Regularity and Unitarity of Bilinear Time-Frequency Signal Representations

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Abstract—Two structural properties of bilinear time-frequency representations (BTFR's) of signals are introduced and studied. The definition of these properties is based on a linear-operator description of BTFR's. The first property, termed *regularity*, has important implications with respect to the recovery of signals from their BTFR outcome, the derivation of other bilinear signal representations from a BTFR, the BTFR's reaction to linear signal transformations, and the construction of bases of induced BTFR-domain spaces. The second property, called *unitarity*, is equivalent to validity of Moyal's formula. Unitarity is thus necessary and sufficient for a closed-form solution of optimal signal synthesis and for a BTFR formulation of optimal detection/estimation methods. Besides, unitarity also allows the systematic construction of BTFR "product relations" like Wigner distribution's interference formula and ambiguity function's self-transformation property. Finally, unitarity permits the construction of induced orthogonal projection operators and guarantees the orthonormality of induced basis functions. The properties of regularity and unitarity are studied both in the general framework of bilinear signal representations and inside the important classes of shift-invariant and shift-scale-invariant BTFR's.

Index Terms—Time-frequency analysis, bilinear signal representations, Cohen class, Wigner distribution.

I. INTRODUCTION

TIME-FREQUENCY signal representations characterize signals via a function of time and frequency, thereby combining conventional time-domain and frequency-domain analyses. They may depend on the signal(s) in a linear, quadratic (bilinear), or otherwise nonlinear manner. The present paper considers the class of *bilinear time-frequency representations* (BTFR's), of which the Wigner distribution (WD), the ambiguity function (AF), and the spectrogram are the most important members [1–8].

From the viewpoint of interpretation, the various BTFR's can be grouped into two subclasses or "domains" [9]. BTFR's of the *energetic domain* ("E-domain") can, at least conceptually and with certain restrictions due to the uncertainty principle, be interpreted as a joint time-frequency

energy density. Among E-domain BTFR's, the WD

$$W_{x,y}(t, f) = \int_{\tau} x\left(t + \frac{\tau}{2}\right) y^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f\tau} d\tau \quad (1.1)$$

is theoretically optimal in a certain sense [3], [5], [6], [10] (in (1.1) and subsequent equations, $x(t)$ and $y(t)$ are the signals to be analyzed, t and f are time and frequency, respectively, and integrations extend from $-\infty$ to ∞). Other E-domain BTFR's are the distributions of Rihaczek, Page, and Levin, and smoothed versions of WD such as the spectrogram, the pseudo WD, the Born-Jordan distribution, and the exponential (Choi-Williams) distribution [1].

BTFR's of the *correlative domain* ("C-domain") are interpreted as a joint time-frequency correlation. Among C-domain BTFR's, the AF

$$A_{x,y}(\tau, \nu) = \int_t x\left(t + \frac{\tau}{2}\right) y^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi \nu t} dt \quad (1.2)$$

can be considered theoretically optimal (here, τ and ν denote time lag and frequency lag, respectively). E-domain and C-domain are related by a Fourier-transform duality [9]: to any E-domain BTFR $T^{(E)}(t, f)$, a dual C-domain BTFR $T^{(C)}(\tau, \nu)$ can be defined by

$$\begin{aligned} T^{(C)}(\tau, \nu) &= \mathcal{F}_{t \rightarrow \nu} \mathcal{F}_{f \rightarrow \tau}^{-1} T^{(E)}(t, f) \\ &= \int_t \int_f T^{(E)}(t, f) e^{-j2\pi(\nu t - \tau f)} dt df. \end{aligned} \quad (1.3)$$

For example, WD and AF are dual in the sense of (1.3) [3], [11].

In the context of BTFR's, many "desirable" properties have been considered which a BTFR should satisfy in order to be accepted as a valid time-frequency description of signals. Examples of such properties are the shift-invariance, scale-invariance, marginal, finite-support, instantaneous-frequency, and group-delay property [3], [5], [6], [12]. These properties essentially express the consistency of a BTFR with conventional time-domain and frequency-domain analysis.

The present paper defines and discusses two BTFR properties, termed *regularity* and *unitarity*, which are somewhat different from the properties previously mentioned since they are only based on a BTFR's bilinear structure and not on any time-frequency interpretation. The definition of both properties makes use of a characterization of BTFR's by means of linear operators. The first property, *regularity*, expresses basically the "reversibility" of a BTFR. The second property, *unitarity*, is equivalent to validity of Moyal's formula

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[13], [14] that expresses essentially the preservation of inner products and norms.

The two properties of regularity and unitarity have many implications that are both theoretically and practically important. Specifically, some implications discussed in this paper concern the following topics: recovery of signals from their BTFR outcomes, derivation of other bilinear signal representations and signal parameters from a BTFR, reaction of a BTFR to linear signal transformations, preservation of inner products and norms, construction of bases and projection operators of induced BTFR-domain spaces, and derivation of “BTFR product relations.”

The paper is organized as follows. Section II provides a unified concept of bilinear signal representations (BSR's) that includes BTFR's as a special case. Two different ways of characterizing BSR's by means of linear operators are then discussed, and the concept of induced spaces is introduced. Concentrating on BTFR's, Sections III and IV define the properties of regularity and unitarity, respectively and discuss some implications of these properties. Finally, Sections V and VI consider the properties of regularity and unitarity inside the classes of shift-invariant and shift-scale-invariant BTFR's.

Throughout the paper, continuous-time signals will be used; however, we stress that all definitions and results can be reformulated for discrete-time signal and BTFR's in a straightforward manner.

II. BILINEAR SIGNAL REPRESENTATIONS

A. The Class of Bilinear Signal Representations

E-domain and C-domain BTFR's constitute subclasses of the general class of *bilinear signal representations* (BSR's). Any BSR can be written as [12], [15], [16]

$$T_{x,y}(\Theta) = \int_{t_1} \int_{t_2} u_T(\Theta; t_1, t_2) x(t_1) y^*(t_2) dt_1 dt_2, \quad (2.1)$$

where Θ is a vector of parameters and $u_T(\Theta; t_1, t_2)$ is a kernel function characterizing the BSR T . (Strictly speaking, the representation $T_{x,y}(\Theta)$ is *sesquilinear* rather than bilinear, owing to the conjugation of $y(t)$ involved in (2.1).) The number and nature of the parameters depend on the respective BSR T . Obviously, BTFR's have two parameters: $\Theta = (t, f)$ for an E-domain BTFR or $\Theta = (\tau, v)$ for a C-domain BTFR. Other BSR's with two parameters are bilinear time-scale representations [17] and the *outer signal product*

$$q_{x,y}(t_1, t_2) = x(t_1) y^*(t_2). \quad (2.2)$$

Examples of BSR's with only one parameter are the instantaneous power $p(t)$ and the correlation $r(\tau)$ defined as

$$p_{x,y}(t) = x(t) y^*(t),$$

$$r_{x,y}(\tau) = \int_t x(t + \tau) y^*(t) dt. \quad (2.3)$$

Finally, there may also be no parameter at all, as in the case of the energy

$$E_{x,y} = \int_t x(t) y^*(t) dt = (x, y).$$

The general BSR $T_{x,y}(\Theta)$ of (2.1) can be considered a “cross representation” of the two signals $x(t)$ and $y(t)$; the corresponding “auto representation” $T_x(\Theta) \triangleq T_{x,x}(\Theta)$ is then obtained by letting $y(t) = x(t)$. Note that the auto BSR $T_x(\Theta)$ is quadratic rather than bilinear.

