be written as a linear combination of valid WD's of orthonormal signals. Because of (1.7), we shall call the $\lambda_k$ and $u_k(t)$ the "eigenvalues and eigensignals of the model $\tilde{W}(t, f)$." The signal synthesis result (1.4) uses only the largest eigenvalue $\lambda_1$ and the corresponding eigensignal $u_1(t)$; these can be calculated selectively by means of the iterative power algorithm (see Section III).

The following two subsections illustrate the application of WD-based signal synthesis to signal design and signal separation, respectively. They also indicate potential problems caused by the occurrence of IT's in the WD.

### B. Application to Signal Design

For a time-frequency design of a signal, we construct the model $\tilde{W}(t, f)$ such that it reflects the time-frequency energy distribution desired. We then generate the signal by means of WD-based signal synthesis, as discussed previously.

Let us apply this strategy to design an $N$-component signal

$$x(t) = \sum_{k=1}^{N} x_k(t). \quad (1.8)$$

The WD of $x(t)$ is given by

$$W_x(t, f) = \sum_{k=1}^{N} W_{x_k}(t, f) + \sum_{\substack{k=1 \\ (l>k)}}^{N} \sum_{l=1}^{N} 2 \operatorname{Re}\{W_{x_k, x_l}(t, f)\} \quad (1.9)$$

where

$$W_{x_k, x_l}(t, f) = \int_\tau x_k\left(t + \frac{\tau}{2}\right) x_l^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau$$

is the cross-WD of the signal components $x_k(t)$ and $x_l(t)$. Equation (1.9) shows that the WD of an $N$-component signal consists of $N$ "signal terms" $W_{x_k}(t, f)$ and $\binom{N}{2} = N(N-1)/2$ "interference terms" (IT's) $2 \operatorname{Re}\{W_{x_k, x_l}(t, f)\}$ (where $l > k$). We assume that the signal components $x_k(t)$ are (approximately) time-frequency disjoint in the sense that their WD's do not overlap significantly, i.e.,

$$W_{x_k}(t, f) W_{x_l}(t, f) \approx 0 \qquad \text{for all } t, f \text{ and } k \neq l. \quad (1.10)$$

With this disjointness assumption, the IT's can be shown to be oscillatory structures whose contribution to signal energy (as measured by their time-frequency integrals) is approximately zero [4], [6]. Fig. 1(a) shows the WD of a simple two-dimensional signal with time-frequency disjoint signal components.

Obviously, for WD-based signal design, the model $\tilde{W}(t, f)$ must be constructed such that it resembles the WD (1.9) of the desired signal. Here, the main difficulty lies in the approximate construction of the IT's $2 \operatorname{Re}\{W_{x_k, x_l}(t, f)\}$ since the IT's are complicated oscillatory functions. Therefore, we would like to use a "signal-terms-only" model in which the IT's are simply omitted, i.e.,

$$\tilde{W}(t, f) \approx \sum_{k=1}^{N} W_{x_k}(t, f). \quad (1.11)$$

This choice of model is justified to some extent since, after all, the signal's energy is contained in the signal terms only. However, it is now doubtful if the result of signal synthesis is the desired $N$-component signal (1.8): if it were, then the residual synthesis error $\epsilon_x$ [cf. (1.3)] would be large since the model does not contain any IT's whereas the WD does. This seems to contradict the fact that, by definition, the signal synthesis result (1.2) achieves the minimum synthesis error. More about this will be said in Section II.

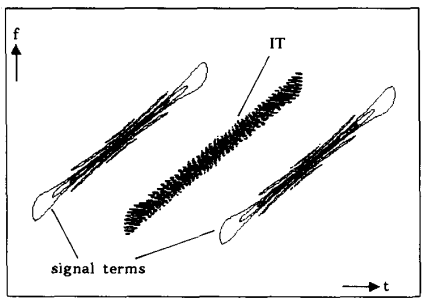
### C. Application to Signal Separation

Time-frequency signal processing can be performed by modifying the WD outcome of the input signal $x(t)$ and then synthesizing the output signal $y(t)$ from the modified WD outcome (the model) $\tilde{W}(t, f)$.

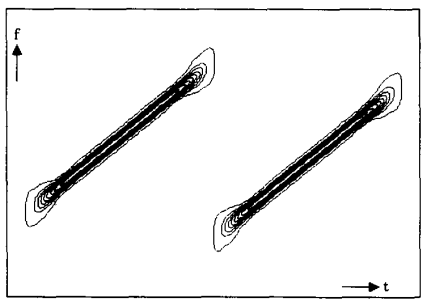
A typical application of this scheme is signal separation [2], [3]. We assume that the input signal $x(t)$ is $N$-component with (approximately) time-frequency disjoint components [see (1.8), (1.10)]. The WD of $x(t)$ is given by (1.9). We seek to isolate some signal component $x_i(t)$ by applying a mask $M_i(t, f)$ to the overall WD outcome such that, after masking, only the corresponding signal term $W_{x_i}(t, f)$ is retained:

$$\tilde{W}(t, f) = M_i(t, f) W_x(t, f) \approx W_{x_i}(t, f).$$

Note that, due to the disjointness assumption (1.10), we can always find a mask $M_i(t, f)$ such that the unwanted signal terms $W_{x_n}(t, f)$ $(n \neq i)$ are masked out. However, it may happen that the IT $2 \operatorname{Re}\{W_{x_m, x_n}(t, f)\}$ correspond-




(a)




(b)

Fig. 1. Two-component signal: (a) WD; and (b) SWD.

ing to two other signal components $x_m(t)$ and $x_n(t)$ is superimposed on the signal term $W_{x_i}(t, f)$. Since IT's are always located midway between the corresponding signal terms [cf. Fig. 1(a)], this situation will occur whenever $W_{x_i}(t, f)$ is located midway between $W_{x_m}(t, f)$ and $W_{x_n}(t, f)$ in the time-frequency plane. Evidently, masking is here incapable of suppressing the parasitic IT 2 Re $\{W_{x_m, x_n}(t, f)\}$. Assuming for simplicity that the IT lies entirely inside the mask $M_i(t, f)$, the model now is

$$\tilde{W}(t, f) = M_i(t, f) W_x(t, f) \approx W_{x_i}(t, f) + 2 \operatorname{Re} \{W_{x_m, x_n}(t, f)\}. \quad (1.12)$$

This model significantly deviates from the WD $W_{x_i}(t, f)$; hence, it is possible that the result of signal synthesis is quite different from the desired signal component $x_i(t)$. This issue will be considered further in Section II.

