Bilinear Signal Synthesis

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***Abstract*—We discuss the signal synthesis problem in the general framework of bilinear signal representations (BSR's), thereby obtaining a unified treatment which encompasses, e.g., Wigner distribution and ambiguity function as special cases. The inclusion of a signal space constraint serves to impart flexibility to the signal synthesis process and to relax mathematical requirements. The characterization of signal spaces either by orthogonal projection operators or by orthonormal bases leads to two different signal synthesis methods. Both methods assume the BSR to be unitary (i.e., satisfy Moyal's formula) on the signal space on which signal synthesis is performed. As an application of the general signal synthesis methods, we consider band-limited signal synthesis in the case of Wigner distribution.**

I. Introduction and Problem Statement

Bilinear signal representations (BSR's) play an important role in signal analysis and signal processing. In this paper, we consider BSR's depending on two parameters σ and ϵ which may be time t, frequency f, time lag τ, and/or frequency lag υ. Any such BSR can be written as [1], [2]

$$T_{x,y}(\sigma, \epsilon) = \int_{t_1}\int_{t_2} u_T(\sigma, \epsilon; t_1, t_2) q_{x,y}(t_1, t_2)\, dt_1\, dt_2 \tag{1.1}$$

where

$$q_{x,y}(t_1, t_2) = x(t_1) y^*(t_2)$$

is the outer product of the signals $x(t)$ and $y(t)$, and $u_T(\sigma, \epsilon; t_1, t_2)$ is a kernel function specifying the BSR T. (All integrations are from $-\infty$ to ∞.) $T_{x,y}(\sigma, \epsilon)$ is the "cross BSR" of two signals $x(t)$, $y(t)$; the corresponding "auto BSR" of a single signal is then defined as $T_x(\sigma, \epsilon) = T_{x,x}(\sigma, \epsilon)$.

Examples of BSR's depending on two parameters are the outer signal product $q_{x,y}(t_1, t_2)$ and, in particular, all bilinear time-frequency signal representations [3]–[6] such as Wigner distribution (WD):

$$W_{x,y}(t, f) = \int_\tau x\left(t + \frac{\tau}{2}\right) y^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau}\, d\tau \tag{1.2}$$

Rihaczek distribution:

$$R_{x,y}(t, f) = \int_\tau x(t + \tau) y^*(t) e^{-j2\pi f\tau}\, d\tau$$

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and the following two versions of ambiguity function (AF):

$$A_{x,y}(\tau, \upsilon) = \int_t x\left(t + \frac{\tau}{2}\right) y^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi\upsilon t}\, dt$$

$$A'_{x,y}(\tau, \upsilon) = \int_t x(t + \tau) y^*(t) e^{-j2\pi\upsilon t}\, dt.$$

The problem considered in this paper is the synthesis of a signal $x(t)$ based on the (approximate) specification of its BSR outcome $T_x(\sigma, \epsilon)$. Let T be a given BSR (e.g., WD) and let $\tilde{T}(\sigma, \epsilon)$ be a given "model function." We wish to find a signal $x(t)$ such that the signal's BSR outcome $T_x(\sigma, \epsilon)$ equals the model function $\tilde{T}(\sigma, \epsilon)$

$$T_x(\sigma, \epsilon) = \tilde{T}(\sigma, \epsilon). \tag{1.3}$$

Unfortunately, the model will not, in general, be a valid BSR outcome of any signal, and therefore (1.3) will not have a solution. In this situation, it is natural to look for the signal $x(t)$ whose BSR outcome $T_x(\sigma, \epsilon)$ is closest to the model $\tilde{T}(\sigma, \epsilon)$ in the sense that it minimizes the "synthesis error" $\epsilon_x = \|\tilde{T} - T_x\|$

$$\epsilon_x = \|\tilde{T} - T_x\| \to \min_x \tag{1.4}$$

where

$$\epsilon_x^2 = \|\tilde{T} - T_x\|^2 \triangleq \int_\sigma\int_\epsilon |\tilde{T}(\sigma, \epsilon) - T_x(\sigma, \epsilon)|^2\, d\sigma\, d\epsilon.$$

The minimization (1.4) will be termed the bilinear signal synthesis problem. We shall also consider a subspace-constrained version of the bilinear signal synthesis problem where the signal $x(t)$ is constrained to be an element of a given linear signal subspace $\mathfrak{S}$, $x(t) \in \mathfrak{S}$:

$$\epsilon_x = \|\tilde{T} - T_x\| \to \min_{x \in \mathfrak{S}}. \tag{1.5}$$

If, in the extreme case, the signal space $\mathfrak{S}$ is the "total signal space" $\mathfrak{X}$ (which we define somewhat loosely as the space containing all signals, although we will usually identify $\mathfrak{X}$ with the space $L_2(\mathbb{R})$ of square-integrable or finite-energy signals), then the constrained synthesis problem (1.5) reduces to the unconstrained ("global") synthesis problem (1.4). Hence global signal synthesis can be treated as a special case of subspace-constrained signal synthesis.

There are two reasons for including a subspace constraint $x(t) \in \mathfrak{S}$ in the formulation of bilinear signal syn-

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thesis. First, the subspace constraint can be used for imposing certain properties on the synthesis result $x(t)$: by prescribing a suitable signal subspace $\mathfrak{S}$, $x(t)$ can be forced to be band-limited, analytic, time limited, causal, symmetric, etc. Second, in some instances, the very structure of the BSR T calls for a signal subspace constraint. For example, discrete-time WD suffers from severe aliasing effects unless the signals are restricted to a suitably defined subspace of band-limited signals [7].

The signal synthesis problem (1.5) could be called "autosynthesis" since the model $\tilde{T}(\sigma, \epsilon)$ is approximated by an auto-BSR outcome. An obvious extension is the "cross synthesis" problem $\epsilon_{x,y} = \|\tilde{T} - T_{x,y}\| \rightarrow$ min with signal space constraints $x(t) \in \mathfrak{S}$, $y(t) \in \mathfrak{S}'$: here, the model is approximated by a cross BSR outcome of signals $x(t)$, $y(t)$ belonging to linear spaces $\mathfrak{S}$, $\mathfrak{S}'$, respectively. Note that autosynthesis is not simply a special case of cross synthesis since the implicit condition $x(t) = y(t)$ of autosynthesis is really an extra constraint which comes in addition to the signal space constraint. While we here restrict our discussion to the (practically more important) problem of autosynthesis, we remark that a similar development and similar solutions can be found for the cross-synthesis problem as well.

There exist two main applications of bilinear signal synthesis. The first application is obviously the BSR-based design of signals. This becomes a time-frequency signal design if the BSR T is a bilinear time-frequency signal representation. For example, the time-frequency design of window functions and filter impulse responses based on WD has been proposed in [8], and AF has been used for the time-frequency design of radar pulses [9], [10]. The second application is a BSR-based scheme for signal processing (see Fig. 1) which consists of the following three steps: i) calculation of the BSR outcome $T_x(\sigma, \epsilon)$ of the input signal $x(t)$; ii) modification of the BSR outcome $T_x(\sigma, \epsilon)$; iii) synthesis of the output signal $y(t)$ from the modified BSR outcome $\tilde{T}(\sigma, \epsilon)$. This scheme has been proposed for WD-based time-varying filtering in [8], [11].

The bilinear signal synthesis problem has been studied for specific BSR's by various authors. The case of AF has been considered by Wilcox [9] and Sussman [10]. Both authors use a characterization of the underlying signal space by an orthonormal basis. The global signal synthesis problem in the context of discrete-time WD has been studied by Boudreaux-Bartels and Parks [8] who used a transformation approach, and by Kumar *et al.* [12] and Yu and Cheng [13] with the basis approach of Sussman. Subspace-constrained signal synthesis for general subspace-unitary BSR's has been considered by Hlawatsch and Krattenthaler [14]. Signal synthesis algorithms for smoothed versions of Wigner distribution such as pseudo-Wigner distribution have been developed by Yu and Cheng [15] (see also the comments given in [16]) and by Krattenthaler and Hlawatsch [16]–[18].

This paper unifies and extends previous work by developing solutions to the bilinear signal synthesis problem for arbitrary (sub-)space $\mathfrak{S}$ and arbitrary BSR's possessing a unitarity property on the space $\mathfrak{S}$. A brief account of the main results of this paper has been given in [14]. The development relies on a linear-operator description and calculus of BSR's which is more extensively discussed in [2], [19]. We here use a continuous-time formulation; however, the theory can easily be reformulated in a discrete-time setting.

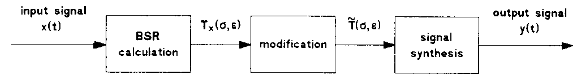

Fig. 1. A general scheme for BSR-based signal processing.

The paper is organized as follows. As a basis for subsequent development, Section II provides a brief review of linear spaces, considers the linear-operator description and unitarity property of BSR's, and discusses the concept of induced spaces. With these fundamentals as a background, Sections III and IV derive two solutions to the signal synthesis problem: the *projection-transformation method*, discussed in Section III, relies on the characterization of spaces by orthogonal projection operators whereas the *basis method*, described in Section IV, uses the characterization of spaces by orthonormal bases. The interrelation of the two methods is studied in Section V, and some special situations are considered in Section VI. Finally, Section VII demonstrates the application of the two synthesis methods to the problem of band-limited signal synthesis in the case of WD.

II. Linear-Operator Description of BSR's, Unitarity, and Induced Spaces

This section briefly summarizes some fundamentals which form a theoretical basis for the signal synthesis methods derived in Sections III and IV.