B. Linear-Operator Descriptions of BSR's

BSR's can be interpreted mathematically in two different ways, each of them corresponding to a linear-operator characterization of BSR's. The first interpretation [15] is obtained by writing (2.1) as a bilinear form

$$T_{x,y}(\Theta) = (v_T^{(\Theta)} x, y) = \int_t (v_T^{(\Theta)} x)(t) y^*(t) dt, \quad (2.4)$$

where

$$(v_T^{(\Theta)} x)(t) = \int_{t'} v_T^{(\Theta)}(t, t') x(t') dt'$$

describes a linear transformation of $x(t)$, the kernel of the linear operator $v_T^{(\Theta)}$ being

$$v_T^{(\Theta)}(t, t') = u_T(\Theta; t', t). \quad (2.5)$$

The family of linear operators $v_T^{(\Theta)}$ characterizes the BSR T . Some BSR properties can easily be expressed via constraints on $v_T^{(\Theta)}$: for example, the auto BSR $T_x(\Theta)$ is real-valued for any (complex-valued) signal $x(t)$, if and only if the operators $v_T^{(\Theta)}$ are self-adjoint for any Θ .

The second interpretation [12] of a BSR is obtained by writing (2.1) as

$$T_{x,y}(\Theta) = \int_{t_1} \int_{t_2} u_T(\Theta; t_1, t_2) q_{x,y}(t_1, t_2) dt_1 dt_2,$$

with the outer signal product $q_{x,y}(t_1, t_2)$ as defined in (2.2), and interpreting the BSR $T_{x,y}(\Theta)$ as the result of a linear transformation of the “elementary BSR” $q_{x,y}(t_1, t_2)$,

$$T_{x,y} = u_T q_{x,y}, \quad (2.6)$$

where the kernel of the linear operator u_T is $u_T(\Theta; t_1, t_2)$. The “BSR operator” u_T again characterizes the BSR T . We assume that u_T is a bounded operator with respect to the L_2 norm [18]; this condition is satisfied by practical BSR's.

The two BSR interpretations are illustrated in Fig. 1. It should be noted that, even though the kernels of the operators $v_T^{(\Theta)}$ and u_T can be equated as in (2.5), the operators themselves are profoundly different.

The BSR properties considered in later sections will be based on the second interpretation (2.6). In the light of (2.6), the BSR outcome $T_{x,y}(\Theta)$ to given signals $x(t)$, $y(t)$ is formed by first calculating the outer signal product $q_{x,y}(t_1, t_2)$ and then applying the linear operator u_T to $q_{x,y}(t_1, t_2)$. In this process, three “domains” can be distinguished: the *signal domain* (signals $x(t)$, $y(t)$), the *q-domain* (signal product $q_{x,y}(t_1, t_2)$), and the *T-domain* (BSR $T_{x,y}(\Theta)$). The operator u_T performs a mapping from *q-domain* into *T-domain*. The kernel $u_T(\Theta; t_1, t_2)$ of u_T can be interpreted as the *impulse response* of the BSR T : from

Fig. 1. Two interpretations and linear-operator characterizations of a BSR $T_{x,y}(\Theta)$: (a) Bilinear form in the signals $x(t), y(t)$; (b) result of linear transformation $\mathbf{u}_T$ of "elementary BSR" $q_{x,y}(t_1, t_2)$.

(2.1), there follows

$$u_T(\Theta; t_1, t_2) = T_{x,y}(\Theta), \quad \text{for } \begin{cases} x(t) = \delta(t - t_1) \\ y(t) = \delta(t - t_2). \end{cases}$$

This fact often permits a simple calculation of $u_T(\Theta; t_1, t_2)$ if the BSR $T_{x,y}(\Theta)$ is given by an explicit expression. For example, the impulse response of WD (1.1) is easily obtained as

$$u_W(t, f; t_1, t_2) = \delta\left(t - \frac{t_1 + t_2}{2}\right) e^{-j2\pi(t_1 - t_2)f}. \quad (2.7)$$

We note that the impulse responses of dual E-domain and C-domain BTFR's $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ are related by a Fourier transform exactly as are the BTFR's themselves (cf. (1.3)),

$$u_{T(C)}(\tau, v; t_1, t_2) = \mathcal{F}_{t \rightarrow v} \mathcal{F}_{f \rightarrow \tau}^{-1} u_{T(E)}(t, f; t_1, t_2).$$

C. Induced T -Domain Spaces

The somewhat vague notion of T -domain can be made mathematically precise by the introduction of "induced T -domain spaces." In the following, we concentrate on E-domain BTFR's for the sake of concreteness. The discussion can be extended to general BSR's in a straightforward manner.

Let us consider the BTFR outcome $T_{x,y}(t, f)$ for given signals $x(t), y(t)$. We assume that $x(t), y(t) \in L_2(\mathbb{R})$, i.e., the signals $x(t), y(t)$ are taken from the linear space of square-integrable (finite-energy) signals, such that inner products and norms can be defined in the usual way. Then, it is easily seen that $T_{x,y}(t, f) \in L_2(\mathbb{R}^2)$, i.e., the BTFR outcome of square-integrable signals is a square-integrable two-dimensional function. Indeed, it follows from $x(t), y(t) \in L_2(\mathbb{R})$ that $q_{x,y}(t_1, t_2) \in L_2(\mathbb{R}^2)$ whence also $T_{x,y}(t, f) \in L_2(\mathbb{R}^2)$ since $T_{x,y} = \mathbf{u}_T q_{x,y}$ with $\mathbf{u}_T$ being a bounded operator.

We next suppose that, more specifically, $x(t), y(t) \in S$ where S is a given linear subspace of $L_2(\mathbb{R})$ (or, as a special case, $L_2(\mathbb{R})$ itself). We may now ask if the BTFR outcome $T_{x,y}(t, f)$ is an element of some "induced T -domain space" $S_T \subseteq L_2(\mathbb{R}^2)$, which is associated with the signal space $S \subseteq L_2(\mathbb{R})$ in some meaningful sense. Now, the set of all

BTFR outcomes $T_{x,y}(t, f)$ with $x(t), y(t) \in S$ does not itself constitute a linear space since a linear combination of BTFR outcomes is not, in general, again a valid BTFR outcome. Therefore, the induced T -domain space must be defined such that it includes also functions $\tilde{T}(t, f)$ which are not valid BTFR outcomes:

Definition 1: For a given BTFR $T(t, f)$, the *induced T -domain space* S_T associated with a linear signal space $S \subseteq L_2(\mathbb{R})$ is defined as the linear space of all two-dimensional functions

$$\tilde{T}(t, f) = \sum_k \sum_l c_{kl} T_{e_k, e_l}(t, f) \quad \text{with } \sum_k \sum_l |c_{kl}|^2 < \infty, \quad (2.8)$$

where $\{e_k(t)\}$ is an arbitrary orthonormal basis of S .