## II. Analysis of Interference Effects in WD-Based Signal Synthesis

Reconsidering the two examples discussed in the previous section, we see that the models $\tilde{W}(t, f)$ of (1.11) and (1.12) feature a large deviation from a valid WD (in particular, from the WD of the signal we want to synthesize). Specifically, the model (1.11) has *too few* IT's, whereas the model (1.12) has one IT *too many*. These two situations can be generalized by defining the "weighted model"

$$\tilde{W}(t, f) = \sum_{k=1}^{N} \alpha_{kk} W_{x_k}(t, f) + \sum_{\substack{k=1 \\ (l>k)}}^{N} \sum_{l=1}^{N} \alpha_{kl} \, 2 \operatorname{Re} \{W_{x_k, x_l}(t, f)\} \quad (2.1)$$

where the weighting factors $\alpha_{kl}$ are 0 or 1. Evidently, the model $\tilde{W}(t, f)$ deviates from the valid WD of the $N$-component signal as given by (1.9) in that certain signal terms and/or IT's are suppressed ($\alpha_{kl} = 0$) while others are retained ($\alpha_{kl} = 1$). Thus, the weighted model provides a general framework for studying IT effects in WD-based signal synthesis.

### A. Signal Synthesis From the Weighted Model

In the following, we derive the signal synthesis result for the weighted model (2.1) [9]. Our analysis will be valid for arbitrary real-valued weighting factors $\alpha_{kl}$. (Although we are here interested in the special case where $\alpha_{kl} = 0$ or 1, we note that weighting factors with values between 0 and 1 may be useful for approximating the attenuation of IT's caused by a WD smoothing.)

Using elementary properties of the WD [1], we can rewrite (2.1) as

$$\tilde{W}(t, f) = \sum_{k=1}^{N} \sum_{l=1}^{N} \beta_{kl} W_{e_k, e_l}(t, f) \quad (2.2)$$

where the $e_k(t)$ are normalized versions of the signal components $x_k(t)$, $e_k(t) = x_k(t)/\|x_k\|$, and the new weighting factors are

$$\beta_{kl} = \alpha_{kl} \|x_k\| \, \|x_l\|$$

with $\beta_{kl} \in \mathbb{R}$ and $\beta_{lk} = \beta_{kl}$. Applying the transformation (1.5) to (2.2) yields

$$\tilde{q}(t_1, t_2) = \sum_{k=1}^{N} \sum_{l=1}^{N} \beta_{kl} e_k(t_1) e_l^*(t_2) = \boldsymbol{e}^+(t_2) \boldsymbol{B} \boldsymbol{e}(t_1). \quad (2.3)$$

Here, the column vector $\boldsymbol{e}(t)$ and the matrix $\boldsymbol{B}$ are defined by $(\boldsymbol{e}(t))_k = e_k(t)$ and $(\boldsymbol{B})_{kl} = \beta_{kl}$, respectively, and "$^+$" denotes complex conjugate transposition. Since the matrix $\boldsymbol{B}$ is symmetric, it allows a spectral decomposition

$$\boldsymbol{B} = \sum_{k=1}^{N} \lambda_k \boldsymbol{c}_k \boldsymbol{c}_k^+ = \boldsymbol{C} \boldsymbol{\Lambda} \boldsymbol{C}^+ \quad (2.4)$$

where $\lambda_k$ and $\boldsymbol{c}_k$ denote, respectively, the real-valued eigenvalues and the orthonormal eigenvectors of $\boldsymbol{B}$, $\boldsymbol{\Lambda}$ is the diagonal eigenvalue matrix, and $\boldsymbol{C}$ is the orthogonal matrix whose columns are the eigenvectors $\boldsymbol{c}_k$. Insertion of (2.4) into (2.3) results in a diagonalization of the bilinear form (2.3)

$$\tilde{q}(t_1, t_2) = \boldsymbol{e}^+(t_2) \boldsymbol{C} \boldsymbol{\Lambda} \boldsymbol{C}^+ \boldsymbol{e}(t_1) = \boldsymbol{u}^+(t_2) \boldsymbol{\Lambda} \boldsymbol{u}(t_1) = \sum_{k=1}^{N} \lambda_k u_k(t_1) u_k^*(t_2) \quad (2.5)$$

with the transformed signal vector

$$\boldsymbol{u}(t) = \boldsymbol{C}^{+}\boldsymbol{e}(t), \qquad u_k(t) = \sum_{l=1}^{N} (\boldsymbol{c}_k)_l e_l(t). \tag{2.6}$$

Applying the transformation (1.6) to (2.5), we finally obtain the following expression for the model $\tilde{W}(t, f)$

$$\tilde{W}(t, f) = \sum_{k=1}^{N} \lambda_k W_{u_k}(t, f). \tag{2.7}$$

This is a diagonalized version of the original model expression (2.2). Comparing with (1.7), we notice that (2.7) is, in fact, the model's eigendecomposition if the signals $u_k(t)$ are orthonormal. Let us assume that, idealizing, the signal components $x_k(t)$ are perfectly time-frequency disjoint such that the equality (1.10) is exact, $W_{x_k}(t, f) W_{x_l}(t, f) \equiv 0$ for $k \neq l$. We then obtain

$$(W_{x_k}, W_{x_l}) = \int_t \int_f W_{x_k}(t, f) W_{x_l}(t, f)\, dt\, df = 0, \quad \text{for } k \neq l$$

and from Moyal's formula [1, part I]

$$(W_{x_k}, W_{x_l}) = |(x_k, x_l)|^2$$

it follows that the signals $x_k(t)$ and $x_l(t)$ are orthogonal for $k \neq l$. Hence the normalized signal versions $e_k(t)$ will be orthonormal, and it is finally easily shown that the transformed signals $u_k(t)$ are orthonormal as well since they are derived from the $e_k(t)$ via the *orthogonal* matrix $\boldsymbol{C}^{+}$ [cf. (2.6)].