A. Linear Spaces, Projection Operators, and Bases [20]

A (linear) signal space $\mathfrak{S}$ is a class of signals satisfying the following linearity property: if $x_1(t) \in \mathfrak{S}$ and $x_2(t) \in \mathfrak{S}$, then $c_1 x_1(t) + c_2 x_2(t) \in \mathfrak{S}$ for arbitrary complex coefficients c_1, c_2. The orthogonal projection $x_{\mathfrak{S}}(t) \in \mathfrak{S}$ of an arbitrary signal $x(t) \in \mathfrak{X}$ on the space $\mathfrak{S}$ (see Fig. 2) is given by

$$x_{\mathfrak{S}}(t) = (\boldsymbol{I}_{\mathfrak{S}} x)(t) = \int_{t'} I_{\mathfrak{S}}(t, t') x(t')\, dt'$$

where $\boldsymbol{I}_{\mathfrak{S}}$ is the (orthogonal) projection operator of $\mathfrak{S}$ and $I_{\mathfrak{S}}(t, t')$ is its kernel. Note that a projection operator satisfies $\boldsymbol{I}_{\mathfrak{S}}^2 = \boldsymbol{I}_{\mathfrak{S}}$ (idempotency) and $\boldsymbol{I}_{\mathfrak{S}}^+ = \boldsymbol{I}_{\mathfrak{S}}$ (self-adjointness; $\boldsymbol{I}_{\mathfrak{S}}^+$ is the adjoint of $\boldsymbol{I}_{\mathfrak{S}}$ with kernel $I_{\mathfrak{S}}^+(t, t') = I_{\mathfrak{S}}^*(t', t)$). Any signal $x(t) \in \mathfrak{S}$ satisfies $\boldsymbol{I}_{\mathfrak{S}} x = x$; the projection operator $\boldsymbol{I}_{\mathfrak{S}}$ is thus an identity operator on $\mathfrak{S}$ and, in fact, it defines the space $\mathfrak{S}$ as the class of signals $x(t)$ satisfying $\boldsymbol{I}_{\mathfrak{S}} x = x$. Alternatively, a space $\mathfrak{S}$ may be defined by an orthonormal basis $\{e_k(t)\}$ spanning $\mathfrak{S}$ such that every $x(t) \in \mathfrak{S}$ can be represented as

$$x(t) = \sum_k \alpha_k e_k(t) \qquad \text{with} \quad \alpha_k = (x, e_k) = \int_t x(t) e_k^*(t)\, dt.$$

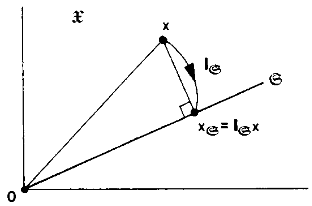

Fig. 2. Orthogonal projection on subspace.

Orthonormality of the basis signals is expressed by $(e_k, e_{k'}) = \delta_{kk'}$, where $\delta_{kk'}$ is the Kronecker delta symbol. In terms of the basis signals $e_k(t)$, the projection $x_{\mathfrak{S}}(t)$ of a signal $x(t) \in \mathfrak{X}$ can be written as

$$x_{\mathfrak{S}}(t) = \sum_k \alpha_k e_k(t) \qquad \text{with} \quad \alpha_k = (x, e_k).$$

B. Linear-Operator Description of BSR's

Any BSR $T(\sigma, \epsilon)$ can be written in the "normal form" (1.1)

$$T_{x,y}(\sigma, \epsilon) = \int_{t_1}\int_{t_2} u_T(\sigma, \epsilon; t_1, t_2) q_{x,y}(t_1, t_2)\, dt_1\, dt_2 \tag{2.1}$$

where $q_{x,y}(t_1, t_2) = x(t_1)y^*(t_2)$ is the outer product of $x(t)$ and $y(t)$, and the kernel $u_T(\sigma, \epsilon; t_1, t_2)$ can be interpreted as the BSR's impulse response, $u_T(\sigma, \epsilon; t_1, t_2) = T_{x,y}(\sigma, \epsilon)$ for $x(t) = \delta(t - t_1)$ and $y(t) = \delta(t - t_2)$. The normal form (2.1) can be considered to express a linear transformation of the signal product $q_{x,y}(t_1, t_2)$ [19]; as such, it will be briefly written as

$$T_{x,y} = \boldsymbol{u}_T q_{x,y}$$

with the linear BSR operator $\boldsymbol{u}_T$ whose kernel is the impulse response $u_T(\sigma, \epsilon; t_1, t_2)$. Note that the BSR operator $\boldsymbol{u}_T$ (or, equivalently, the BSR's impulse response $u_T(\sigma, \epsilon; t_1, t_2)$) fully characterizes the BSR T.

C. Domains and Spaces

In the process of BSR calculation as expressed by (2.1), three "domains" can be distinguished: the signal domain (signals $x(t)$, $y(t)$), the q-domain (signal product $q_{x,y}(t_1, t_2)$), and the T-domain (resulting BSR outcome $T_{x,y}(\sigma, \epsilon)$). We now associate linear spaces with these domains. The total signal space $\mathfrak{X}$ has already been introduced as the linear space containing all signals. In the q-domain, we define the total q-domain space $\mathfrak{Q}$ as the linear space containing all two-dimensional functions $\tilde{q}(t_1, t_2)$. Note that not every element $\tilde{q}(t_1, t_2) \in \mathfrak{Q}$ is a valid signal product $q_{x,y}(t_1, t_2)$: all signal product outcomes $q_{x,y}(t_1, t_2)$ are contained in $\mathfrak{Q}$ but they do not themselves form a linear space. Finally, we define the total T-domain space $\mathfrak{T}$ as the linear space containing all two-dimensional functions $\tilde{T}(\sigma, \epsilon)$. Again, there exist elements of $\mathfrak{T}$ (e.g., the model $\tilde{T}(\sigma, \epsilon)$ of the signal synthesis problem (1.5)) which are not valid BSR outcomes $T_{x,y}(\sigma, \epsilon)$. We note that the spaces $\mathfrak{Q}$ and $\mathfrak{T}$ are in fact identical as they both consist of all two-dimensional functions. We shall nevertheless distinguish between $\mathfrak{Q}$ and $\mathfrak{T}$ since these spaces are different when a discrete-time formulation is used.

We next define inner products, norms, and identity operators for the spaces $\mathfrak{X}$, $\mathfrak{Q}$, and $\mathfrak{T}$. For this, we identify $\mathfrak{X}$ with $L_2(\mathbb{R})$ and $\mathfrak{Q}$ and $\mathfrak{T}$ with $L_2(\mathbb{R}^2)$. In the signal domain (space $\mathfrak{X}$), inner product $(\cdot, \cdot)$, norm $\|\cdot\|$, and identity operator $\boldsymbol{I}$ are then given by

$$(x, y) \triangleq \int_t x(t)y^*(t)\, dt, \qquad \|x\|^2 \triangleq (x, x),$$

$$I(t, t') \triangleq \delta(t - t')$$

where $I(t, t')$ denotes the kernel of the identity operator $\boldsymbol{I}$. For the q-domain (space $\mathfrak{Q}$), the definitions of inner product $(\cdot, \cdot)$, norm $\|\cdot\|$, and identity operator $\boldsymbol{I}_{\mathfrak{Q}}$ are

$$(\tilde{q}_a, \tilde{q}_b) \triangleq \int_{t_1}\int_{t_2} \tilde{q}_a(t_1, t_2)\tilde{q}_b^*(t_1, t_2)\, dt_1\, dt_2,$$

$$\|\tilde{q}\|^2 \triangleq (\tilde{q}, \tilde{q})$$

$$I_{\mathfrak{Q}}(t_1, t_2; t_1', t_2') \triangleq \delta(t_1 - t_1')\,\delta(t_2 - t_2').$$

The T-domain definitions are analogous.

D. The Induced q-Domain Space

A linear signal space $\mathfrak{S} \subseteq \mathfrak{X} = L_2(\mathbb{R})$ "induces" [10] a corresponding linear q-domain space $\mathfrak{S}_q \subseteq \mathfrak{Q} = L_2(\mathbb{R}^2)$. Loosely speaking, the induced q-domain space $\mathfrak{S}_q$ is defined as the linear space of all linear combinations of outer signal products $q_{x,y}(t_1, t_2)$ with $x(t), y(t) \in \mathfrak{S}$. If $x(t) \in \mathfrak{S}$, then $q_x(t_1, t_2) \in \mathfrak{S}_q$ and vice versa. Projecting a signal $x(t)$ on the signal space $\mathfrak{S}$ corresponds to projecting the signal product $q_x(t_1, t_2)$ on $\mathfrak{S}_q$: there is

$$q_{I_{\mathfrak{S}}x} = \boldsymbol{I}_{\mathfrak{S}q} q_x$$

where $\boldsymbol{I}_{\mathfrak{S}q}$ is the orthogonal projection operator of the induced q-domain space $\mathfrak{S}_q$. The kernel of this "induced q-domain projection operator" is given by

$$I_{\mathfrak{S}q}(t_1, t_2; t_1', t_2') = I_{\mathfrak{S}}(t_1, t_1') I_{\mathfrak{S}}^*(t_2, t_2'). \tag{2.2}$$

The induced q-domain space $\mathfrak{S}$ is spanned by an "induced q-domain basis" $\{q_{kl}(t_1, t_2)\}$ with

$$q_{kl}(t_1, t_2) \triangleq q_{e_k, e_l}(t_1, t_2) = e_k(t_1)e_l^*(t_2) \tag{2.3}$$

where $\{e_k(t)\}$ is any orthonormal basis of $\mathfrak{S}$. The induced q-domain basis $\{q_{kl}(t_1, t_2)\}$ is again orthonormal, $(q_{kl}, q_{k'l'}) = \delta_{kk'}\delta_{ll'}$.

E. The Induced T-Domain Space [19]

The linear signal space $\mathfrak{S}$ also induces a corresponding linear T-domain space $\mathfrak{S}_T \subseteq \mathfrak{T} = L_2(\mathbb{R}^2)$ which is again defined as the linear space of all linear combinations of cross BSR outcomes $T_{x,y}(\sigma, \epsilon)$ with $x(t), y(t) \in \mathfrak{S}$. If $x(t) \in \mathfrak{S}$, then $T_x(\sigma, \epsilon) \in \mathfrak{S}_T$. To find the orthogonal projection operator and an orthonormal basis of the induced T-domain space $\mathfrak{S}_T$, we now assume that the BSR T satisfies a unitarity property on the signal space $\mathfrak{S}$.

F. BSR Unitarity [19]

We shall call a BSR T *unitary on the space* $\mathfrak{S}$ (briefly $\mathfrak{S}$-unitary) if it satisfies Moyal's formula [21], [3] on $\mathfrak{S}$, i.e.,

$$(T_{x_1,y_1}, T_{x_2,y_2}) = (q_{x_1,y_1}, q_{x_2,y_2}) = (x_1, x_2)(y_1, y_2)^*$$

for $x_1(t), x_2(t), y_1(t), y_2(t) \in \mathfrak{S}$. If $\mathfrak{S} = L_2(\mathbb{R})$, i.e., Moyal's formula holds on the entire space of finite-energy signals, then the BSR T will be called *globally unitary*. Evidently, a globally unitary BSR is also $\mathfrak{S}$-unitary for arbitrary signal subspaces $\mathfrak{S}$.