The following statements are easily shown:

- a) The induced T -domain space S_T is a subspace of $L_2(\mathbb{R}^2)$,

$$S_T \subseteq L_2(\mathbb{R}^2).$$

- b) If two signals are elements of the signal space S , then their BTFR outcome is an element of the induced T -domain space S_T ,

$$x(t), y(t) \in S \Rightarrow T_{x,y}(t, f) \in S_T.$$

This statement can be extended to finite linear combinations of BTFR outcomes:

$$x_k(t), y_l(t) \in S \Rightarrow \sum_{k=1}^K \sum_{l=1}^L a_{kl} T_{x_k, y_l}(t, f) \in S_T.$$

- c) An element $\tilde{T}(t, f)$ of the induced T -domain S_T is a valid BTFR outcome $T_{x,y}(t, f)$ of some signals $x(t), y(t) \in S$, if and only if the coefficients c_{kl} in the expansion (2.8) are *separable*,

$$c_{kl} = \alpha_k \beta_l^* \quad \text{with } \sum_k |\alpha_k|^2 < \infty \quad \text{and} \quad \sum_l |\beta_l|^2 < \infty;$$

the signals are here given by

$$x(t) = C \sum_k \alpha_k e_k(t), \quad y(t) = \frac{1}{C^*} \sum_l \beta_l e_l(t),$$

where $\{e_k(t)\}$ is the basis used in (2.8) and C is an arbitrary, nonzero, complex constant.

The induced T -domain space $[L_2(\mathbb{R})]_T$ associated with the space $L_2(\mathbb{R})$ of square-integrable (finite-energy) signals is of particular importance. In general, $[L_2(\mathbb{R})]_T$ is a subspace of $L_2(\mathbb{R}^2)$. For the elementary BSR (outer signal product) $q(t_1, t_2)$, however, it is easily shown that $[L_2(\mathbb{R})]_q = L_2(\mathbb{R}^2)$. The class of BTFR's T for which $[L_2(\mathbb{R})]_T = L_2(\mathbb{R}^2)$ will be considered in Section III.

D. Domain and Range of the BTFR Operator $\mathbf{u}_T$

Due to $T_{x,y} = \mathbf{u}_T q_{x,y}$, the BTFR operator $\mathbf{u}_T$ of a given BTFR T constitutes a mapping from the q -domain into the T -domain. To make this more precise, we suppose that the underlying signal space is $L_2(\mathbb{R})$, that is $x(t), y(t) \in L_2(\mathbb{R})$. It follows that $q_{x,y}(t_1, t_2) \in [L_2(\mathbb{R})]_q = L_2(\mathbb{R}^2)$, and it is thus reasonable to define the *domain* [18] of the BTFR

operator u_T to be the linear space $L_2(\mathbb{R}^2)$ of square-integrable two-dimensional functions.

Theorem 1: If the domain of the BTFR operator u_T is defined as $L_2(\mathbb{R}^2)$, then the range [18] of u_T is the induced T -domain space $[L_2(\mathbb{R})]_T$ associated with $L_2(\mathbb{R})$.

A proof of this theorem is given in Appendix A. Due to Theorem 1, the BTFR operator u_T is a mapping of $L_2(\mathbb{R}^2)$ onto $[L_2(\mathbb{R})]_T \subseteq L_2(\mathbb{R}^2)$ (see Fig. 2). Two consequences of this fact are the following.

- a) If $\tilde{q}(t_1, t_2) \in L_2(\mathbb{R}^2)$, then $\tilde{T}(t, f) = (u_T \tilde{q})(t, f) \in [L_2(\mathbb{R})]_T$.
- b) Conversely, if $\tilde{T}(t, f) \in [L_2(\mathbb{R})]_T$, then there exists a $\tilde{q}(t_1, t_2) \in L_2(\mathbb{R}^2)$ such that $\tilde{T} = u_T \tilde{q}$. Note that, in general, $\tilde{q}(t_1, t_2)$ need not be unique (it will be unique if the BTFR is *regular*—see Section III).

III. REGULARITY

In this and the next section, we concentrate on the special case of E-domain BTFR's for the sake of concreteness. All results can be reformulated for C-domain BTFR's simply by replacing t, f by τ, v .

A. Definition of Regularity

The linear BTFR operator u_T was shown in the previous section to perform a mapping of $L_2(\mathbb{R}^2)$ into $L_2(\mathbb{R}^2)$, with range $[L_2(\mathbb{R})]_T \subseteq L_2(\mathbb{R}^2)$. If this mapping is one-to-one, then it can be inverted on the range $[L_2(\mathbb{R})]_T$, i.e., there exists a (left) inverse BTFR operator u_T^{-1} with domain $[L_2(\mathbb{R})]_T$ (the range of u_T) and range $L_2(\mathbb{R}^2)$ (the domain of u_T) [18].

Definition 2 [12], [19]: A BTFR $T(t, f)$ is said to be *regular* if its operator u_T possesses a bounded left inverse u_T^{-1} such that

$$u_T^{-1} u_T = I, \quad (3.1a)$$

which is short for

$$\begin{aligned} \int_t \int_f u_T^{-1}(t_1, t_2; t, f) u_T(t, f; t'_1, t'_2) dt df \\ = \delta(t_1 - t'_1) \delta(t_2 - t'_2). \end{aligned} \quad (3.1b)$$

A BTFR that is not regular will be called *singular*. In the case of a regular BTFR T , the relation $T_{x,y} = u_T q_{x,y}$ can be inverted as

$$q_{x,y} = u_T^{-1} T_{x,y}, \quad (3.2a)$$

$$q_{x,y}(t_1, t_2) = \int_t \int_f u_T^{-1}(t_1, t_2; t, f) T_{x,y}(t, f) dt df. \quad (3.2b)$$

This has some far-reaching implications [12], [19], which are presented in the following theorems.

B. Uniqueness

Theorem 2: Let T be a regular BTFR. Then

$$\begin{aligned} T_{x_1, y_1}(t, f) &= T_{x_2, y_2}(t, f) = \\ x_1(t) &= C x_2(t), \quad y_1(t) = \frac{1}{C^*} y_2(t), \end{aligned}$$

Fig. 2. The BTFR operator u_T is a mapping of $L_2(\mathbb{R}^2)$ into $L_2(\mathbb{R}^2)$, with range $[L_2(\mathbb{R})]_T$.

where C is an arbitrary, nonzero, complex constant. Similarly,

$$T_{x_1}(t, f) = T_{x_2}(t, f) \Rightarrow x_1(t) = e^{j\varphi} x_2(t),$$

where φ is an arbitrary constant phase.

Proof: Since the BTFR operator u_T is one-to-one, $T_{x_1, y_1}(t, f) = T_{x_2, y_2}(t, f)$ entails $q_{x_1, y_1}(t_1, t_2) = q_{x_2, y_2}(t_1, t_2)$. Using (2.2), it is easily seen that the latter identity is only possible if the signals are related as stated in the theorem. $\square$

C. Recovery of Signals

Theorem 3: Let T be a regular BTFR. Then a signal $x(t)$ can be recovered from its auto BTFR outcome $T_x(t, f)$ up to a constant phase factor by the following steps:

- a) $q_x = u_T^{-1} T_x$,
- b) $x(t) = e^{j\varphi_o} P q_x(t, t_o) / \sqrt{q_x(t_o, t_o)} P$,

where t_o is an arbitrary time point for which $q_x(t_o, t_o) = |x(t_o)|^2 \neq 0$, and φ_o is an unknown constant phase.