We have thus verified that (2.7) is indeed the eigendecomposition (1.7) of the weighted model $\tilde{W}(t, f)$. The following conclusions can now be drawn:

i) The "rank" of the weighted model $\tilde{W}(t, f)$ [see (2.7)] may not exceed the number $N$ of signal components $x_k(t)$.

ii) The $N$ nonzero eigenvalues $\lambda_k$ of $\tilde{W}(t, f)$ equal the eigenvalues of the matrix $\boldsymbol{B}$, which depends only on the model's weighting factors $\alpha_{kl}$ and the signal components' norms $\|x_k\|$.

iii) From (2.6), the eigensignals $u_k(t)$ of $\tilde{W}(t, f)$ are seen to be linear combinations of the original signal components $x_k(t)$

$$u_k(t) = \sum_{l=1}^{N} d_{kl} x_l(t), \qquad \text{with } d_{kl} = \frac{(\boldsymbol{c}_k)_l}{\|x_l\|} \tag{2.8}$$

where the coefficients $d_{kl}$ depend only on the weighting factors $\alpha_{kl}$ and the signal components' norms $\|x_k\|$.

iv) Assuming $\lambda_1 \geq 0$, the result of WD-based signal synthesis from the weighted model $\tilde{W}(t, f)$ is given by (1.4) as

$$x_{\text{opt}}(t) = e^{j\varphi} \sqrt{\lambda_1}\, u_1(t).$$

With (2.8), this becomes a linear combination of the signal components $x_k(t)$

$$x_{\text{opt}}(t) = \sum_{k=1}^{N} \gamma_k x_k(t), \qquad \text{with } \gamma_k = \frac{e^{j\varphi} \sqrt{\lambda_1}\, (\boldsymbol{c}_1)_k}{\|x_k\|} \tag{2.9}$$

where the coefficients $\gamma_k$ again depend only on the model's weighting factors $\alpha_{kl}$ and the signal components' norms $\|x_k\|$.

### B. Signal-Terms-Only Model

We now specialize our results to the two situations discussed in Section I.

The "signal-terms-only" model (1.11) corresponds to weighting factors $\alpha_{kl} = \delta_{kl}$, so that $\boldsymbol{B}$ is a diagonal matrix with elements $\beta_{kl} = \|x_k\|^2 \delta_{kl}$. Thus, the eigenvalues of $\boldsymbol{B}$ are simply the energies of the signal components, $\lambda_k = \|x_k\|^2$, and the eigenvectors are the unit vectors, $(\boldsymbol{c}_k)_l = \delta_{kl}$. Inserting into (2.9), the signal synthesis result is here obtained as

$$x_{\text{opt}}(t) = e^{j\varphi} x_1(t) \tag{2.10}$$

where $x_1(t)$ denotes the signal component with maximum energy. Therefore, instead of the $N$-component signal $x(t) = \sum_{k=1}^{N} x_k(t)$ that we would like to design, signal synthesis from the "signal-terms-only" model (1.11) merely yields the single signal component $x_1(t)$ with maximum energy.

### C. Model with Parasitic IT

In the case of the model (1.12) consisting of one signal term $W_{x_i}(t, f)$ and a parasitic IT 2 Re $\{W_{x_m, x_n}(t, f)\}$, all elements of $\boldsymbol{B}$ are zero except $\beta_{ii} = \|x_i\|^2$ and $\beta_{mn} = \beta_{nm} = \|x_m\| \|x_n\|$. It is easily shown that there exist three nonzero eigenvalues $\lambda = \|x_i\|^2$, $\|x_m\| \|x_n\|$, and $-\|x_m\| \|x_n\|$, with the corresponding eigenvectors given by $(\boldsymbol{c})_k = \delta_{ki}$, $(\delta_{km} + \delta_{kn})/\sqrt{2}$, and $(\delta_{km} - \delta_{kn})/\sqrt{2}$, respectively. Since the signal synthesis result is controlled by the largest eigenvalue, the following two cases must be distinguished:

i) If $\|x_i\|^2 > \|x_m\| \|x_n\|$, then the largest eigenvalue is $\lambda_1 = \|x_i\|^2$ and the corresponding eigenvector is $(\boldsymbol{c}_1)_k = \delta_{ki}$. From (2.9), the signal synthesis result here follows as

$$x_{\text{opt}}(t) = e^{j\varphi} x_i(t)$$

which is the desired signal component up to a constant phase factor.

ii) If $\|x_i\|^2 < \|x_m\| \|x_n\|$, then the largest eigenvalue is $\lambda_1 = \|x_m\| \|x_n\|$ and the corresponding eigenvector is $(\boldsymbol{c}_1)_k = (\delta_{km} + \delta_{kn})/\sqrt{2}$. Inserting into (2.9), we obtain

$$x_{\text{opt}}(t) = e^{j\varphi} \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{\|x_n\|}{\|x_m\|}}\, x_m(t) + \sqrt{\frac{\|x_m\|}{\|x_n\|}}\, x_n(t) \right] \tag{2.11}$$

which is obviously not the desired signal.

These results show the existence of an interesting threshold effect: if the energy of the desired signal component is larger than the geometric mean of the energies of the interfering signal components, then WD-based signal synthesis will indeed yield the desired signal component. In the opposite case, however, the signal synthesis result is a linear combination of the *interfering* signal

components, and the masking-synthesis strategy fails completely.

## III. Signal Synthesis Based on Smoothed Wigner Distributions

In many cases, the IT effects discussed previously can be avoided by suitable extensions of WD-based signal synthesis [3]. For example, an $N$-component signal may be designed by applying WD-based signal synthesis to the individual WD signal term models and, thus, synthesizing the signal components separately. We then need $N$ synthesis procedures instead of one. Note, however, that this approach cannot be used for avoiding IT effects when designing a monocomponent signal (cf. Fig. 4). In the case of signal separation, we may start by masking a signal term *not* overlaid by an IT, synthesize the corresponding signal component, subtract it from the overall $N$-component signal, calculate the WD of the resulting $(N-1)$-component signal, and proceed by masking the next signal term, etc. [3], [10]. Alternatively, signal separation may also be performed by applying an iterative cross-WD synthesis technique using a reference signal which is improved in each step [3], [10], [11].

Here, we propose an alternative synthesis method that features a "built-in" immunity to IT effects. Clearly, the troublesome interference effects studied in the previous sections would be avoided if we based the formulation of signal synthesis not on the WD but on some other time-frequency representation in which IT's do not occur. In the case of signal design, we could then use the "signal-terms-only" model without incurring any penalty due to the absence of IT's in our model. In the case of signal separation, there would not be parasitic IT's falling inside our mask.