To formulate a condition for $\mathfrak{S}$-unitarity, we first note that the BSR operator $\boldsymbol{u}_T$ can be replaced by the operator

$$\boldsymbol{u}_{T\mathfrak{S}} \triangleq \boldsymbol{u}_T \boldsymbol{I}_{\mathfrak{S}q}$$

$$u_{T\mathfrak{S}}(\sigma, \epsilon; t_1, t_2) = \int_{t_1'}\int_{t_2'} u_T(\sigma, \epsilon; t_1', t_2') \cdot I_{\mathfrak{S}q}(t_1', t_2'; t_1, t_2)\, dt_1'\, dt_2'$$

if the signals $x(t)$, $y(t)$ are elements of $\mathfrak{S}$:

$$T_{x,y} = \boldsymbol{u}_T q_{x,y} = \boldsymbol{u}_{T\mathfrak{S}} q_{x,y} \qquad \text{for} \quad x(t), y(t) \in \mathfrak{S}.$$

This alternative expression for the BSR T does not require T to be $\mathfrak{S}$-unitary. The kernel of the operator $\boldsymbol{u}_{T\mathfrak{S}}$ can be interpreted in a way similar to the impulse response $u_T(\sigma, \epsilon; t_1, t_2)$ since $u_{T\mathfrak{S}}(\sigma, \epsilon; t_1, t_2) = T_{x,y}(\sigma, \epsilon)$ for $x(t) = I_{\mathfrak{S}}(t, t_1)$ and $y(t) = I_{\mathfrak{S}}(t, t_2)$ (note that $I_{\mathfrak{S}}(t, t')$ is the projection of $\delta(t - t')$ on $\mathfrak{S}$). A necessary and sufficient condition for $\mathfrak{S}$-unitarity can now be expressed in terms of the new operator $\boldsymbol{u}_{T\mathfrak{S}}$ as

$$\boldsymbol{u}_{T\mathfrak{S}}^+ \boldsymbol{u}_{T\mathfrak{S}} = \boldsymbol{I}_{\mathfrak{S}q} \tag{2.4a}$$

$$\int_\sigma\int_\epsilon u_{T\mathfrak{S}}^+(t_1, t_2; \sigma, \epsilon) u_{T\mathfrak{S}}(\sigma, \epsilon; t_1', t_2')\, d\sigma\, d\epsilon = I_{\mathfrak{S}q}(t_1, t_2; t_1', t_2') \tag{2.4b}$$

where $\boldsymbol{u}_{T\mathfrak{S}}^+$, the adjoint of $\boldsymbol{u}_{T\mathfrak{S}}$, is characterized by the kernel $u_{T\mathfrak{S}}^+(t_1, t_2; \sigma, \epsilon) = u_{T\mathfrak{S}}^*(\sigma, \epsilon; t_1, t_2)$. In the case of global unitarity where $\mathfrak{S} = L_2(\mathbb{R})$, condition (2.4) reduces to

$$\boldsymbol{u}_T^+ \boldsymbol{u}_T = \boldsymbol{I}_{\mathfrak{Q}} \tag{2.5a}$$

$$\int_\sigma\int_\epsilon u_T^+(t_1, t_2; \sigma, \epsilon) u_T(\sigma, \epsilon; t_1' t_2')\, d\sigma\, d\epsilon = \delta(t_1 - t_1')\,\delta(t_2 - t_2'). \tag{2.5b}$$

We note that all BSR's listed in Section I are globally unitary. The discrete-time WD [7], on the other hand, is not globally unitary but unitary on all "halfband subspaces" [14]. Smoothed versions of WD (such as the pseudo-WD, the spectrogram, and the Choi-Williams distribution) are nonunitary on any signal space.

G. Implications of Unitarity

The $\mathfrak{S}$-unitarity of a BSR T has some important implications which will be utilized in Sections III and IV. First, if a BSR T is $\mathfrak{S}$-unitary, then the orthogonal projection operator of the induced T-domain space $\mathfrak{S}_T$ is given by

$$\boldsymbol{I}_{\mathfrak{S}T} = \boldsymbol{u}_{T\mathfrak{S}} \boldsymbol{u}_{T\mathfrak{S}}^+ \tag{2.6a}$$

$$I_{\mathfrak{S}T}(\sigma, \epsilon; \sigma', \epsilon') = \int_{t_1}\int_{t_2} u_{T\mathfrak{S}}(\sigma, \epsilon; t_1, t_2) \cdot u_{T\mathfrak{S}}^+(t_1, t_2; \sigma', \epsilon')\, dt_1\, dt_2 \tag{2.6b}$$

and an orthonormal basis $\{T_{kl}(\sigma, \epsilon)\}$ of $\mathfrak{S}_T$ is

$$T_{kl}(\sigma, \epsilon) = T_{e_k, e_l}(\sigma, \epsilon) \tag{2.7}$$

where $\{e_k(t)\}$ is any orthonormal basis of $\mathfrak{S}$. Table I summarizes the corresponding spaces, projection operators, and orthonormal bases in signal domain, q-domain, and T-domain.

TABLE I
INDUCED SPACES, PROJECTION OPERATORS, AND BASES

	Space	Projection Operator	Basis
Signal domain	$\mathfrak{S}$	$I_{\mathfrak{S}}$	$\{e_k(t)\}$
q-domain	$\mathfrak{S}_q$	$I_{\mathfrak{S}q}$ Eq. (2.2)	$\{q_{kl}(t_1, t_2)\}$ Eq. (2.3)
T-domain	$\mathfrak{S}_T$	$I_{\mathfrak{S}T}$ Eq. (2.6)	$\{T_{kl}(\sigma, \epsilon)\}$ Eq. (2.7)

Second, the induced spaces $\mathfrak{S}_q$ and $\mathfrak{S}_T$ are related by a unitary mapping which is given by the operator $\boldsymbol{u}_{T\mathfrak{S}}$ and inverted by its adjoint $\boldsymbol{u}_{T\mathfrak{S}}^+$. Indeed, any element $\tilde{T}(\sigma, \epsilon) \in \mathfrak{S}_T$ can be represented as

$$\tilde{T} = \boldsymbol{u}_{T\mathfrak{S}} \tilde{q}$$

with a unique $\tilde{q}(t_1, t_2) \in \mathfrak{S}_q$ given by

$$\tilde{q} = \boldsymbol{u}_{T\mathfrak{S}}^+ \tilde{T}. \tag{2.8}$$

The mapping between $\tilde{q}(t_1, t_2)$ and $\tilde{T}(\sigma, \epsilon)$ is unitary, i.e., inner products and norms of corresponding elements $\tilde{T}(\sigma, \epsilon) \in \mathfrak{S}_T$ and $\tilde{q}(t_1, t_2) \in \mathfrak{S}_q$ are equal

$$(\tilde{T}_a, \tilde{T}_b) = (\tilde{q}_a, \tilde{q}_b), \qquad \|\tilde{T}\| = \|\tilde{q}\|.$$

The unitary mapping relating $\mathfrak{S}_q$ and $\mathfrak{S}_T$ is illustrated in Fig. 3. We note that if $\tilde{T}(\sigma, \epsilon) \in \mathfrak{S}_T$ is a valid BSR outcome of signals $x(t), y(t) \in \mathfrak{S}$, $\tilde{T}(\sigma, \epsilon) = T_{x,y}(\sigma, \epsilon)$, then the corresponding $\tilde{q}(t_1, t_2)$ is a valid signal product of the same signals, $\tilde{q}(t_1, t_2) = q_{x,y}(t_1, t_2)$.

Finally, if the BSR T is globally unitary (unitarity on $\mathfrak{S}$ is here not sufficient), then the projection of an arbitrary signal $x(t)$ on $\mathfrak{S}$ corresponds to the projection of $T_x(\sigma, \epsilon)$ on $\mathfrak{S}_T$

$$T_{I_{\mathfrak{S}}x} = \boldsymbol{I}_{\mathfrak{S}T} T_x. \tag{2.9}$$

H. Ambiguity of the Signal Synthesis Solution

The solution to the signal synthesis problem (1.5) is not uniquely defined. Let us assume that the BSR T is invariant with respect to some signal transformation $\boldsymbol{a}$ such that $T_{ax}(\sigma, \epsilon) = T_x(\sigma, \epsilon)$ for all $x(t) \in \mathfrak{S}$. We furthermore suppose that the signal transformation $\boldsymbol{a}$ is "space preserving" with respect to the given signal space $\mathfrak{S}$ in the sense that $x(t) \in \mathfrak{S}$ entails $(\boldsymbol{a}x)(t) \in \mathfrak{S}$. Then, if $\hat{x}(t) \in \mathfrak{S}$

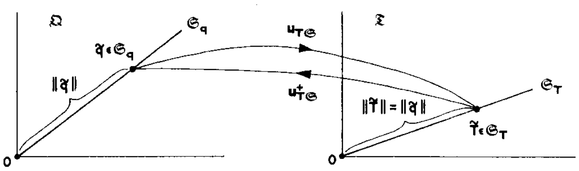

Fig. 3. Unitary mapping between induced spaces $\mathfrak{S}_q$ and $\mathfrak{S}_T$.

is a solution to (1.5), $(\boldsymbol{a}\hat{x})(t)$ is a solution as well since it is also an element of $\mathfrak{S}$ and achieves the same (minimal) synthesis error

$$\epsilon_{a\hat{x}} = \|\tilde{T} - T_{a\hat{x}}\| = \|\tilde{T} - T_{\hat{x}}\| = \epsilon_{\hat{x}}.$$

We thus see that any invariance of the BSR with respect to a space-preserving signal transformation entails a corresponding ambiguity of the signal synthesis solution. Now, in the case of an $\mathfrak{S}$-unitary BSR, only a very trivial and comparatively harmless invariance exists. In fact, it can be shown that $T_{x_1}(\sigma, \epsilon) = T_{x_2}(\sigma, \epsilon)$ (with $x_1(t)$, $x_2(t) \in \mathfrak{S}$) then implies $x_2(t) = x_1(t)e^{j\varphi}$ with φ being an arbitrary phase constant. Hence the only transformation $\boldsymbol{a}$ for which an $\mathfrak{S}$-unitary BSR is invariant is the multiplication of the signal by a constant phase factor $e^{j\varphi}$, and the synthesis solution is thus unique up to a constant phase factor. (For very special models, however, there exists another ambiguity of the synthesis solution which is discussed in Section VI.)