Proof: Step a) is possible since T is regular (cf. (3.2)). Step b) is easily shown by inserting (2.2) and $x(t) = |x(t)| \exp(j\varphi(t))$ into the right-hand side of Step b). $\square$

Theorem 3 can be generalized to cross BTFR's provided that there exists a time point t_o for which $q_{x,y}(t_o, t_o) = x(t_o)y^*(t_o) \neq 0$ [12], [19].

D. Derivation of Other BSR's

Theorem 4: From a BTFR $T(t, f)$, any other BSR $T'(\Theta)$ can be derived by a bounded linear transformation,

$$T' = LT, \quad T'(\Theta) = \int_t \int_f L(\Theta; t, f) T(t, f) dt df, \quad (3.3)$$

if and only if $T(t, f)$ is regular. The conversion operator L is given by

$$L = u_T u_T^{-1}, \quad (3.4a)$$

$$L(\Theta; t, f) = \int_{t_1} \int_{t_2} u_T(\theta; t_1, t_2) u_T^{-1}(t_1, t_2; t, f) dt_1 dt_2. \quad (3.4b)$$

Proof:

If: With (3.2), $T' = \mathbf{u}_{T'} \mathbf{q} = \mathbf{u}_{T'} (\mathbf{u}_T^{-1} T) = (\mathbf{u}_{T'} \mathbf{u}_T^{-1}) T = \mathbf{L} T$. The boundedness of $\mathbf{L}$ follows from the boundedness of $\mathbf{u}_{T'}$ (this was assumed in Section II) and the boundedness of $\mathbf{u}_T^{-1}$ (see Definition 2).

Only if: If any BSR can be derived from T via a bounded linear operator, then also $\mathbf{q}$ can be thus derived; hence, (3.2) is valid and T must be regular. $\square$

Two simple consequences of Theorem 4 are the following: a) all regular BTFR's are interrelated by reversible, bounded, linear transformations; b) if a BTFR is related with a regular BTFR by a reversible, bounded, linear transformation, then the first BTFR is regular as well.

Example: WD can be shown to be regular, with inverse operator

$$\mathbf{u}_W^{-1}(t_1, t_2; t, f) = \delta\left(t - \frac{t_1 + t_2}{2}\right) e^{j2\pi(t_1 - t_2)f}. \quad (3.5)$$

Inserting into (3.4), it follows that any BSR $T'(\Theta)$ can be derived from WD by means of the conversion operator

$$L(\Theta; t, f) = \int_{\tau} \mathbf{u}_{T'}\left(\Theta; t + \frac{\tau}{2}, t - \frac{\tau}{2}\right) e^{j2\pi f \tau} d\tau. \quad (3.6)$$

We apply this result to derive the instantaneous power $p(t)$ (see (2.3)) from WD. The BSR operator (impulse response) of $p(t)$ is easily calculated; we obtain $u_p(t; t_1, t_2) = \delta(t_1 - t)\delta(t_2 - t)$. Inserting $u_p(t; t_1, t_2)$ for $\mathbf{u}_{T'}(\Theta; t_1, t_2)$ in (3.6) then yields

$$\begin{aligned} L(t; t', f') &= \int_{\tau} \delta\left(t' + \frac{\tau}{2} - t\right) \delta\left(t - \frac{\tau}{2} - t\right) e^{j2\pi f' \tau} d\tau \\ &= \dots = \delta(t' - t). \end{aligned}$$

Inserting into (3.3), we finally obtain the well-known marginal property of WD,

$$\begin{aligned} p(t) &= \int_{t'} \int_{f'} \delta(t' - t) W(t', f') dt' df' \\ &= \int_{f'} W(t, f') df'. \end{aligned}$$

E. Reaction to Linear Signal Transformations

Theorem 5: Let $T(t, f)$ be a regular BTFR. Then, the BTFR outcome for linearly transformed signals

$$\begin{aligned} (gx)(t) &= \int_{t'} g(t, t') x(t') dt', \\ (hy)(t) &= \int_{t'} h(t, t') y(t') dt' \end{aligned}$$

is a linearly transformed BTFR outcome of the original signals,

$$T_{gx, hy} = \mathbf{L}_{g, h} T_{x, y}, \quad (3.7a)$$

$$T_{gx, hy}(t, f) = \int_{t'} \int_{f'} L_{g, h}(t, f; t', f') T_{x, y}(t', f') dt' df', \quad (3.7b)$$

where

$$\mathbf{L}_{g, h} = \mathbf{u}_T \mathbf{z}_{g, h} \mathbf{u}_T^{-1}, \quad (3.8)$$

$$\begin{aligned} L_{g, h}(t, f; t', f') &= \int_{t_1} \int_{t_2} \int_{t'_1} \int_{t'_2} \mathbf{u}_T(t, f; t_1, t_2) \\ &\quad \cdot \mathbf{z}_{g, h}(t_1, t_2; t'_1, t'_2) \mathbf{u}_T^{-1}(t'_1, t'_2; t', f') dt_1 dt_2 dt'_1 dt'_2 \end{aligned}$$

with

$$\mathbf{z}_{g, h}(t_1, t_2; t'_1, t'_2) = g(t_1, t'_1) h^*(t_2, t'_2). \quad (3.9)$$

Proof: It is straightforward to show that $q_{gx, hy} = \mathbf{z}_{g, h} q_{x, y}$. With this and (3.2),

$$\begin{aligned} T_{gx, hy} &= \mathbf{u}_T q_{gx, hy} = \mathbf{u}_T (\mathbf{z}_{g, h} q_{x, y}) \\ &= (\mathbf{u}_T \mathbf{z}_{g, h}) (\mathbf{u}_T^{-1} T_{x, y}) \\ &= (\mathbf{u}_T \mathbf{z}_{g, h} \mathbf{u}_T^{-1}) T_{x, y} = \mathbf{L}_{g, h} T_{x, y}. \quad \square \end{aligned}$$

The operator $\mathbf{L}_{g, h}$ describes the effect of the signal-domain operators $\mathbf{g}, \mathbf{h}$ in the T -domain. We shall call $\mathbf{L}_{g, h}$ the *induced T -domain operator* associated with the signal-domain operators $\mathbf{g}, \mathbf{h}$.

Example: In the case of WD, the induced T -domain transformation associated with signal transformations $\mathbf{g}, \mathbf{h}$ is obtained from (3.8), (3.9), and (2.7), (3.5) as

$$\begin{aligned} L_{g, h}(t, f; t', f') &= \int_{\tau} \int_{\tau'} g\left(t + \frac{\tau}{2}, t' + \frac{\tau'}{2}\right) \\ &\quad \cdot h^*\left(t - \frac{\tau}{2}, t' - \frac{\tau'}{2}\right) e^{-j2\pi(f\tau - f'\tau')} d\tau d\tau'. \end{aligned}$$

Apart from a sign factor in the exponent, this is the WD of the (two-dimensional) functions $g(t, t')$ and $h(t, t')$ [7].