### A. Smoothed Versions of the WD

While a quadratic (i.e., energy-distribution type) time-frequency representation without IT's evidently does not exist, we can use a *smoothed version of the WD* (SWD) which features an attenuation of IT's as compared to the WD [5], [6]. This attenuation is due to the fact that IT's are oscillatory and smoothing amounts to a low-pass filtering. The attenuation of an IT in the SWD will be larger when the corresponding signal terms are more distant in the time-frequency plane and/or when the smoothing in the SWD is stronger. Fig. 1(b) shows an SWD for the signal of Fig. 1(a); we notice that the IT is here essentially suppressed.

In this section, we consider SWD-based signal synthesis and present an iterative signal-synthesis algorithm for the SWD [7], [12], [13]. With a view toward practical implementation, *we now use a discrete-time formulation*.

An SWD is obtained by convolving the WD with a time-frequency smoothing function. An equivalent "time-domain" formulation of the discrete-time SWD is [7]

$$W_x^{(\varphi)}(n, \theta) = 2 \sum_m c_x^{(\varphi)}(n, m) e^{-j4\pi\theta m} \tag{3.1a}$$

where

$$c_x^{(\varphi)}(n, m) = \sum_{n'} \varphi(n - n', m) x(n' + m) x^*(n' - m). \tag{3.1b}$$

Here, $x(n)$ is a discrete-time signal, $n$ is a discrete time index, and $\theta$ is a normalized frequency variable. We note that the SWD is $\frac{1}{2}$-periodic with respect to $\theta$. The kernel $\varphi(n, m)$ must be chosen such that its Fourier transform with respect to $m$ (the WD smoothing function) is real valued and smooth. Specific choices of $\varphi(n, m)$ yield well-known SWD's such as the pseudo-WD [1, part II], the smoothed pseudo-WD [5], [6], [14], the Choi–Williams distribution [15], the Born–Jordan distribution [16], or the cone-kernel representation [17]. Formally, also the discrete-time WD [1, part II] can be obtained from (3.1) by letting $\varphi(n, m) = \delta(n)$, where $\delta(n)$ is the unit sample.

### B. SWD-Based Signal Synthesis

The SWD-based signal synthesis problem reads

$$x_{\text{opt}}(n) = \arg\min_x \epsilon_x^{(\varphi)} \tag{3.2}$$

where the synthesis error to be minimized is now defined in terms of the SWD,

$$\begin{aligned} \epsilon_x^{(\varphi)2} &\triangleq \|\tilde{W} - W_x^{(\varphi)}\|^2 \\ &= \sum_n \int_{-1/4}^{1/4} [\tilde{W}(n, \theta) - W_x^{(\varphi)}(n, \theta)]^2 \, d\theta. \end{aligned} \tag{3.3}$$

It is not difficult to derive a necessary condition for $x_{\text{opt}}(n)$ [7], [12], [13]. Using Parseval's theorem, $\epsilon_x^{(\varphi)2}$ can be reformulated as

$$\epsilon_x^{(\varphi)2} = 2 \|\tilde{c} - c_x^{(\varphi)}\|^2 = 2 \sum_n \sum_m |\tilde{c}(n, m) - c_x^{(\varphi)}(n, m)|^2$$

where [cf. (1.5)]

$$\tilde{c}(n, m) = \int_{-1/4}^{1/4} \tilde{W}(n, \theta) e^{j4\pi m\theta} \, d\theta.$$

Setting the derivatives of $\epsilon_x^{(\varphi)2}$ with respect to the signal samples $x(n)$ equal to zero, we obtain the following set of equations as a necessary condition for $x_{\text{opt}}(n)$:

$$\sum_m x(n - 2m) C_x(n - m, m) = 0, \qquad \text{for all } n \tag{3.4}$$

with

$$C_x(n, m) \triangleq \sum_{n'} \varphi^*(n' - n, m) [\tilde{c}(n', m) - c_x^{(\varphi)}(n', m)]. \tag{3.5}$$

We can separate these equations into a subset for $n$ even $(n = 2k)$ and a subset for $n$ odd $(n = 2k + 1)$. This gives

$$\boldsymbol{Q}'_{e,x} \boldsymbol{x}_e = \mathbf{o}, \qquad \boldsymbol{Q}'_{o,x} \boldsymbol{x}_o = \mathbf{o} \tag{3.6}$$

where the matrices $\boldsymbol{Q}'_{e,x}$ and $\boldsymbol{Q}'_{o,x}$ are defined as

$$(\boldsymbol{Q}'_{e,x})_{kl} = C_x(k + l, k - l), \tag{3.7a}$$

$$(\boldsymbol{Q}'_{o,x})_{kl} = C_x(k + l + 1, k - l) \tag{3.7b}$$

and $\boldsymbol{x}_e$ and $\boldsymbol{x}_o$ denote the vectors of even- and odd-indexed signal samples, respectively, $(\boldsymbol{x}_e)_k = x(2k)$, and $(\boldsymbol{x}_o)_k = x(2k + 1)$. Note that, in general, both the matrices $\boldsymbol{Q}'_{e,x}$ and $\boldsymbol{Q}'_{o,x}$ contain all signal samples $x(n)$; hence, (3.6) does not imply that even- and odd-indexed signal samples can be synthesized independently of each other. It is easily shown that $\boldsymbol{Q}'_{e,x}$ and $\boldsymbol{Q}'_{o,x}$ will be hermitian for a real-valued model $\tilde{W}(n, \theta)$.

### C. The Quasi-Power Algorithm

The necessary condition (3.4) or, equivalently, (3.6) is a set of third-order equations that apparently do not allow a closed-form solution. Hence we develop an iterative algorithm for solving (3.6).