III. The Projection-Transformation Method

Using the framework of induced spaces developed in the previous section, and assuming the BSR T to be unitary on the prescribed space $\mathfrak{S}$, a solution to the subspace-constrained bilinear signal synthesis problem (1.5)

$$\epsilon_x = \|\tilde{T} - T_x\| \rightarrow \min_{x \in \mathfrak{S}} \tag{3.1}$$

can be derived in two different ways. According to the characterization of signal spaces by either an orthogonal projection operator or an orthonormal basis, there exist two equivalent signal synthesis methods which we shall term *projection-transformation method* and *basis method.* The projection-transformation method, to be discussed in this section, derives its name from the fact that it involves a projection of the model $\tilde{T}(\sigma, \epsilon)$ on the induced T-domain space $\mathfrak{S}_T$ and a subsequent transformation into the q-domain. In the following, we assume that the BSR T and the space $\mathfrak{S}$ are given by the BSR operator $\boldsymbol{u}_T$ and the projection operator $\boldsymbol{I}_{\mathfrak{S}}$, respectively. The model $\tilde{T}(\sigma, \epsilon)$ is required to be square-integrable, $\tilde{T} \in L_2(\mathbb{R}^2)$. Finally, and most important, the BSR T is supposed to be $\mathfrak{S}$-unitary.

A. Model Decomposition ("Projection" Step)

To solve (3.1), we first note that $T_x \in \mathfrak{S}_T$ due to our constraint $x \in \mathfrak{S}$. While the model $\tilde{T} \in L_2(\mathbb{R}^2)$ will generally lie outside $\mathfrak{S}_T$, it can be decomposed as

$$\tilde{T} = \tilde{T}_{\mathfrak{S}} + \tilde{T}_{\perp} \tag{3.2}$$

where

$$\tilde{T}_{\mathfrak{S}} \triangleq \boldsymbol{I}_{\mathfrak{S}T}\tilde{T}, \qquad \tilde{T}_{\mathfrak{S}} \in \mathfrak{S}_T$$

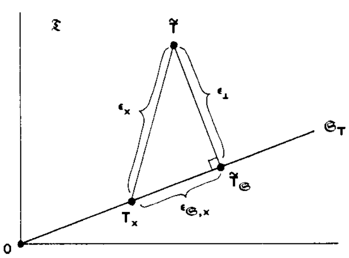

Fig. 4. Decomposition of T-domain model and synthesis error.

is the model's projection on $\mathfrak{S}_T$ and thus an element of $\mathfrak{S}_T$, and $\tilde{T}_{\perp} = \tilde{T} - \tilde{T}_{\mathfrak{S}}$ is orthogonal to $\mathfrak{S}_T$. With (3.2), the (squared) synthesis error ϵ_x^2 can then be decomposed as

$$\begin{aligned}\epsilon_x^2 &= \|\tilde{T} - T_x\|^2 = \|(\tilde{T}_{\mathfrak{S}} + \tilde{T}_{\perp}) - T_x\|^2 \\ &= \|(\tilde{T}_{\mathfrak{S}} - T_x) + \tilde{T}_{\perp}\|^2 \\ &= \|\tilde{T}_{\mathfrak{S}} - T_x\|^2 + \|\tilde{T}_{\perp}\|^2 = \epsilon_{\mathfrak{S},x}^2 + \epsilon_{\perp}^2\end{aligned}$$

where we have applied the Pythagorean theorem [22] to the orthogonal functions $(\tilde{T}_{\mathfrak{S}} - T_x) \in \mathfrak{S}_T$ and $\tilde{T}_{\perp} \perp \mathfrak{S}_T$ (see Fig. 4). Since the "orthogonal" error component $\epsilon_{\perp} = \|\tilde{T}_{\perp}\|$ does not depend on the signal $x(t)$, it can be disregarded for minimization. Note that $\epsilon_{\perp}$ is altogether zero if the model $\tilde{T}$ happens to be an element of $\mathfrak{S}_T$.

B. Q-Domain Formulation ("Transformation" Step)

We now use the unitary mapping relating the induced spaces $\mathfrak{S}_T$ and $\mathfrak{S}_q$ to reformulate the relevant error component $\epsilon_{\mathfrak{S},x}$ in the q-domain

$$\epsilon_{\mathfrak{S},x} = \|\tilde{T}_{\mathfrak{S}} - T_x\| = \|\tilde{q}_{\mathfrak{S}} - q_x\|$$

where (cf. (2.8))

$$\tilde{q}_{\mathfrak{S}} = \boldsymbol{u}_{T\mathfrak{S}}^{+}\tilde{T}_{\mathfrak{S}} \qquad \text{and} \qquad q_x = \boldsymbol{u}_{T\mathfrak{S}}^{+}T_x$$

with $\tilde{q}_{\mathfrak{S}}$, $q_x \in \mathfrak{S}_q$ (see Fig. 5). Hence the bilinear signal synthesis problem is expressed as

$$\epsilon_{\mathfrak{S},x} = \|\tilde{q}_{\mathfrak{S}} - q_x\| \rightarrow \min_{x \in \mathfrak{S}}.$$

At this point, a simple indirect proof shows that the constraint $x(t) \in \mathfrak{S}$ can be dropped since the *unconstrained* minimization problem will itself assume its solution in $\mathfrak{S}$. Suppose that the solution $\hat{x}(t)$ to the unconstrained problem $\epsilon_{\mathfrak{S},x} \rightarrow \min$ lies outside $\mathfrak{S}$, $\hat{x}(t) \notin \mathfrak{S}$. From Fig. 6, it is then clear that the projection $\hat{x}_{\mathfrak{S}}(t)$ of $\hat{x}(t)$ on $\mathfrak{S}$ will achieve a smaller synthesis error than $\hat{x}(t)$ itself; hence $\hat{x}(t) \notin \mathfrak{S}$ cannot be the solution to the unconstrained problem.

C. Another Model Decomposition

The auto signal product $q_x(t_1, t_2)$ is always a Hermitian function, i.e., $q_x(t_1, t_2) = q_x^*(t_2, t_1)$. While the q-domain model $\tilde{q}_{\mathfrak{S}}(t_1, t_2)$ will not be Hermitian in general, it can be split into a Hermitian component $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ and an anti-Hermitian component $\tilde{q}_{\mathfrak{S}A}(t_1, t_2)$,

$$\tilde{q}_{\mathfrak{S}}(t_1, t_2) = \tilde{q}_{\mathfrak{S}H}(t_1, t_2) + \tilde{q}_{\mathfrak{S}A}(t_1, t_2).$$

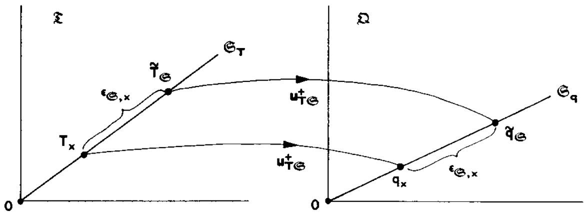

Fig. 5. Transformation into the q-domain.

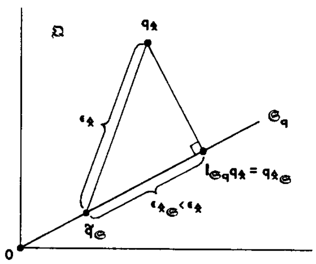

Fig. 6. Dropping of subspace constraint.

The (squared) synthesis error $\epsilon^2_{\mathfrak{S},x}$ can then be shown to allow the following decomposition:

$$\begin{aligned}\epsilon^2_{\mathfrak{S},x} &= \|\tilde{q}_{\mathfrak{S}} - q_x\|^2 = \|(\tilde{q}_{\mathfrak{S}H} + \tilde{q}_{\mathfrak{S}A}) - q_x\|^2 \\ &= \|(\tilde{q}_{\mathfrak{S}H} - q_x) + \tilde{q}_{\mathfrak{S}A}\|^2 \\ &= \|\tilde{q}_{\mathfrak{S}H} - q_x\|^2 + \|\tilde{q}_{\mathfrak{S}A}\|^2 = \epsilon^2_{\mathfrak{S}H,x} + \epsilon^2_{\mathfrak{S}A}\end{aligned}$$

where the "anti-Hermitian" error component $\epsilon_{\mathfrak{S}A} = \|\tilde{q}_{\mathfrak{S}A}\|$ does not depend on the signal $x(t)$ and can be disregarded for minimization.

D. The Basic Approximation Problem

There finally remains to minimize $\epsilon_{\mathfrak{S}H,x}$ in the absence of any constraint

$$\begin{aligned}\epsilon^2_{\mathfrak{S}H,x} &= \|\tilde{q}_{\mathfrak{S}H} - q_x\|^2 \\ &= \int_{t_1}\int_{t_2} |\tilde{q}_{\mathfrak{S}H}(t_1, t_2) - q_x(t_1, t_2)|^2 \, dt_1 \, dt_2 \to \min_x.\end{aligned} \tag{3.3}$$

This minimization amounts to the approximation of a two-dimensional Hermitian function $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ by a separable function $q_x(t_1, t_2) = x(t_1)x^*(t_2)$. As is shown in the Appendix, this basic approximation problem is solved by

$$x_{\text{opt}}(t) = e^{j\varphi}\sqrt{\lambda_1}\,u_1(t) \tag{3.4}$$

where $\lambda_1 \in \mathbb{R}$ is the largest eigenvalue of the Hermitian function $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ (or, to be more precise, of the self-adjoint integral operator with kernel $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ [20]), $u_1(t)$ is the corresponding (normalized) eigenfunction, and φ is an arbitrary phase constant. It has been assumed that λ_1 is nonnegative; the case $\lambda_1 < 0$ is discussed in Section VI. Inserting (3.4) and the spectral decomposition (see the Appendix)

$$\tilde{q}_{\mathfrak{S}H}(t_1, t_2) = \sum_{k=1}^{\infty} \lambda_k u_k(t_1) u_k^*(t_2) \tag{3.5}$$

into (3.3), the residual synthesis error is obtained as

$$\epsilon^2_{\mathfrak{S}H,\min} = \epsilon^2_{\mathfrak{S}H,x_{\text{opt}}} = \left\| \sum_{k=2}^{\infty} \lambda_k u_k(t_1) u_k^*(t_2) \right\|^2 = \sum_{k=2}^{\infty} \lambda_k^2$$

where the orthonormality of the eigenfunctions $u_k(t)$ (see the Appendix) has been used.