F. Construction of Induced T -Domain Basis

Theorem 6: Let $\{s_k(t)\}$ be a basis of a linear signal space S . If a BTFR $T(t, f)$ is regular, then the BTFR outcomes $T_{s_k, s_l}(t, f)$ constitute a basis of the induced T -domain space S_T , i.e., the BTFR outcomes $T_{s_k, s_l}(t, f)$ are linearly independent and complete in S_T .

A proof of this theorem is given in Appendix B. An important consequence of Theorem 6 is the following: in the case of a regular BTFR $T(t, f)$, any element of the induced T -domain space S_T can be expanded in a unique manner in terms of the valid BTFR outcomes $T_{s_k, s_l}(t, f)$, where the $s_k(t)$ are a given linearly independent and complete (but not necessarily orthogonal) set of elements of the signal space S ,

$$\tilde{T}(t, f) \in S_T \Rightarrow \tilde{T}(t, f) = \sum_k \sum_l a_{kl} T_{s_k, s_l}(t, f).$$

We shall call $\{T_{s_k, s_l}(t, f)\}$ the *induced T -domain basis* associated with the signal-domain basis $\{s_k(t)\}$. Note that, in general, orthonormality of the signal-domain basis $\{s_k(t)\}$ does not entail orthonormality or orthogonality of the induced T -domain basis $\{T_{s_k, s_l}(t, f)\}$. This point is further discussed in Section IV.

G. Strict Regularity

In some applications (in particular, BTFR-based signal synthesis), there arises the question whether any square-integrable function $\tilde{T}(t, f) \in L_2(\mathbb{R}^2)$ is an element of the induced T -domain space $[L_2(\mathbb{R})]_T$ associated with the space $L_2(\mathbb{R})$ of square-integrable (finite-energy) signals. To guarantee this, we need a further assumption regarding the BTFR $T(t, f)$.

Definition 3: A BTFR $T(t, f)$ is said to be *strictly regular* if it is regular and, in addition, satisfies

$$u_T u_T^{-1} = I, \quad (3.10a)$$

$$\int_{t_1} \int_{t_2} u_T(t, f; t_1, t_2) u_T^{-1}(t_1, t_2; t', f') dt_1 dt_2 = \delta(t' - t) \delta(f' - f). \quad (3.10b)$$

Strict regularity means that the inverse (left and right inverse) of the BTFR operator u_T exists. Consequently, the range of u_T (which equals the induced T -domain space $[L_2(\mathbb{R})]_T$ and, in general, is a subspace of $L_2(\mathbb{R}^2)$), must be $L_2(\mathbb{R}^2)$ [18].

Theorem 7: If a BTFR $T(t, f)$ is strictly regular, then the induced T -domain space associated with $L_2(\mathbb{R})$ is $L_2(\mathbb{R}^2)$,

$$[L_2(\mathbb{R})]_T = L_2(\mathbb{R}^2).$$

It follows from Theorem 7 that, in the case of a strictly regular BTFR $T(t, f)$, any square-integrable function $\tilde{T}(t, f)$ is an element of the induced T -domain space $[L_2(\mathbb{R})]_T$. Combining with Theorem 6, we have the following corollary.

Corollary 1: If a BTFR $T(t, f)$ is strictly regular, then the induced T -domain basis associated with a signal-domain basis of $L_2(\mathbb{R})$ is a basis of $L_2(\mathbb{R}^2)$.

Thus, any square-integrable function $\tilde{T}(t, f)$ can be expanded in terms of valid BTFR outcomes $T_{s_k, s_l}(t, f)$, where $\{s_k(t)\}$ is any basis of $L_2(\mathbb{R})$.

Example: Inserting (2.7) and (3.5) into (3.10), it is easily shown that WD is strictly regular.

H. Dual BTFR's

The relation between dual E-domain and C-domain BTFR's being given by the reversible Fourier transform (1.3), it is finally simple to show the following.

Theorem 8: If an E-domain BTFR $T^{(E)}(t, f)$ is strictly regular/regular/singular, then the dual C-domain BTFR $T^{(C)}(\tau, v)$ is strictly regular/regular/singular as well, and vice versa. In the case of regularity, the inverse BTFR operators of $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ are related as

$$u_{T^{(C)}}^{-1}(t_1, t_2; \tau, v) = \mathcal{F}_{\tau \rightarrow v}^{-1} \mathcal{F}_{f \rightarrow t} u_{T^{(E)}}^{-1}(t_1, t_2; t, f).$$

Example: It follows from the strict regularity of WD that AF (1.2) is strictly regular as well since WD and AF are dual BTFR's.

IV. UNITARITY

A. Definition of Unitarity

In the previous section, the property of regularity, which expresses the reversibility of a BTFR, was shown to have rather fundamental implications with respect to signal recovery, derivation of other BSR's, the BTFR's reaction to linear signal transformations, and the construction of induced bases. In some applications of BTFR's, however, it is essential to relate the inner product of signals with the inner product of the associated BTFR outcomes. This relation, known as *Moyal's formula* [13], [14], is particularly important for optimal methods for time-frequency signal synthesis [20]–[24] and detection/estimation [25]–[27]. Here, the BTFR property of regularity is not sufficient. A necessary and sufficient property is expressed as follows.

Definition 4 [12], [28], [29]: A BTFR $T(t, f)$ is said to be *unitary* if its operator satisfies

$$u_T^+ u_T = I \quad (4.1a)$$

$$\int_t \int_f u_T^+(t_1, t_2; t, f) u_T(t, f; t'_1, t'_2) dt df = \delta(t_1 - t'_1) \delta(t_2 - t'_2), \quad (4.1b)$$

where u_T^+ denotes the adjoint of the BTFR operator u_T defined as

$$u_T^+(t_1, t_2; t, f) = u_T^*(t, f; t_1, t_2).$$

In the case of a unitary BTFR T , the mapping u_T relating q -domain and T -domain preserves inner products and norms. This property is exploited by the “projection-transformation method” for BTFR-based signal synthesis [24].

Comparing Definition 4 with Definition 2, we note that a *unitary BTFR is regular*, with left inverse operator given by

$$u_T^{-1} = u_T^+,$$

$$u_T^{-1}(t_1, t_2; t, f) = u_T^+(t_1, t_2; t, f) = u_T^*(t, f; t_1, t_2).$$

Thus, all results derived in Section III for regular BTFR's apply equally well to unitary BTFR's. Obviously, it is rather straightforward to check the unitarity criterion (4.1) for a given BTFR. Also, the calculation of the inverse operator u_T^{-1} is trivial if the BTFR T is unitary.

B. Moyal's Formula

An important implication of unitarity is the validity of Moyal's formula [13], [14] that relates the inner product of two BTFR outcomes with the inner products of the signals involved. Introducing inner products in signal domain and T -domain as

$$(x, y) = \int_t x(t) y^*(t) dt, \\ (T_1, T_2) = \int_t \int_f T_1(t, f) T_2^*(t, f) dt df,$$

we have the following theorem.

Theorem 9: A BTFR $T(t, f)$ satisfies Moyal's formula

$$(T_{x_1, y_1}, T_{x_2, y_2}) = (x_1, x_2)(y_1, y_2)^*,$$

if and only if it is unitary.