Since the two equations of (3.6) are analogous, we only consider the "even" equation in the following. Adding $\|\boldsymbol{x}_e\|^2\boldsymbol{x}_e$ to both sides (where $\|\cdot\|$ denotes the Frobenius matrix norm), we obtain

$$\boldsymbol{Q}_{e,x}\boldsymbol{x}_e = \lambda_e\boldsymbol{x}_e, \qquad \text{where } \lambda_e = \|\boldsymbol{x}_e\|^2 = \sum_k |x(2k)|^2 \tag{3.8}$$

with the hermitian matrix

$$\boldsymbol{Q}_{e,x} \triangleq \boldsymbol{Q}'_{e,x} + \boldsymbol{x}_e\boldsymbol{x}_e^+. \tag{3.9}$$

At this point, we temporarily consider the special case of the WD given by $\varphi(n, m) = \delta(n)$. Here, $\boldsymbol{Q}_{e,x}$ is easily shown to reduce to $(\boldsymbol{Q}_{e,x})_{kl} = \tilde{c}(k + l, k - l)$ and thus does not depend on the signal $x(n)$, $\boldsymbol{Q}_{e,x} = \boldsymbol{Q}_e$. Hence, (3.8) reduces to the eigenequation

$$\boldsymbol{Q}_e\boldsymbol{x}_e = \lambda_e\boldsymbol{x}_e, \qquad \text{where } \lambda_e = \|\boldsymbol{x}_e\|^2.$$

It is well known [2] [cf. (1.4)] that the signal synthesis result in the WD case is $x_{\text{opt}}(n) = e^{j\varphi}\sqrt{\lambda_{e,1}}\,\boldsymbol{u}_{e,1}$ where $\lambda_{e,1}$ is the largest eigenvalue of $\boldsymbol{Q}_e$ ($\lambda_{e,1}$ has been assumed nonnegative), $\boldsymbol{u}_{e,1}$ is the corresponding normalized eigenvector, and $\varphi$ is an arbitrary phase constant. Both $\lambda_{e,1}$ and $\boldsymbol{u}_{e,1}$ can be calculated iteratively by means of the *power algorithm* [18]: for $i = 1, 2, \cdots$,

$$\text{a)}\quad \boldsymbol{v}_e^{(i)} = \boldsymbol{Q}_e\boldsymbol{u}_e^{(i-1)} \tag{3.10a}$$

$$\text{b)}\quad \lambda_e^{(i)} = \|\boldsymbol{v}_e^{(i)}\| \tag{3.10b}$$

$$\text{c)}\quad \boldsymbol{u}_e^{(i)} = \boldsymbol{v}_e^{(i)}/\lambda_e^{(i)}. \tag{3.10c}$$

If $\lambda_{e,1} > |\lambda_{e,k}|$ for $k \geq 2$, then (for largely arbitrary start vector $\boldsymbol{u}_e^{(0)}$) $\lambda_e^{(i)}$ and $\boldsymbol{u}_e^{(i)}$ are guaranteed to converge toward $\lambda_{e,1}$ and $\boldsymbol{u}_{e,1}$, respectively.

After this excursion to the special case of the WD, we now return to the general SWD case (3.8). Unfortunately, the necessary-condition equation (3.8) is, in general, no longer an eigenequation since $\boldsymbol{Q}_{e,x}$ depends on the signal $x(n)$. In this situation, it seems to be natural to apply a modification of the power algorithm where the matrix $\boldsymbol{Q}_{e,x}$ is updated at each iteration using the current iteration signal. The $i$th iteration of the resulting "quasi-power algorithm" (QPA) is formulated as follows:

1) Using the signal $x^{(i-1)}(n)$ calculated at the previous iteration, form the matrices $\boldsymbol{Q}_{e,x(i-1)}$ and $\boldsymbol{Q}_{o,x(i-1)}$ according to (3.9), (3.7), (3.5), and (3.1).

2) Perform the three steps of the power algorithm (3.10):

$$\text{a)}\quad \boldsymbol{v}_e^{(i)} = \boldsymbol{Q}_{e,x(i-1)}\boldsymbol{u}_e^{(i-1)} \tag{3.11a}$$

$$\text{b)}\quad \lambda_e^{(i)} = \|\boldsymbol{v}_e^{(i)}\| \tag{3.11b}$$

$$\text{c)}\quad \boldsymbol{u}_e^{(i)} = \boldsymbol{v}_e^{(i)}/\lambda_e^{(i)} \tag{3.11c}$$

with analogous operations to obtain $\lambda_o^{(i)}$ and $\boldsymbol{u}_o^{(i)}$.

3) Form the signal $x^{(i)}(n)$ as

$$x^{(i)}(n) = \begin{cases} \sqrt{\lambda_e^{(i)}}(\boldsymbol{u}_e^{(i)})_k, & n = 2k \\ \sqrt{\lambda_o^{(i)}}(\boldsymbol{u}_o^{(i)})_k, & n = 2k + 1. \end{cases} \tag{3.12}$$

Upon convergence ($i \to \infty$), the synthesis result $x(n)$ is finally obtained as

$$\begin{aligned} x(2k) &= e^{j\varphi_e}x^{(\infty)}(2k), \\ x(2k + 1) &= e^{j\varphi_o}x^{(\infty)}(2k + 1) \end{aligned} \tag{3.13}$$

where $\varphi_e$ and $\varphi_o$ are arbitrary phase constants (these can be determined using specific phase-matching algorithms [2], [19]).

### D. Properties of the QPA

A partial justification of the QPA is given by the following two properties [7], [13]:

1) In the special case of the WD (i.e., $\varphi(n, m) = \delta(n)$), the QPA reduces to the conventional power algorithm (3.10), which is guaranteed to converge and produce the optimal signal.

2) If the QPA converges, then the resulting signal $x(n)$ is guaranteed to satisfy the necessary-condition equation (3.4).

While the convergence of the QPA has not been proven theoretically, it has always been observed in experiments. The convergence speed of the QPA depends mainly on the start signal [$\boldsymbol{u}_e^{(0)}$ and $\boldsymbol{u}_o^{(0)}$] used and on the amount of smoothing in the SWD. We briefly discuss this dependence in the following:

i) Convergence of the QPA will be slow if the start signal assumes most of its energy outside the model's essential time-frequency support. This property of the QPA conforms to an analogous property of the power algorithm in the WD case: here, a start signal that is time-frequency disjoint with the model will be orthogonal to the dominant eigenvectors $\boldsymbol{u}_{e,1}$ and $\boldsymbol{u}_{o,1}$, in which case, theoretically, the power algorithm does not converge. We note that this situation can easily be avoided by using a noise-type (pseudorandom) start signal whose energy will be spread over a large region of the time-frequency plane.

ii) Convergence of the QPA will be slower for an SWD with more smoothing. Thus, better attenuation of IT's in the SWD (achieved by stronger smoothing, i.e., a kernel $\varphi(n, m)$ that is broader with respect to $n$ and/or narrower with respect to $m$) is paid for by slower convergence of the QPA. This behavior of the QPA can easily be explained: since a smoothing of the WD causes a smearing of the signal's time-frequency structure, more smoothing causes the signal synthesis problem (3.2), (3.3) to be increasingly ill-posed.