E. Algorithm Summary

Based on the above derivation, the projection-transformation method can now be summarized as follows.

Step 1: Form the projection $\tilde{T}_{\mathfrak{S}} = \boldsymbol{I}_{\mathfrak{S}T}\tilde{T}$ of the model $\tilde{T}(\sigma, \epsilon)$ on the induced T-domain space $\mathfrak{S}_T$

$$\tilde{T}_{\mathfrak{S}}(\sigma, \epsilon) = \int_{\sigma'}\int_{\epsilon'} I_{\mathfrak{S}T}(\sigma, \epsilon; \sigma', \epsilon')\tilde{T}(\sigma', \epsilon') \, d\sigma' \, d\epsilon'.$$

Step 2: Transform the projected model $\tilde{T}_{\mathfrak{S}}(\sigma, \epsilon)$ into the q-domain

$$\tilde{q}_{\mathfrak{S}}(t_1, t_2) = \int_{\sigma}\int_{\epsilon} u^+_{T\mathfrak{S}}(t_1, t_2; \sigma, \epsilon)\tilde{T}_{\mathfrak{S}}(\sigma, \epsilon) \, d\sigma \, d\epsilon. \tag{3.6}$$

Step 3: Take the Hermitian component $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ of $\tilde{q}_{\mathfrak{S}}(t_1, t_2)$

$$\tilde{q}_{\mathfrak{S}H}(t_1, t_2) = \tfrac{1}{2}[\tilde{q}_{\mathfrak{S}}(t_1, t_2) + \tilde{q}^*_{\mathfrak{S}}(t_2, t_1)].$$

Step 4: Calculate the largest eigenvalue λ_1 and the associated (normalized) eigenfunction $u_1(t)$ of $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$; if $\lambda_1 \geq 0$, then the synthesis solution is given by

$$x_{\text{opt}}(t) = e^{j\varphi}\sqrt{\lambda_1}\,u_1(t)$$

where φ is an arbitrary phase constant.

We finally note that step 1 (projection) and step 2 (transformation) can be combined into a single transformation step: as it is easily shown that

$$\boldsymbol{u}^+_{T\mathfrak{S}}\boldsymbol{I}_{\mathfrak{S}T} = \boldsymbol{u}^+_{T\mathfrak{S}},$$

$\tilde{q}_{\mathfrak{S}}(t_1, t_2)$ can be directly calculated from the original model $\tilde{T}(\sigma, \epsilon)$ as $\tilde{q}_{\mathfrak{S}} = \boldsymbol{u}^+_{T\mathfrak{S}}\tilde{T}$ or

$$\tilde{q}_{\mathfrak{S}}(t_1, t_2) = \int_{\sigma}\int_{\epsilon} u^+_{T\mathfrak{S}}(t_1, t_2; \sigma, \epsilon)\tilde{T}(\sigma, \epsilon) \, d\sigma \, d\epsilon. \tag{3.7}$$

IV. The Basis Method

The basis method [14] derived in this section is an extension of the signal synthesis method for AF described in [9] [10]. We here suppose that the signal space $\mathfrak{S}$ is characterized by an orthonormal basis $\{e_k(t)\}$. The BSR T is again assumed to be unitary on the signal space $\mathfrak{S}$.

A. Derivation of the Basis Method

Using the initial result in Section III, we have to minimize $\epsilon_{\mathfrak{S},x} = \|\tilde{T}_{\mathfrak{S}} - T_x\|$ subject to the subspace constraint $x(t) \in \mathfrak{S}$. We incorporate this constraint by representing the signal $x(t)$ in terms of the basis $\{e_k(t)\}$ spanning $\mathfrak{S}$

$$x(t) = \sum_{k=1}^{N} \alpha_k e_k(t) \tag{4.1}$$

where N, the dimension of the space $\mathfrak{S}$, may be infinite. This induces a corresponding T-domain representation of $T_x(\sigma, \epsilon)$ in terms of the induced T-domain basis $\{T_{kl}(\sigma, \epsilon)\}$

$$T_x(\sigma, \epsilon) = \sum_{k=1}^{N} \sum_{l=1}^{N} \gamma_{kl} T_{kl}(\sigma, \epsilon) \qquad \text{with} \quad \gamma_{kl} = \alpha_k \alpha_l^*. \tag{4.2}$$

The model projection $\tilde{T}_{\mathfrak{S}}(\sigma, \epsilon)$, too, is an element of the induced T-domain space $\mathfrak{S}_T$ and can thus be represented in terms of the induced T-domain basis $\{T_{kl}(\sigma, \epsilon)\}$

$$\tilde{T}_{\mathfrak{S}}(\sigma, \epsilon) = \sum_{k=1}^{N} \sum_{l=1}^{N} \tilde{\gamma}_{kl} T_{kl}(\sigma, \epsilon) \qquad \text{with} \quad \tilde{\gamma}_{kl} = (\tilde{T}, T_{kl}). \tag{4.3}$$

Using (4.2), (4.3), and the orthonormality of the induced T-domain basis $\{T_{kl}(\sigma, \epsilon)\}$, the (squared) synthesis error can be developed as

$$\begin{aligned}\epsilon_{\mathfrak{S},x}^2 &= \|\tilde{T}_{\mathfrak{S}} - T_x\|^2 = \left\| \sum_{k=1}^{N} \sum_{l=1}^{N} (\tilde{\gamma}_{kl} - \alpha_k \alpha_l^*) T_{kl} \right\|^2 \\ &= \sum_{k=1}^{N} \sum_{l=1}^{N} |\tilde{\gamma}_{kl} - \alpha_k \alpha_l^*|^2 = \|\tilde{\mathbf{\Gamma}} - \boldsymbol{\alpha}\boldsymbol{\alpha}^+\|_F^2\end{aligned}$$

with the $(N \times N)$-dimensional coefficient matrix $\tilde{\mathbf{\Gamma}} = (\tilde{\gamma}_{kl})$ and the N-dimensional coefficient vector $\boldsymbol{\alpha} = (\alpha_k)$; $\|\cdot\|_F$ denotes the Euclidean matrix norm (Frobenius norm), and "$^+$" stands for complex transposition. The dyadic-product matrix $\boldsymbol{\alpha}\boldsymbol{\alpha}^+$ is Hermitian and rank 1; the matrix $\tilde{\mathbf{\Gamma}}$, on the other hand, is generally not Hermitian but can be split into a Hermitian component $\tilde{\mathbf{\Gamma}}_H$ and an anti-Hermitian component $\tilde{\mathbf{\Gamma}}_A$. With this, the squared synthesis error can be decomposed as

$$\begin{aligned}\epsilon_{\mathfrak{S},x}^2 &= \|\tilde{\mathbf{\Gamma}} - \boldsymbol{\alpha}\boldsymbol{\alpha}^+\|_F^2 = \|(\tilde{\mathbf{\Gamma}}_H + \tilde{\mathbf{\Gamma}}_A) - \boldsymbol{\alpha}\boldsymbol{\alpha}^+\|_F^2 \\ &= \|(\tilde{\mathbf{\Gamma}}_H - \boldsymbol{\alpha}\boldsymbol{\alpha}^+) + \tilde{\mathbf{\Gamma}}_A\|_F^2 \\ &= \|\tilde{\mathbf{\Gamma}}_H - \boldsymbol{\alpha}\boldsymbol{\alpha}^+\|_F^2 + \|\tilde{\mathbf{\Gamma}}_A\|_F^2 = \epsilon_{\mathfrak{S}H,x}^2 + \epsilon_{\mathfrak{S}A}^2\end{aligned}$$

where the "anti-Hermitian" error component $\epsilon_{\mathfrak{S}A} = \|\tilde{\mathbf{\Gamma}}_A\|_F$ does not depend on $x(t)$ (i.e., on $\boldsymbol{\alpha}$) and can hence be disregarded for minimization. Thus, it remains to minimize the "Hermitian" error component

$$\epsilon_{\mathfrak{S}H,x} = \|\tilde{\mathbf{\Gamma}}_H - \boldsymbol{\alpha}\boldsymbol{\alpha}^+\|_F \tag{4.4}$$

in the absence of any constraint (note that the constraint $x(t) \in \mathfrak{S}$ has been taken account of by representing $x(t)$ according to (4.1)). Minimization of (4.4) amounts to the approximation of the Hermitian matrix $\tilde{\mathbf{\Gamma}}_H$ by a dyadic product $\boldsymbol{\alpha}\boldsymbol{\alpha}^+$; this is a discrete version of the basic approximation problem (3.3). With μ_k and $\boldsymbol{v}_k$ denoting, respectively, the real-valued eigenvalues and the corresponding orthonormal eigenvectors of the Hermitian matrix $\tilde{\mathbf{\Gamma}}_H$, a derivation analogous to that of the Appendix yields the solution

$$\boldsymbol{\alpha}_{\text{opt}} = e^{j\varphi} \sqrt{\mu_1}\, \boldsymbol{v}_1 \tag{4.5}$$

where we have again assumed that the largest eigenvalue μ_1 is nonnegative, and φ is an arbitrary phase constant. Inserting (4.5) and the spectral decomposition

$$\tilde{\mathbf{\Gamma}}_H = \sum_{k=1}^{N} \mu_k \boldsymbol{v}_k \boldsymbol{v}_k^+ \tag{4.6}$$

into (4.4), the residual synthesis error is obtained as

$$\epsilon_{\mathfrak{S}H,\min}^2 = \epsilon_{\mathfrak{S}H,x_{\text{opt}}}^2 = \left\| \sum_{k=2}^{N} \mu_k \boldsymbol{v}_k \boldsymbol{v}_k^+ \right\|_F^2 = \sum_{k=2}^{N} \mu_k^2$$

where the orthonormality of the eigenvectors $\boldsymbol{v}_k$ has been used.