Proof:

If: Let T be unitary. Then,

$$\begin{aligned} (T_{x_1, y_1}, T_{x_2, y_2}) &= (\mathbf{u}_T q_{x_1, y_1}, \mathbf{u}_T q_{x_2, y_2}) \\ &= (q_{x_1, y_1}, (\mathbf{u}_T^\dagger \mathbf{u}_T) q_{x_2, y_2}) \\ &= (q_{x_1, y_1}, q_{x_2, y_2}) = (x_1, x_2)(y_1, y_2)^*. \end{aligned}$$

Only if: Let Moyal's formula be satisfied by T . Then, inserting the signals

$$\begin{aligned} x_1(t) &= \delta(t - t_1), \quad y_1(t) = \delta(t - t_2), \\ x_2(t) &= \delta(t - t'_1), \quad y_2(t) = \delta(t - t'_2) \end{aligned}$$

into Moyal's formula, condition (4.1b) is obtained, which shows that T is unitary. $\square$

We note that three special versions of Moyal's formula which are often encountered in applications are

$$\begin{aligned} (T_x, T_y) &= |(x, y)|^2, \\ \|T_{x, y}\| &= \|x\| \|y\|, \quad \|T_x\| = \|x\|^2, \end{aligned}$$

with signal-domain and T -domain norms defined by $\|x\|^2 = (x, x)$ and $\|T\|^2 = (T, T)$, respectively.

Example: Inserting (2.7) into (4.1), it is easily checked that WD is unitary. Indeed, it is well known that WD satisfies Moyal's formula [13], [14].

C. Relations Between Unitary BTFR's

Combining Theorem 4 and Definition 4, we obtain the following theorem.

Theorem 10: All unitary BTFR's are interrelated by unitary linear transformations.

Proof: Let T and T' be two unitary BTFR's. According to Theorem 4, there is $T' = LT$ where $L = \mathbf{u}_T \mathbf{u}_T^{-1} = \mathbf{u}_T \mathbf{u}_T^\dagger$. The operator L is a unitary mapping of $[L_2(\mathbb{R})]_T$ onto $[L_2(\mathbb{R})]_{T'}$, if and only if $(L\tilde{T}_1, L\tilde{T}_2) = (\tilde{T}_1, \tilde{T}_2)$ for all $\tilde{T}_1(t, f), \tilde{T}_2(t, f) \in [L_2(\mathbb{R})]_T$. This equation is proved by noting that $(L\tilde{T}_1, L\tilde{T}_2) = (\tilde{T}_1, (L^\dagger L)\tilde{T}_2)$, where it is easily shown that $L^\dagger L = \mathbf{u}_T \mathbf{u}_T^\dagger$ and thus $(L^\dagger L)\tilde{T} = (\mathbf{u}_T \mathbf{u}_T^\dagger)\tilde{T} = \tilde{T}$ for all $\tilde{T}(t, f) \in [L_2(\mathbb{R})]_T$. $\square$

Conversely, if T is a unitary BTFR and T' is derived from T via a unitary/nonunitary linear transformation, then T' must be unitary/nonunitary. As a consequence of Theorem 10, we also note the equality of inner products and norms

$$(T_{x_1, y_1}, T_{x_2, y_2}) = (T'_{x_1, y_1}, T'_{x_2, y_2}), \quad \|T_{x, y}\| = \|T'_{x, y}\|,$$

which holds for arbitrary unitary BTFR's T and T' .

D. Product Relations

There exists a class of BTFR relations that express the product of two BSR's $T^{(1)}(\Theta_1), T^{(2)}(\Theta_2)$ as a bilinear form in a BTFR $T(t, f)$. The next theorem discusses the existence and systematic construction of such "product relations" [12].

Theorem 11: The product of arbitrary BSR's $T^{(1)}(\Theta_1), T^{(2)}(\Theta_2)$ can be written in terms of any unitary BTFR $T(t, f)$ according to

$$T_{x_1, y_1}^{(1)}(\Theta_1) T_{x_2, y_2}^{(2)*}(\Theta_2) = (L^{(\Theta_1, \Theta_2)} T_{x_1, x_2}, T_{y_1, y_2}), \quad (4.2)$$

where the operator $L^{(\Theta_1, \Theta_2)}$ is given by

$$L^{(\Theta_1, \Theta_2)} = \mathbf{u}_T \mathbf{z}^{(\Theta_1, \Theta_2)} \mathbf{u}_T^\dagger \quad (4.3)$$

with

$$\mathbf{z}^{(\Theta_1, \Theta_2)}(t_1, t_2; t'_1, t'_2) = \mathbf{u}_{T(1)}(\Theta_1; t'_1, t_1) \mathbf{u}_{T(2)}^*(\Theta_2; t'_2, t_2). \quad (4.4)$$

Proof: Using, in turn, (2.4), Moyal's formula (which applies since T is assumed unitary), and Theorem 5, we have

$$\begin{aligned} T_{x_1, y_1}^{(1)}(\Theta_1) T_{x_2, y_2}^{(2)*}(\Theta_2) &= (\mathbf{v}_{T(1)}^{(\Theta_1)} x_1, y_1) (\mathbf{v}_{T(2)}^{(\Theta_2)} x_2, y_2)^* \\ &= T_{\mathbf{v}_{T(1)}^{(\Theta_1)} x_1, \mathbf{v}_{T(2)}^{(\Theta_2)} x_2}(y_1, y_2) \\ &= (L^{(\Theta_1, \Theta_2)} T_{x_1, x_2}, T_{y_1, y_2}), \end{aligned}$$

where, due to Theorem 5 and (2.5), $L^{(\Theta_1, \Theta_2)}$ is given by (4.3), (4.4). $\square$

Examples: The following three well-known relations involving WD and AF [6], [11], [30] are special cases of the general product relation (4.2):

$$\begin{aligned} |W_{x, y}(t, f)|^2 &= \int_{\tau} \int_{\nu} W_x\left(t + \frac{\tau}{2}, f + \frac{\nu}{2}\right) \\ &\quad \cdot W_y\left(t - \frac{\tau}{2}, f - \frac{\nu}{2}\right) d\tau d\nu, \end{aligned}$$

$$|A_x(\tau, \nu)|^2 = \int_{\tau'} \int_{\nu'} |A_x(\tau', \nu')|^2 e^{-j2\pi(\nu\tau' - \tau\nu')} d\tau' d\nu',$$

$$|A_{x, y}(\tau, \nu)|^2 = \int_t \int_f W_x(t + \tau, f + \nu) W_y(t, f) dt df.$$

The first two relations are known as WD's interference formula and AF's self-transformation property, respectively. Another special case of (4.2) is Moyal's formula itself,

$$E_{x_1, y_1} E_{x_2, y_2}^* = (T_{x_1, x_1}, T_{y_1, y_2}),$$

where $E_{x, y} = (x, y)$ is the cross energy, and T is any unitary BTFR.

E. Construction of Induced T -Domain Basis and Induced T -Domain Projection Operator

In some applications (in particular, the "basis method" [24] for BTFR-based signal synthesis), the existence of an orthonormal induced T -domain basis is of importance [20]–[22], [24], [31]. In the case of a BTFR that is merely regular, the induced T -domain basis $\{T_{e_k, e_l}(t, f)\}$ associated with an orthonormal signal-domain basis $\{e_k(t)\}$ need not be orthonormal (or orthogonal) as well. This situation is changed if the BTFR is unitary.