### E. Half-Band-Constrained Signal Synthesis

In many cases, it is desirable that the result of signal synthesis be a *half-band signal* (i.e., a signal whose spectrum is limited to one-half of the spectral period 1 of discrete-time signals) since only then the signal's SWD will be essentially nonaliased [7], [1, part II]. A half-band-constrained version of the QPA can be developed [7], [12], [13] but is computationally expensive. Therefore we propose a "low-cost" algorithm where the $i$th iteration is as follows [7]:

1) The even-indexed signal samples $x^{(i)}(2k)$ are calculated as usual [see (3.11), (3.12)].

2) The odd-indexed signal samples are derived from the even-indexed samples via the interpolation

$$x^{(i)}(2k+1) = 2 \sum_{k'} h(2k+1-2k')x^{(i)}(2k') \quad (3.14)$$

where

$$h(n) = \frac{1}{2} \operatorname{sinc}\left(\frac{1}{2}n\right) e^{j2\pi\theta_o n}, \text{ with } \operatorname{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

is the impulse response of an idealized half-band filter corresponding to the half-band prescribed (this half-band is specified by its center frequency $\theta_o$; e.g., $\theta_o = \frac{1}{4}$ defines the half-band $[0, \frac{1}{2}]$ of analytic signals).

Due to the interpolation (3.14), it is guaranteed that at each iteration (and thus after convergence as well) the signal $x^{(i)}(n)$ is a half-band signal. Also, since even- and odd-indexed signal samples are strictly coupled by (3.14), the synthesis result features only a single phase ambiguity (affecting the entire signal) instead of the separate phase ambiguities $\varphi_e$, $\varphi_o$ present in (3.13) [19]. Note, however, that the synthesis result is not optimal in any sense. While the algorithm presented is thus heuristic, it has yet been observed to yield satisfactory results (cf. Section IV).

## IV. Simulation Results

In this section, we compare the results of WD- and SWD-based signal synthesis obtained in some simple signal-design and signal-separation experiments. The SWD used is a *smoothed pseudo-WD* [5], [6], [14] defined by the separable kernel $\varphi(n, m) = g(n)h^2(m)$, where $g(n)$ and $h(m)$ are two window functions with lengths $L_g$ and $L_h$, respectively. [Here, we use Hamming windows for $g(n)$ and $h(m)$.] The choice of the window lengths $L_g$ and $L_h$ permits a simple and independent control of the amounts of smoothing in the time and frequency directions, respectively. More smoothing corresponds to a larger $L_g$ and/or a smaller $L_h$. The number of QPA iterations used for each experiment is denoted below by $N_I$.

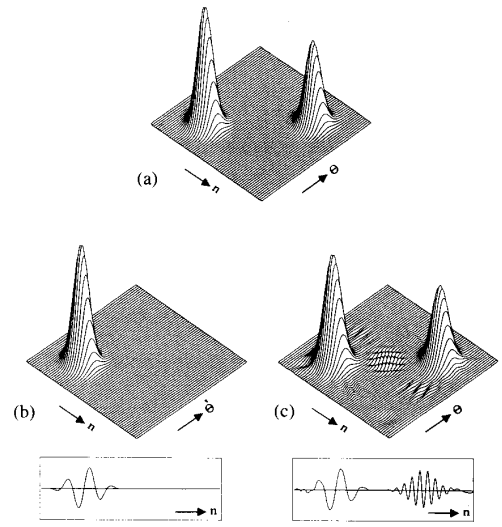



Fig. 2. Design of a two-component signal: (a) model; (b) real part and WD of result of WD-based signal synthesis; and (c) real part and SWD of result of SWD-based signal synthesis.

For SWD-based signal synthesis, the reduced-cost half-band QPA with $\theta_o = \frac{1}{4}$ (corresponding to the half-band of analytic signals) is employed. An analogous half-band version of the power algorithm is used for WD-based signal synthesis. The iterations are initialized by noise signals.

Fig. 2 considers the design of a two-component signal. The model is of the "signal-terms-only" type (1.11). As predicted by (2.10), the result of WD-based signal synthesis is not the desired two-component signal but simply the single signal component with maximum energy. In contrast, the result of SWD-based signal synthesis (with $L_g = 7$, $L_h = 47$, $N_I = 50$) is indeed the desired two-component signal.

Fig. 3 illustrates the convergence of the QPA for the example of Fig. 2. The QPA is intialized by a noise signal whose energy is irregularly spread over the time-frequency plane. With increasing iteration index $i$, the signal's energy becomes more and more concentrated in the model's time-frequency support, the signal's SWD shows increasing similarity to the model, and the synthesis error decreases. A nonzero residual synthesis error remains due to the fact that the model is not a valid SWD.

The design of a *monocomponent* FM signal is considered in Fig. 4. Even in the case of monocomponent signals, the WD often contains an oscillatory "inner" IT [4], [6]. The model deviates from a valid WD in that such an IT is not present; this corresponds to a monocomponent "signal-term-only" model. As a consequence, the