B. Algorithm Summary

We can finally summarize the basis method as follows:

Step 1: Calculate the expansion coefficients of the model projection $\tilde{T}_{\mathfrak{S}}(\sigma, \epsilon)$

$$\tilde{\gamma}_{kl} = (\tilde{T}, T_{kl}) = \int_\sigma \int_\epsilon \tilde{T}(\sigma, \epsilon) T_{kl}^*(\sigma, \epsilon)\, d\sigma\, d\epsilon, \qquad 1 \le k, l \le N. \tag{4.7}$$

Step 2: Form the matrix $\tilde{\mathbf{\Gamma}} = (\tilde{\gamma}_{kl})$ and take its Hermitian component

$$\tilde{\mathbf{\Gamma}}_H = \tfrac{1}{2}(\tilde{\mathbf{\Gamma}} + \tilde{\mathbf{\Gamma}}^+).$$

Step 3: Calculate the largest eigenvalue μ_1 and the associated (normalized) eigenvector $\boldsymbol{v}_1$ of $\tilde{\mathbf{\Gamma}}_H$; if $\mu_1 \ge 0$, then the synthesis solution is given by (cf. (4.1))

$$x_{\text{opt}}(t) = \sum_{k=1}^{N} \alpha_{\text{opt},k}\, e_k(t)$$

where

$$\boldsymbol{\alpha}_{\text{opt}} = e^{j\varphi} \sqrt{\mu_1}\, \boldsymbol{v}_1$$

with φ being an arbitrary phase constant.

V. Relation Between Projection-Transformation Method and Basis Method

There are striking similarities between the projection-transformation method (PTM) and the basis method (BM). Indeed, both methods perform an orthogonal projection of the model $\tilde{T}(\sigma, \epsilon)$ onto the induced T-domain space $\mathfrak{S}_T$ and a transformation either into the q-domain (PTM) or into a discrete coefficient domain (BM). Then, the Hermitian component of the q-domain model $\tilde{q}_{\mathfrak{S}}(t_1, t_2)$ (PTM) or of the model coefficient matrix $\tilde{\mathbf{\Gamma}}$ (BM) is calculated, and finally a continuous (PTM) or discrete (BM) version of the basic approximation problem (approximation by a separable function or matrix) is solved.

To further demonstrate the relation between PTM and BM, we note that the PTM makes implicit use of the spectral decomposition (3.5)

$$\tilde{q}_{\mathfrak{S}H}(t_1, t_2) = \sum_{k=1}^{\infty} \lambda_k u_k(t_1) u_k^*(t_2) \tag{5.1}$$

From (6.3), we readily obtain the spectral decomposition (cf. (5.1))

$$\tilde{q}_{\mathfrak{S}H}(t_1, t_2) = E_x u_1(t_1) u_1^*(t_2) + E_x u_2(t_1) u_2^*(t_2)$$

where (apart from arbitrary constant phase factors) $u_1(t) = x_1(t)/\sqrt{E_x}$ and $u_2(t) = x_2(t)/\sqrt{E_x}$. It follows that $\lambda_1 = \lambda_2 = E_x$ and $\lambda_k = 0$ for $k > 2$; the largest eigenvalue $\lambda_1 = \lambda_2 = E_x$ is thus seen to have multiplicity $K = 2$.

In general, if the largest eigenvalue λ_1 has multiplicity K, then K orthonormal eigenfunctions $u_{1,k}(t)$ $(k = 1, 2, \cdots, K)$ can be associated with λ_1, and any linear combination

$$u_1(t) = \sum_{k=1}^{K} c_k u_{1,k}(t) \qquad \text{with} \qquad \sum_{k=1}^{K} |c_k|^2 = 1$$

is again a normalized eigenfunction corresponding to λ_1. Hence the signal synthesis solution $x_{\text{opt}}(t) = e^{j\varphi}\sqrt{\lambda_1}\, u_1(t)$ is ambiguous in that $u_1(t)$ may be any (normalized) element of the K-dimensional signal subspace spanned by the basis $\{u_{1,k}(t)\}$.

D. Pathological Model

For a given signal space $\mathfrak{S}$, we shall call a model $\tilde{T}(\sigma, \epsilon)$ pathological if the largest eigenvalue λ_1 of $q_{\mathfrak{S}H}(t_1, t_2)$ (or, equivalently, of $\tilde{\mathbf{\Gamma}}_H$) is zero or negative. An example is $\tilde{T}(\sigma, \epsilon) = -T_x(\sigma, \epsilon)$ with $x(t) \in \mathfrak{S}$; we here obtain the spectral decomposition $\tilde{q}_{\mathfrak{S}H}(t_1, t_2) = -E_x u(t_1) u^*(t_2)$ where $E_x = \|x\|^2$ and (apart from an arbitrary constant phase factor) $u(t) = x(t)/\sqrt{E_x}$. It follows that all eigenvalues λ_k are zero apart from one eigenvalue which is $-E_x$ and thus negative.

To find the synthesis solution for a general pathological model, we decompose the squared synthesis error (3.3) as

$$\epsilon^2_{\mathfrak{S}H,x} = \|\tilde{q}_{\mathfrak{S}H} - q_x\|^2 = \|\tilde{q}_{\mathfrak{S}H}\|^2 + \|q_x\|^2 - 2(\tilde{q}_{\mathfrak{S}H}, q_x) \tag{6.4}$$

where

$$\|q_x\| = \|x\|^2 \tag{6.5}$$

and

$$(\tilde{q}_{\mathfrak{S}H}, q_x) = \int_{t_1}\int_{t_2} \tilde{q}_{\mathfrak{S}H}(t_1, t_2) q_x^*(t_1, t_2)\, dt_1\, dt_2$$
$$= \int_{t_1}\int_{t_2} \tilde{q}_{\mathfrak{S}H}(t_1, t_2) x^*(t_1) x(t_2)\, dt_1\, dt_2 \tag{6.6}$$

is recognized as a Hermitian form. For a pathological model, the largest eigenvalue of $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ is zero or negative; hence all eigenvalues are zero or negative and the Hermitian form (6.6) is negative semidefinite [20]

$$(\tilde{q}_{\mathfrak{S}H}, q_x) \le 0 \qquad \text{for all } x(t) \tag{6.7a}$$

and

$$(\tilde{q}_{\mathfrak{S}H}, q_x) = 0 \qquad \text{for } x(t) \equiv 0. \tag{6.7b}$$

With (6.5) and (6.7), (6.4) shows that the zero signal $x(t) \equiv 0$ minimizes the error $\epsilon_{\mathfrak{S}H,x}$; since it also satisfies our signal space constraint $x(t) \in \mathfrak{S}$, we conclude that the zero signal is the synthesis solution for a pathological model, $x_{\text{opt}}(t) \equiv 0$.

VII. Example: Band-Limited Signal Synthesis for Wigner Distribution

In the previous sections, we have derived and discussed two general methods for subspace-constrained signal synthesis. These methods are applicable to arbitrary linear signal spaces $\mathfrak{S}$ and to arbitrary $\mathfrak{S}$-unitary BSR's T. To demonstrate the application of both methods, we now specialize to a specific BSR T, namely, Wigner distribution (WD) $W(t, f)$, and to a specific signal space $\mathfrak{S}$, namely, the space $\mathfrak{B}$ of signals which are band limited on a predefined frequency band $[f_1, f_2]$.

From the definition (1.2) of WD and the fact that $u_W(t, f; t_1, t_2) = W_{x,y}(t, f)$ for $x(t) = \delta(t - t_1)$ and $y(t) = \delta(t - t_2)$, WD's impulse response $u_W(t, f; t_1, t_2)$ is obtained as

$$u_W(t, f; t_1, t_2) = \delta\left(t - \frac{t_1 + t_2}{2}\right) \exp\left[-j2\pi(t_1 - t_2)f\right].$$

Inserting into (2.5b), it is easily shown that WD is globally unitary and thus also $\mathfrak{S}$-unitary for arbitrary signal subspace $\mathfrak{S}$; hence the projection-transformation method (PTM) or the basis method (BM) can be applied for global signal synthesis as well as subspace signal synthesis on arbitrary signal subspaces $\mathfrak{S}$. We now specialize to the given subspace $\mathfrak{B}$ of band-limited signals. It is easily shown that the kernel of the orthogonal projection operator $\boldsymbol{I}_{\mathfrak{B}}$ of $\mathfrak{B}$ is

$$I_{\mathfrak{B}}(t, t') = h(t - t')$$

where

$$h(t) = e^{j2\pi f_0 t} \upsilon \operatorname{sinc}(\upsilon t)$$

is the impulse response of an ideal bandpass filter with passband $[f_1, f_2]$; here, $f_0 = (f_1 + f_2)/2$ and $\upsilon = f_2 - f_1$ denote the passband's center frequency and bandwidth, respectively. Also, it follows from the sampling theorem that an orthonormal basis $\{e_k(t)\}$ spanning $\mathfrak{B}$ is given by

$$e_k(t) = h(t - k/\upsilon), \qquad -\infty < k < \infty. \tag{7.1}$$

A. Projection-Transformation Method

To apply the PTM, we have first to calculate the operator $\boldsymbol{u}_{W\mathfrak{B}}$. With $u_{W\mathfrak{B}}(t, f; t_1, t_2) = W_{x,y}(t, f)$ for $x(t) = I_{\mathfrak{B}}(t, t_1) = h(t - t_1)$ and $y(t) = I_{\mathfrak{B}}(t, t_2) = h(t - t_2)$ (cf. Section II), it is easily shown that

$$u_{W\mathfrak{B}}(t, f; t_1, t_2) = W_h\left(t - \frac{t_1 + t_2}{2}, f\right) \cdot \exp\left[-j2\pi(t_1 - t_2)f\right]$$

whereas the BM uses the spectral decomposition (4.6)

$$\tilde{\boldsymbol{\Gamma}}_H = \sum_{k=1}^{N} \mu_k \boldsymbol{v}_k \boldsymbol{v}_k^+. \tag{5.2}$$