Theorem 12: Let $T(t, f)$ be a unitary BTFR. If $\{e_k(t)\}$ is an orthonormal basis of a linear signal space S , then the induced T -domain basis $\{T_{e_k, e_l}(t, f)\}$ spanning the induced T -domain space S_T is orthonormal as well,

$$(T_{e_k, e_l}, T_{e_{k'}, e_{l'}}) = \delta_{kk'} \delta_{ll'}.$$

Proof: $\{T_{e_k, e_l}(t, f)\}$ is a basis of the induced T -domain space S_T since T is regular and Theorem 6 applies. The orthonormality of the $T_{e_k, e_l}(t, f)$ for orthonormal $e_k(t)$ is a direct consequence of Moyal's formula:

$$(T_{e_k, e_l}, T_{e_{k'}, e_{l'}}) = (e_k, e_{k'})(e_l, e_{l'})^* = \delta_{kk'} \delta_{ll'}. \quad \square$$

A second concept that is important in the context of subspace-constrained signal synthesis [24] is the orthogonal projection on an induced T -domain space S_T . In the following, P_S and P_{S_T} denote the orthogonal projection operators [18] of the signal space S and the induced T -domain space S_T , respectively.

Theorem 13: Let T be a unitary BTFR. Then the orthogonal projection operator P_{S_T} of the induced T -domain space S_T is the induced T -domain operator associated with the signal-domain projection operator P_S ,

$$P_{S_T} = u_T z_P u_T^+ \quad (4.5)$$

with

$$z_P(t_1, t_2; t'_1, t'_2) = P_S(t_1, t'_1) P_S^*(t_2, t'_2).$$

A proof of this theorem is given in Appendix C. Since P_{S_T} is the induced T -domain operator (3.8) associated with the signal-domain projection operator P_S , there must be (see (3.7))

$$T_{P_S x, P_S y}(t, f) = (P_{S_T} T_{x, y})(t, f),$$

which shows that the BTFR outcome of the orthogonal projections of arbitrary signals on S equals the orthogonal projection of the BTFR outcome of the original signals on the induced T -domain space S_T .

F. Strict Unitarity

In order to combine the property of unitarity with the property of strict regularity, we introduce the following.

Definition 5: A BTFR $T(t, f)$ is said to be *strictly unitary* if it is unitary and, in addition, satisfies

$$u_T u_T^+ = I. \quad (4.6)$$

Obviously, a strictly unitary BTFR is also strictly regular. Combining with Corollary 1 and Theorem 12, we note that, in the case of a strictly unitary BTFR, an orthonormal basis of $L_2(\mathbb{R})$ induces an equally orthonormal basis of $L_2(\mathbb{R}^2)$.

Example: Inserting (2.7) into (4.6), it is easily shown that WD is strictly unitary.

G. Dual BTFR's

All results of this section hold equally well for C-domain BTFR's. Moreover, owing to the unitary Fourier-transform relation (1.3) connecting E-domain and C-domain BTFR's, there exists a duality theorem analogous to Theorem 8.

Theorem 14: If an E-domain BTFR $T^{(E)}(t, f)$ is strictly unitary/unitary/nonunitary, then the dual C-domain BTFR $T^{(C)}(\tau, v)$ is also strictly unitary/unitary/nonunitary, and vice versa.

Example: It follows from the strict unitarity of WD that AF (1.2) is strictly unitary as well since WD and AF are dual BTFR's.

H. Subspace Unitarity

There exist BTFR's which, while being nonunitary in general, possess a unitary structure for signals taken from some linear signal subspace. Here, a prominent example is the discrete-time WD that is unitary only on subspaces of "halfband signals" [32], [33]. The concept of *subspace unitarity*, which is an extension of the concept of unitarity, has been introduced and discussed in [24]. We stress that most of the results derived so far for unitary BTFR's can be reformulated for subspace-unitary BTFR's; however, they are valid only for signals taken from the signal subspace on which the BTFR is unitary. The concept of subspace unitarity provides a theoretical basis for subspace-constrained signal synthesis (in particular, halfband signal synthesis in the case of discrete-time WD) [24], [33].

V. THE SHIFT-INVARIANT CLASS

A. Shift-Invariant BTFR's

So far, the BTFR properties of regularity and unitarity have been studied without making any *a priori* assumptions regarding the BTFR $T(t, f)$ (apart from bilinearity). In this section, we specialize to *time-frequency shift-invariant* BTFR's. It will be seen that the assumption of shift invariance substantially simplifies the BTFR description by means of kernel functions.

An E-domain BTFR $T^{(E)}(t, f)$ is said to be *time-frequency shift-invariant* if

$$T_{g x, g y}^{(E)}(t, f) = T_{x, y}^{(E)}(t - t_o, f - f_o),$$

where g is the time-frequency shift operator,

$$(gx)(t) = x(t - t_o) e^{j2\pi f_o t}.$$

If an E-domain BTFR $T^{(E)}(t, f)$ is shift-invariant, then the dual C-domain BTFR $T^{(C)}(\tau, v)$ satisfies a dual "C-domain shift-invariance property" [9],

$$T_{g x, g y}^{(C)}(\tau, v) = T_{x, y}^{(C)}(\tau, v) e^{j2\pi(f_o \tau - t_o v)}.$$

It can be shown [9], [12], [15] that the impulse responses of dual shift-invariant BTFR's $T^{(E)}(t, f)$, $T^{(C)}(\tau, v)$ can be written as

$$u_{T^{(E)}}(t, f; t_1, t_2) = \varphi_T\left(t - \frac{t_1 + t_2}{2}, t_1 - t_2\right) \cdot e^{-j2\pi(t_1 - t_2)f}, \quad (5.1)$$

$$u_{T^{(C)}}(\tau, v; t_1, t_2) = \Psi_T(\tau, v) \delta(\tau - (t_1 - t_2)) \cdot e^{-j2\pi((t_1 + t_2)/2)v}, \quad (5.2)$$

where $\varphi_T(t, \tau)$ and $\Psi_T(\tau, v)$ are related by a Fourier transform,

$$\Psi_T(\tau, v) = \mathcal{F}_{t \rightarrow v} \varphi_T(t, \tau). \quad (5.3)$$

Each of the two-dimensional kernel functions $\varphi_T(t, \tau)$ and $\Psi_T(\tau, v)$ characterizes both $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$. Inserting (5.1) into (2.1), we obtain

$$T_{x,y}^{(E)}(t, f) = \int_{t_1} \int_{t_2} \varphi_T \left(t - \frac{t_1 + t_2}{2}, t_1 - t_2 \right) \cdot e^{-j2\pi(t_1 - t_2)f} x(t_1) y^*(t_2) dt_1 dt_2,$$

which can be rewritten in terms of the WD as

$$T_{x,y}^{(E)}(t, f) = \int_{t'} \int_{f'} \psi_T(t - t', f - f') W_{x,y}(t', f') dt' df', \quad (5.4)$$

where

$$\psi_T(t, f) = \mathcal{F}_{\tau \rightarrow f} \varphi_T(t, \tau) = \mathcal{F}_{\tau \rightarrow f} \mathcal{F}_{v \rightarrow t}^{-1} \Psi_T(\tau, v). \quad (5.5)$$

Similarly, inserting (5.2) into (2.1), we obtain

$$T_{x,y}^{(C)}(\tau, v) = \Psi_T(\tau, v) A_{x,y}(\tau, v). \quad (5.6)$$

Equations (5.4) and (5.6) show that any shift-invariant E-domain (C-domain) BTFR can be derived from WD (AF) via a convolution (multiplication). We note that the class of shift-invariant E-domain BTFR's equals the well-known *Cohen class* of BTFR's [1], [34], [35].