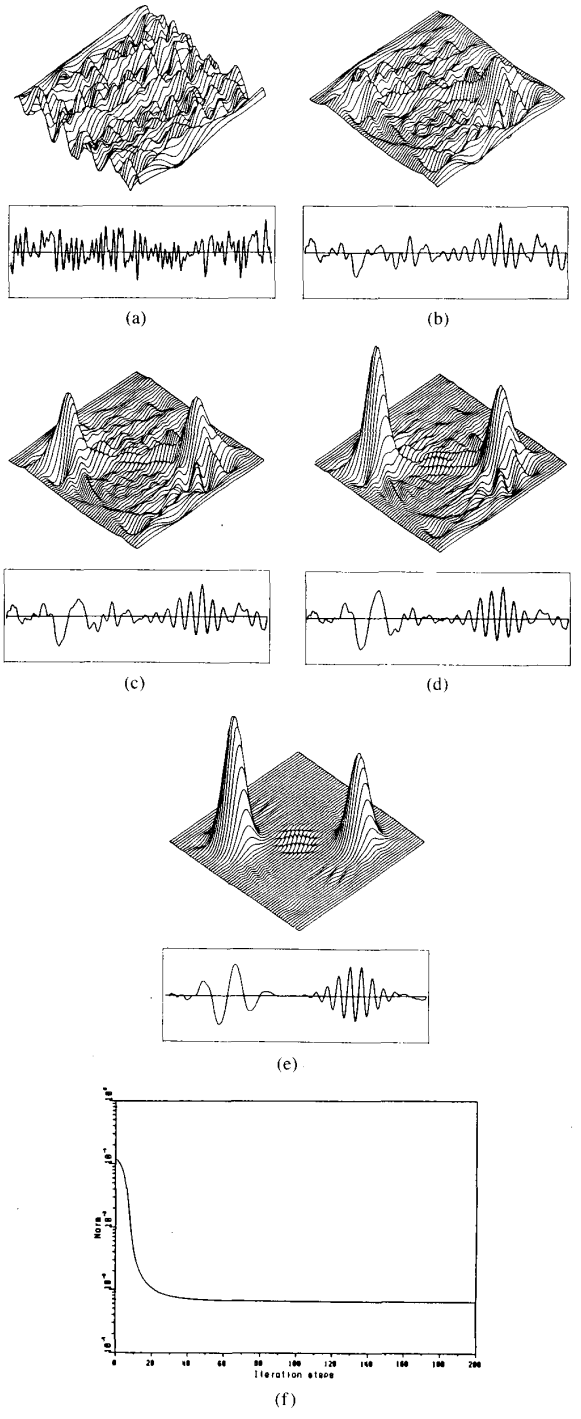



Fig. 3. Convergence of the QPA: (a) real part and SWD of start signal $x^{(0)}(n)$; (b)–(e) real parts and SWD's of iteration signals $x^{(i)}(n)$ for $i = 5$, 7, 10, and 50; and (f) synthesis errors $\epsilon_{x(i)}^{(\varphi)}$ of iteration signals $x^{(i)}(n)$.

result of WD-based signal synthesis is seen to feature an amplitude roll-off, which corresponds to a compromise between the conflicting requirements of i) matching the signal term present in the model, and ii) avoiding an inner IT not present in the model. The result of SWD-based signal synthesis (with $L_g = 7$, $L_h = 47$, $N_I = 100$) does not feature a similar roll-off since, due to the IT attenuation in the SWD, an IT penalty is essentially avoided and the SWD is better able to match the signal-term-only model.

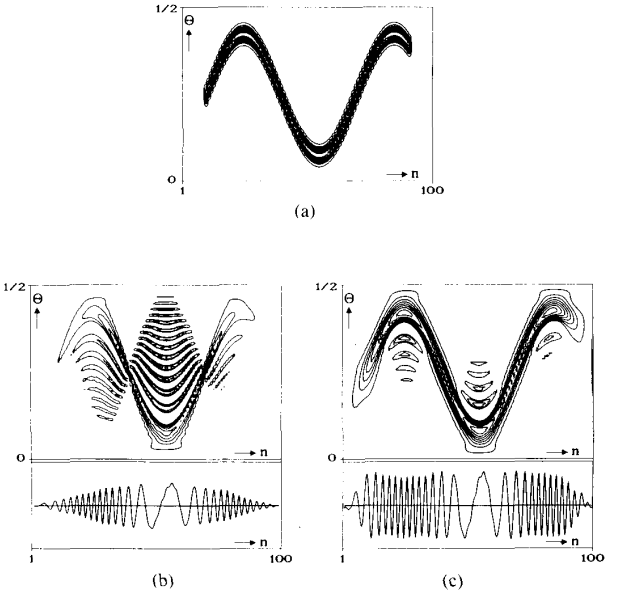



Fig. 4. Design of a monocomponent signal: (a) model; (b) real part and WD of result of WD-based signal synthesis; and (c) real part and SWD of result of SWD-based signal synthesis.

Finally, Figs. 5 and 6 consider a signal-separation problem. The application of WD-based signal synthesis is shown in Fig. 5. The three-component input signal consists of two chirp signals $c_1(t)$ and $c_2(t)$ and a Gaussian signal $g(t)$. In the WD of the input signal, the IT of $c_1(t)$ and $g(t)$ is seen to be superimposed on the signal term of $c_2(t)$. In order to isolate the signal component $c_2(t)$, a mask is applied to the WD. After masking, the IT of $c_1(t)$ and $g(t)$ is still superimposed on the signal term of $c_2(t)$; hence, the result of masking corresponds to the model (1.12) including a parasitic IT. Since the signal components' energies are chosen such that $\|c_1\| \, \|g\| > \|c_2\|^2$, (2.11) applies. As predicted from (2.11), the result of WD-based signal synthesis is indeed seen to be a linear combination of the components $c_1(t)$ and $g(t)$ to be suppressed, rather than the desired component $c_2(t)$.

The application of SWD-based synthesis to the same problem is illustrated in Fig. 6. In the SWD of the input signal, the IT of $c_1(t)$ and $g(t)$ is essentially suppressed due to the smoothing employed. Hence, after masking, the model does not feature a parasitic IT. The result of SWD-based signal synthesis (with $L_g = 7$, $L_h = 63$, $N_I = 200$) is indeed essentially equal to the desired signal component $c_2(t)$.