These spectral decompositions can be related to each other by inserting (4.3) into (3.6); straightforward manipulation then yields

$$\tilde{q}_{\mathfrak{S}H}(t_1, t_2) = \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{\gamma}_{H,ij} q_{ij}(t_1, t_2)$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{\gamma}_{H,ij} e_i(t_1) e_j^*(t_2).$$

We now insert (5.1) for $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ and (5.2) for $\tilde{\gamma}_{H,ij}$ and obtain

$$\sum_{k=1}^{\infty} \lambda_k u_k(t_1) u_k^*(t_2) = \sum_{k=1}^{N} \mu_k \left[\sum_{i=1}^{N} v_{k,i} e_i(t_1) \right] \cdot \left[\sum_{j=1}^{N} v_{k,j} e_j(t_2) \right]^* \tag{5.3}$$

where $v_{k,i}$ denotes the ith element of the kth eigenvector $\boldsymbol{v}_k$. Based on the orthonormality of the eigenvectors $\boldsymbol{v}_k$ and the orthonormality of the basis signals $e_i(t)$, it is easily shown that the signals $\Sigma_i v_{k,i} e_i(t)$ are orthonormal as well. From (5.3), it then follows that

$$\lambda_k = \begin{cases} \mu_k, & 1 \le k \le N \\ 0, & k > N \end{cases}$$

and

$$u_k(t) = \sum_{i=1}^{N} v_{k,i} e_i(t), \qquad 1 \le k \le N. \tag{5.4}$$

We have thus shown that 1) the number of nonzero eigenvalues λ_k of $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ is not greater than N, the dimension of the space $\mathfrak{S}$; 2) the nonzero eigenvalues λ_k of $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ equal the eigenvalues μ_k of $\tilde{\boldsymbol{\Gamma}}_H$; 3) the eigenfunctions $u_k(t)$ of $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ and the eigenvectors $\boldsymbol{v}_k$ of $\tilde{\boldsymbol{\Gamma}}_H$ are related by (5.4), i.e., the elements $v_{k,i}$ of the kth eigenvector $\boldsymbol{v}_k$ of $\tilde{\boldsymbol{\Gamma}}_H$ are the expansion coefficients of the kth eigenfunction $u_k(t)$ of $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ in terms of the basis $\{e_i(t)\}$ spanning the space $\mathfrak{S}$.

VI. Special Cases

This section considers some special situations which call for further discussion.

A. Global Signal Synthesis

Global (unconstrained) signal synthesis can be considered a special case of constrained signal synthesis where the signal space $\mathfrak{S}$ equals the entire space of finite-energy signals, $\mathfrak{S} = L_2(\mathbb{R})$. In this case, $\mathfrak{S}_q = L_2(\mathbb{R}^2)$ and $\boldsymbol{I}_{\mathfrak{S}q} = \boldsymbol{I}_{\mathfrak{Q}}$ whence $\boldsymbol{u}_{T\mathfrak{S}} = \boldsymbol{u}_T$. With this, the transformation (3.7) reduces to $\tilde{q} = \boldsymbol{u}_T^+ \tilde{T}$ or

$$\tilde{q}(t_1, t_2) = \int_\sigma \int_\epsilon u_T^+(t_1, t_2; \sigma, \epsilon) \tilde{T}(\sigma, \epsilon) \, d\sigma \, d\epsilon.$$

Note, however, that our solution to the global signal synthesis problem is valid only if the BSR $T(\sigma, \epsilon)$ is globally unitary, i.e., satisfies $\boldsymbol{u}_T^+ \boldsymbol{u}_T = \boldsymbol{I}_{\mathfrak{Q}}$.

B. Valid Model

Assuming $\mathfrak{S}$-unitarity of the BSR T, we next investigate the case where the model $\tilde{T}(\sigma, \epsilon)$ is a valid BSR outcome of some signal $y(t)$, $\tilde{T}(\sigma, \epsilon) = T_y(\sigma, \epsilon)$.

Case A: Let us first assume that $y(t)$ is an element of the signal space $\mathfrak{S}$ on which signal synthesis is performed, $y(t) \in \mathfrak{S}$. Then $y(t)$ is itself a particular solution of the signal synthesis problem since 1) it satisfies our signal space constraint, and 2) it achieves zero (and thus minimal) synthesis error ϵ_x. Since the synthesis solution is unique apart from an unknown phase factor $e^{j\varphi}$ (cf. Section II), the general synthesis solution is then given by $x_{\text{opt}}(t) = e^{j\varphi} y(t)$.

Case B: If $y(t) \notin \mathfrak{S}$, then $y(t)$ cannot be a solution since it does not satisfy our signal space constraint. In this case, the solution is found quite easily if the BSR T is globally unitary. According to Section III, we have to minimize $\|\tilde{T}_{\mathfrak{S}} - T_x\|$ subject to $x(t) \in \mathfrak{S}$. Now, for a globally unitary BSR T, there is $\boldsymbol{I}_{\mathfrak{S}T} T_y = T_{I_{\mathfrak{S}} y}$ (cf. (2.9)) and thus

$$\tilde{T}_{\mathfrak{S}} = \boldsymbol{I}_{\mathfrak{S}T} \tilde{T} = \boldsymbol{I}_{\mathfrak{S}T} T_y = T_{I_{\mathfrak{S}} y} = T_{y_{\mathfrak{S}}}.$$

This means that the signal is in fact synthesized from the valid model $T_{y_{\mathfrak{S}}}(\sigma, \epsilon)$, where $y_{\mathfrak{S}}(t)$ is the projection of $y(t)$ on $\mathfrak{S}$. Since $y_{\mathfrak{S}}(t) \in \mathfrak{S}$, this reduces to case A whence the solution is found as $x_{\text{opt}}(t) = e^{j\varphi} y_{\mathfrak{S}}(t)$. This solution can be given an interesting interpretation by noting that the projection $y_{\mathfrak{S}}(t)$ is the solution to the classical signal-domain approximation problem [22]

$$\epsilon_x' = \|y - x\| \to \min_{x \in \mathfrak{S}}. \tag{6.1}$$

On the other hand, $e^{j\varphi} y_{\mathfrak{S}}(t)$ has just been shown to be the solution to the T-domain approximation problem (signal synthesis problem)

$$\epsilon_x = \|T_y - T_x\| \to \min_{x \in \mathfrak{S}}. \tag{6.2}$$

This shows that the solution to the signal-domain approximation problem (6.1) and the solution to the T-domain approximation problem (6.2) are equal apart from an unknown phase constant. This equivalence has been noted in [10] for the special case of AF.

C. Ambiguous Model

For a given signal space $\mathfrak{S}$, a model $\tilde{T}(\sigma, \epsilon)$ will be called ambiguous (with respect to the synthesis solution) if the largest eigenvalue λ_1 of $\tilde{q}_{\mathfrak{S}H}(t_1, t_2)$ (or, equivalently, of $\tilde{\boldsymbol{\Gamma}}_H$) has multiplicity $K > 1$. An example of an ambiguous model is

$$\tilde{T}(\sigma, \epsilon) = T_{x_1}(\sigma, \epsilon) + T_{x_2}(\sigma, \epsilon) \tag{6.3}$$

with $x_1(t) \in \mathfrak{S}$, $x_2(t) \in \mathfrak{S}$, $(x_1, x_2) = 0$ and $\|x_1\|^2 = \|x_2\|^2 = E_x$, i.e., the signals $x_1(t)$ and $x_2(t)$ are both elements of $\mathfrak{S}$, are orthogonal and have identical norms (energies).

where the WD of $h(t)$ is [3]

$$W_h(t, f) = \begin{cases} 2(v - 2|f - f_0|) \operatorname{sinc} [2(v - 2|f - f_0|)t], \\ \qquad |f - f_0| < v/2 \\ 0, \\ \qquad |f - f_0| \geq v/2. \end{cases} \quad (7.2)$$

Inserting into (3.7), the projected q-domain model is obtained as

$$\tilde{q}_{\mathfrak{B}}(t_1, t_2) = \int_t \int_f W_h\left(t - \frac{t_1 + t_2}{2}, f\right) \cdot \exp [j2\pi(t_1 - t_2)f] \tilde{W}(t, f) \, dt \, df \quad (7.3)$$

where $\tilde{W}(t, f)$ is the original W-domain model. According to (7.3), $\tilde{q}_{\mathfrak{B}}(t_1, t_2)$ can be calculated by first performing the W-domain convolution

$$\tilde{W}_{\mathfrak{B}}(t, f) = \int_{t'} W_h(t - t', f) \tilde{W}(t', f) \, dt' \quad (7.4)$$

(this can be shown to be the projection of $\tilde{W}(t, f)$ on the induced W-domain subspace $\mathfrak{B}_W$), and then performing a Fourier transform plus a coordinate transform

$$\tilde{q}_{\mathfrak{B}}(t_1, t_2) = \int_f \tilde{W}_{\mathfrak{B}}\left(\frac{t_1 + t_2}{2}, f\right) \exp [j2\pi(t_1 - t_2)f] \, df.$$

Note that $\tilde{q}_{\mathfrak{B}}(t_1, t_2)$ will be Hermitian if the original W-domain model $\tilde{W}(t, f)$ is real-valued. Due to (7.2), the convolution (7.4) can be interpreted as follows: inside the passband $[f_1, f_2]$, the model $\tilde{W}(t, f)$ is filtered with respect to the time parameter t by an ideal low-pass filter with cutoff frequency $f_c(f) = v - 2|f - f_0|$ (which depends on f); outside $[f_1, f_2]$, the model $\tilde{W}(t, f)$ is replaced by zero.

B. Basis Method

To apply the BM, we have first to calculate the induced W-domain basis functions according to (2.7). With (7.1), we obtain

$$W_{kl}(t, f) = W_{e_k, e_l}(t, f) = W_h\left(t - \frac{k + l}{2} \frac{1}{v}, f\right) \cdot \exp [-j2\pi(k - l)f/v] \quad (7.5)$$

and we notice the interesting relation

$$W_{kl}(t, f) = u_{W\mathfrak{B}}(t, f; k/v, l/v).$$

Inserting (7.5) into (4.7), the model expansion coefficients are obtained as

$$\tilde{\gamma}_{kl} = \int_t \int_f \tilde{W}(t, f) W_h\left(t - \frac{k + l}{2} \frac{1}{v}, f\right) \cdot \exp [j2\pi(k - l)f/v] \, dt \, df, \quad -\infty < k, l < \infty. \quad (7.6)$$

The coefficients $\tilde{\gamma}_{kl}$ will be Hermitian if the model $\tilde{W}(t, f)$ is real-valued. Comparing (7.6) and (7.3), we notice that $\tilde{\gamma}_{kl}$ is a sampled version of $\tilde{q}_{\mathfrak{B}}(t_1, t_2)$

$$\tilde{\gamma}_{kl} = \tilde{q}_{\mathfrak{B}}(k/v, l/v).$$

In the case of band-limited signal synthesis and WD, the BM is thus simply a discrete version of the PTM.

We conclude this section by presenting a computer simulation example in Fig. 7. The model (Fig. 7(a)) was designed to resemble the WD of a chirp signal whose instantaneous frequency extends well beyond the band $[f_1, f_2]$ on which signal synthesis is performed. The synthesis result, shown in Fig. 7(b), was derived using a discrete-time verson of the BM. From the WD of the synthesized signal (Fig. 7(c)) it is seen that the signal is indeed band-limited on $[f_1, f_2]$.

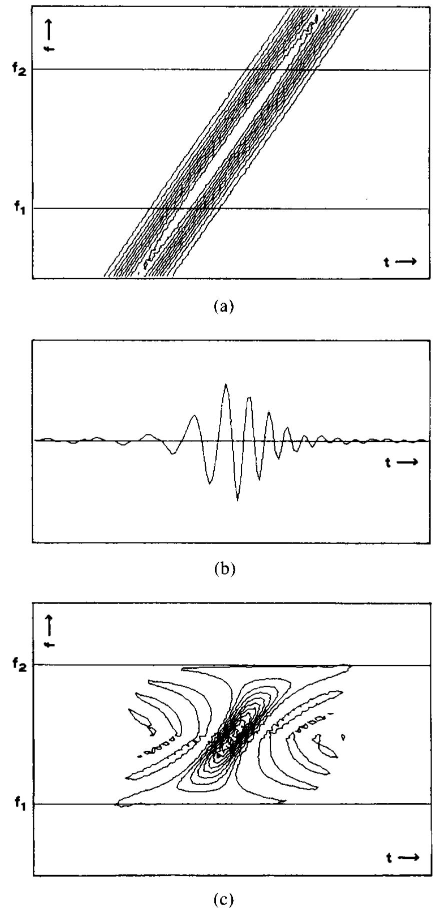

Fig. 7. Band-limited signal synthesis for Wigner distribution—computer simulation. (a) Model $\tilde{W}(t, f)$; (b) real part of synthesized signal $x_{\text{opt}}(t)$; (c) WD $W_{x_{\text{opt}}}(t, f)$ of synthesized signal.

VIII. Conclusion

Using the linear-operator description of bilinear signal representations (BSR's) and the concepts of BSR unitarity and induced spaces [19], we have derived the projection-transformation method (PTM) and the basis method (BM) as two general methods of bilinear signal synthesis with a signal space constraint. The methods are general with respect to both the BSR $T(\sigma, \epsilon)$ and the signal space $\mathfrak{S}$ on which signal synthesis is performed. An essential requirement, however, is that the BSR $T(\sigma, \epsilon)$ is unitary (i.e., satisfies Moyal's formula) on the signal space $\mathfrak{S}$. This requirement is, for example, met by (continuous-time) WD and AF.

The PTM and the BM rely on the description of the signal space $\mathfrak{S}$ by an orthogonal projection operator or an orthonormal basis, respectively. The methods are theoretically equivalent with respect to their results but generally different with respect to computation and storage requirements. The BM is particularly advantageous when the signal space $\mathfrak{S}$ has small dimension.

The signal space constraint can be utilized to enforce practically important signal properties. The synthesis of band-limited signals has been considered as a simple example. We note that a band-limitation constraint is particularly relevant in the case of discrete-time WD where it assures unitarity and serves to avoid aliasing effects and a troublesome phase ambiguity in the synthesized signal [8], [14], [23]. Other signal properties can be accommodated in a likewise fashion provided that they correspond to a linear signal space and an orthogonal operator or an orthonormal basis of the respective space is available. A particularly interesting class of signal spaces is given by the "time-frequency subspaces" introduced in [24]. These spaces are concentrated in a prescribed time-frequency region; when used in the context of subspace-constrained signal synthesis, the synthesized signal is itself forced to be concentrated in this time-frequency region, which corresponds to a "time-frequency selective" mode of signal synthesis.

While most of the theoretically attractive BSR's (like WD, Rihaczek distribution, and AF) satisfy the unitarity requirement of the PTM and the BM, it should be noted that smoothed versions of WD (such as the pseudo-WD [3], the spectrogram [25] or the Choi-Williams distribution [26]) are inherently nonunitary. For smoothed WD versions, heuristic extensions of the PTM and BM have been proposed; these methods are iterative and have been shown to produce satisfactory results [17], [18].

We finally note that the signal synthesis methods developed in this paper can be extended to the optimum time-frequency synthesis of linear signal spaces [27], linear time-varying systems [28], and nonstationary random processes [29].

Appendix
Solution of the Basic Approximation Problem

We derive the solution to the basic approximation problem (3.3)

$$\epsilon_x^2 = \|\tilde{q} - q_x\|^2 = \int_{t_1}\int_{t_2} |\tilde{q}(t_1, t_2) - q_x(t_1, t_2)|^2 \, dt_1 \, dt_2 \rightarrow \min_x. \qquad \text{(A.1)}$$

Here, $\tilde{q}(t_1, t_2)$ is a Hermitian square-integrable function which allows a spectral representation [20]

$$\tilde{q}(t_1, t_2) = \sum_{k=1}^{\infty} \lambda_k u_k(t_1) u_k^*(t_2) = \sum_{k=1}^{\infty} \lambda_k q_{u_k}(t_1, t_2) \qquad \text{(A.2)}$$

where the real-valued eigenvalues λ_k and the orthonormal eigenfunctions $u_k(t)$ are the solutions to the eigenvalue-eigenfunction equation

$$\int_{t_2} \tilde{q}(t_1, t_2) u(t_2) \, dt_2 = \lambda u(t_1).$$

The system $\{u_k(t)\}$ of eigenfunctions need not be complete in $L_2(\mathbb{R})$; in general, it spans some linear space $\mathfrak{U} \subseteq L_2(\mathbb{R})$ which is either $L_2(\mathbb{R})$ (case of completeness) or some subspace of $L_2(\mathbb{R})$. Due to (A.2), $\tilde{q}(t_1, t_2)$ is an element of the induced q-domain subspace $\mathfrak{U}_q$. As shown in Section III (cf. Fig. 6), the solution $x(t)$ of (A.1) must then itself be an element of $\mathfrak{U}$ and can thus be represented in terms of the orthonormal eigenfunction basis $\{u_k(t)\}$ spanning $\mathfrak{U}$,

$$x(t) = \sum_{k=1}^{\infty} \alpha_k u_k(t). \qquad \text{(A.3)}$$

Inserting (A.2) and (A.3) into (A.1) and using the orthonormality of the $u_k(t)$, the (squared) aproximation error can then be developed as

$$\begin{aligned} \epsilon_x^2 &= \int_{t_1}\int_{t_2} \left| \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (\lambda_k \delta_{kl} - \alpha_k \alpha_l^*) \cdot u_k(t_1) u_l^*(t_2) \right|^2 dt_1 \, dt_2 \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |\lambda_k \delta_{kl} - \alpha_k \alpha_l^*|^2 \\ &= \sum_{k=1}^{\infty} (\lambda_k - a_k)^2 + \sum_{k=1}^{\infty} \sum_{\substack{l=1 \\ (k \neq l)}}^{\infty} a_k a_l \end{aligned} \qquad \text{(A.4)}$$

with

$$a_k \triangleq |\alpha_k|^2$$

where the a_k are realvalued and nonnegative. The necessary conditions for a minimum are obtained as

$$\frac{\partial \epsilon_x^2}{\partial a_k} = 0 \qquad \text{for all } k;$$

this gives the system of equations

$$\sum_{l=1}^{\infty} a_l = \lambda_k \qquad \text{for all } k.$$

These equations are incompatible (unless all λ_k are identical); hence only a marginal minimum is possible. As $0 \leq a_k < \infty$ and $\epsilon_x^2 \rightarrow \infty$ for $a_k \rightarrow \infty$ (with k arbitrary), there remains the margin 0 to be investigated. We let $a_m = 0$ for some m are rederive the necessary conditions

$$\left.\frac{\partial \epsilon_x^2}{\partial a_k}\right|_{a_m = 0} = 0 \qquad \text{for all } k \neq m.$$

The resulting equations

$$\sum_{\substack{l=1 \\ (l \neq m)}}^{\infty} a_l = \lambda_k \qquad \text{for all } k \neq m$$

are seen to be still incompatible. We continue by setting one more a_k to zero and reexamining the necessary equations. This process can be repeated with the necessary equations being incompatible at each stage. If we assume all eigenvalues λ_k to be distinct, then the only way to obtain compatible equations is to let all but one a_k be zero ($a_k = 0$ for all $k \neq n$; n arbitrary) whence the necessary condition becomes

$$a_n = \lambda_n. \tag{A.5}$$

Inserting (A.5) and $a_k = 0$ for $k \neq n$ into (A.3) yields

$$x(t) = e^{j\varphi}\sqrt{\lambda_n}u_n(t) \qquad \text{with } n, \varphi \text{ arbitrary.} \tag{A.6}$$

The form (A.6) is necessary for the solution of (A.1); we now have to ask which n actually minimizes the error ϵ_x. Inserting (A.5) and $a_k = 0$ for $k \neq n$ into (A.4), we obtain

$$\epsilon_x^2 = \sum_{k=1}^{\infty} \lambda_k^2 - \lambda_n^2$$

whence it appears that the optimal λ_n is the eigenvalue with maximum magnitude; however, since λ_n cannot be negative due to $\lambda_n = a_n$ and $a_n \geq 0$, it follows that the correct λ_n is simply the maximum eigenvalue provided that the maximum eigenvalue is nonnegative (the case of a negative maximum eigenvalue is discussed in Section VI).

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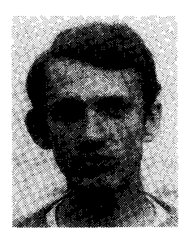

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