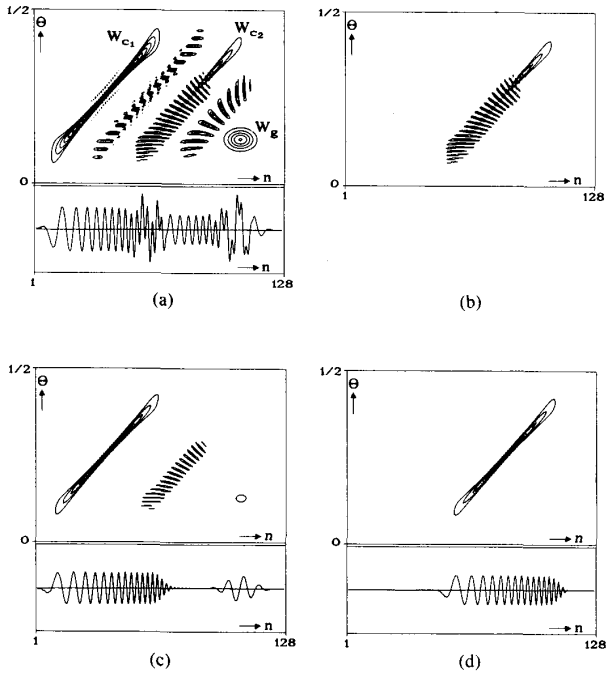



Fig. 5. Signal separation using WD-based synthesis: (a) real part and WD of three-component input signal; (b) result of WD masking; (c) real part and WD of result of WD-based signal synthesis; and (d) real part and WD of desired result.

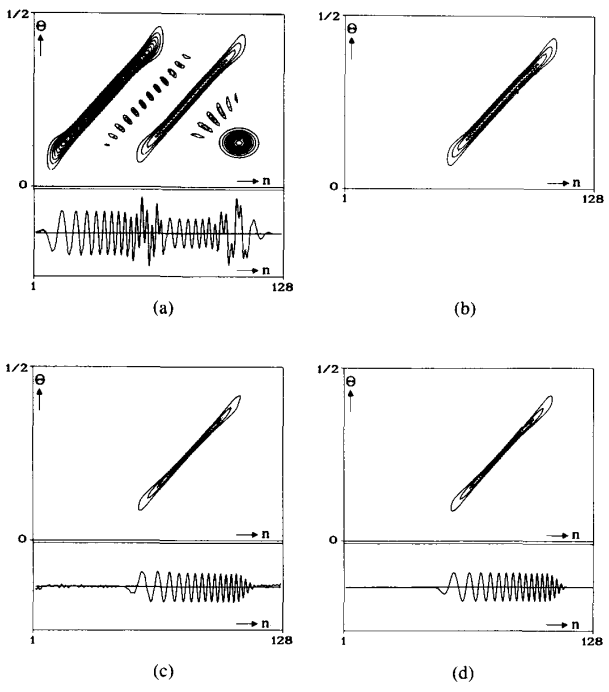



Fig. 6. Signal separation using SWD-based synthesis: (a) real part and SWD of three-component input signal; (b) result of SWD masking; (c) real part and WD of result of SWD-based signal synthesis; and (d) real part and WD of desired result.

## V. Conclusion

It has been shown that the application of WD-based signal synthesis to signal design and signal processing is often hampered by the occurrence of interference terms (IT's) in the WD. In the case of signal design, it is difficult to adequately model the IT's of the signal to be designed; we would hence want to use a "signal-term(s)-only" model without IT's. In the case of signal separation, IT's may be superimposed on the signal term to be isolated; this then results in a model containing parasitic IT's. In both cases, the model used for signal synthesis is "wrong" in that it contains too few (case of signal design) or too many IT's (case of signal separation).

These situations can be generalized by the "weighted model," which provides a (slightly idealized) basis for studying IT effects in WD-based signal synthesis. It has been shown that the signal-synthesis result for the weighted model is a linear combination of the signal components involved, with the coefficients of this linear combination depending on the model's weighting factors and the signal components' energies. In the special case of the "signal-terms-only" model, this linear combination reduces to the signal component with maximum energy. In the case of a model containing a parasitic IT, the linear combination only contains the interfering signal components if, loosely speaking, the interfering components are stronger than the component to be isolated. In both cases, the result of WD-based signal synthesis is drastically different from the desired result.

The IT effects summarized above can be essentially avoided if signal synthesis is based on a smoothed WD (SWD) rather than on the WD. Unfortunately, the solution of SWD-based signal synthesis cannot be reduced to an eigenproblem as in the WD case. It is, however, possible to develop an iterative algorithm for SWD-based signal synthesis. This algorithm is a natural extension of the power algorithm used for WD-based signal synthesis and is consequently termed the "quasi-power algorithm" (QPA). A simple modification of the QPA allows the synthesis of half-band signals whose SWD is essentially nonaliased. Convergence of the QPA will be slower for SWDs with more smoothing; thus, the limiting case of no smoothing (i.e., WD-based signal synthesis using the power algorithm) corresponds to the case of fastest convergence.

Using the QPA, the substantial reduction of IT effects by means of SWD-based signal synthesis has been verified experimentally both for signal design and signal separation applications. It should be noted, however, that increased immunity to IT effects (produced by an SWD with more smoothing) is paid for by slower convergence of the QPA.

SWD-based signal synthesis using the QPA is an offline method since all signal samples are synthesized simultaneously. The time length of the model $\tilde{W}(n, \theta)$ equals the time length of the synthesized signal $x(n)$ and

also determines the dimension of the matrices and vectors involved. For large model (and signal) lengths, both computational expense and storage requirements will become prohibitively large. The resulting restriction of signal length (which applies to WD-based signal synthesis as well) can be removed if the SWD employed is a pseudo-WD [1] since in this case on-line versions of the QPA with significantly reduced computation and storage requirements can be developed [7], [13], [20]. We note, however, that the pseudo-WD is incapable of attenuating IT's which oscillate only in the time direction [5], [6].

We finally remark that a signal-synthesis problem which is formally similar to SWD-based signal synthesis is $\|\tilde{W} - W_{M,x}\| \rightarrow \min$ where $W_{M,x}(n, \theta) \triangleq M(n, \theta) W_x(n, \theta)$ is a *masked* version of the WD. If the mask $M(n, \theta)$ is zero outside a finite time-frequency region $R$, then this type of synthesis is insensitive to model values outside $R$. This allows for models that are unknown or corrupted (e.g., by parasitic IT's or noise) in certain time-frequency regions (cf. [11]). Since here the necessary-condition equations can again be formulated as "quasi-eigenequations" of the type (3.8), a QPA-type algorithm can be applied to this problem as well [21].

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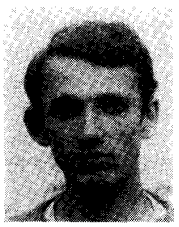


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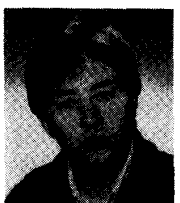


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