B. Regularity

Let $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ be two dual shift-invariant BTFR's. If the E-domain BTFR $T^{(E)}(t, f)$ is *regular*, then the dual C-domain BTFR is regular as well (see Theorem 8), and it is easily shown [12] that the inverse operators of $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ are given by

$$u_{T^{(E)}}^{-1}(t_1, t_2; t, f) = \varphi_T^{-1} \left(\frac{t_1 + t_2}{2} - t, t_1 - t_2 \right) \cdot e^{j2\pi(t_1 - t_2)f}, \quad (5.7)$$

$$u_{T^{(C)}}^{-1}(t_1, t_2; \tau, v) = \Psi_T^{-1}(\tau, v) \delta(\tau - (t_1 - t_2)) \cdot e^{j2\pi((t_1 - t_2)/2)v}, \quad (5.8)$$

where the "inverse kernel functions" $\varphi_T^{-1}(t, \tau)$ and $\Psi_T^{-1}(\tau, v)$ are again related by a Fourier transform according to (5.3). By inserting (5.1), (5.7) and (5.2), (5.8) into (3.1), it follows that the kernels $\varphi_T(t, \tau)$ and $\Psi_T(\tau, v)$ and the inverse kernels $\varphi_T^{-1}(t, \tau)$ and $\Psi_T^{-1}(\tau, v)$ are related as

$$\int_{t'} \varphi_T^{-1}(t - t', \tau) \varphi_T(t', \tau) dt' = \delta(t),$$

$$\Psi_T^{-1}(\tau, v) \Psi_T(\tau, v) \equiv 1, \quad (5.9)$$

hence, in particular [1],

$$\Psi_T^{-1}(\tau, v) = \frac{1}{\Psi_T(\tau, v)}.$$

An inverse kernel $\psi_T^{-1}(t, f)$ can also be defined for $\psi_T(t, f)$; it is related with the other inverse kernels $\varphi_T^{-1}(t, \tau)$ and $\Psi_T^{-1}(\tau, v)$ according to (5.5) and with $\psi_T(t, f)$ as

$$\int_{t'} \int_{f'} \psi_T^{-1}(t - t', f - f') \psi_T(t', f') dt' df' = \delta(t) \delta(f). \quad (5.10)$$

Example: For WD and AF, the kernels and inverse kernels are

$$\begin{aligned} \varphi_{W/A}(t, \tau) &= \varphi_{W/A}^{-1}(t, \tau) = \delta(t), \\ \Psi_{W/A}(\tau, v) &= \Psi_{W/A}^{-1}(\tau, v) = 1, \\ \psi_{W/A}(t, f) &= \psi_{W/A}^{-1}(t, f) = \delta(t) \delta(f). \end{aligned}$$

C. Singularity of Smoothed WD's

In practical signal analysis applications, some sort of smoothing is often applied to WD in order to attenuate WD's interference terms [30]. A smoothed version of WD (with time-frequency invariant smoothing) can be written according to (5.4), with the restriction that the kernel $\psi_T(t, f)$ is a low-pass function, i.e., its Fourier transform

$$\Psi_T(\tau, v) = \mathcal{F}_{t \rightarrow v} \mathcal{F}_{f \rightarrow \tau}^{-1} \psi_T(t, f)$$

tends to zero for increasing $|\tau|$ and/or $|v|$. This, however, implies that the inverse kernel $\Psi_T^{-1}(\tau, v) = 1/\Psi_T(\tau, v)$, if it exists at all, will tend to infinity for increasing $|\tau|$ and/or $|v|$. It follows, in turn, that the inverse operator (5.7) is not bounded, and, recalling Definition 2, we recognize that a *smoothed WD version is singular*. Indeed, the inversion of a smoothed WD version amounts to an ill-conditioned deconvolution since the effect of smoothing has to be removed.

D. Relations of Shift-Invariant BTFR's

If two shift-invariant E-domain BTFR's $T^{(E)}(t, f)$ and $T'^{(E)}(t, f)$ are linearly related, then this relation is a convolution [9],

$$T'^{(E)}(t, f) = \int_{t'} \int_{f'} \lambda(t - t', f - f') T^{(E)}(t', f') dt' df'. \quad (5.11)$$

The dual C-domain shift-invariant BTFR's $T^{(C)}(\tau, v)$ and $T'^{(C)}(\tau, v)$ are then related via a multiplication,

$$T'^{(C)}(\tau, v) = \Lambda(\tau, v) T^{(C)}(\tau, v), \quad (5.12)$$

where

$$\Lambda(\tau, v) = \mathcal{F}_{t \rightarrow v} \mathcal{F}_{f \rightarrow \tau}^{-1} \lambda(t, f).$$

Now, if $T^{(E)}(t, f)$, $T^{(C)}(\tau, v)$ are *regular*, then relations (5.11), (5.12) exist for any pair of dual shift-invariant BTFR's $T'^{(E)}(t, f)$ and $T'^{(C)}(\tau, v)$ according to Theorem 4. From (3.4), we easily obtain that the transformation kernels $\lambda(t, f)$

and $\Lambda(\tau, v)$ can then be constructed according to [1], [12]

$$\lambda(t, f) = \int_{t'} \int_{f'} \psi_T(t - t', f - f') \psi_T^{-1}(t', f') dt' df',$$

$$\Lambda(\tau, v) = \Psi_T(\tau, v) \Psi_T^{-1}(\tau, v) = \frac{\Psi_T(\tau, v)}{\Psi_T(\tau, v)}.$$

E. Strict Regularity

In the case of an (E-domain or C-domain) shift-invariant BTFR T , it is easily shown that $\mathbf{u}_T^{-1} \mathbf{u}_T = \mathbf{I}$ entails $\mathbf{u}_T \mathbf{u}_T^{-1} = \mathbf{I}$, whence we conclude that a regular, shift-invariant BTFR is always *strictly* regular.

F. Unitarity

For a pair of dual shift-invariant BTFR's $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$, the general unitarity condition (4.1) can be simplified. The following equivalent conditions are easily obtained [12]:

$$\int_{t'} \varphi_T^*(t' - t, \tau) \varphi_T(t', \tau) dt' = \delta(t),$$

$$\Psi_T^*(\tau, v) \Psi_T(\tau, v) \equiv 1, \quad (5.13)$$

$$\int_{t'} \int_{f'} \psi_T^*(t' - t, f' - f) \psi_T(t', f') dt' df' = \delta(t) \delta(f).$$

$$(5.14)$$

In particular, the second condition states that a pair of dual shift-invariant BTFR's $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ are unitary, if and only if the kernel $\Psi_T(\tau, v)$ is unimodular,

$$|\Psi_T(\tau, v)| \equiv 1,$$

which is well known as a condition for validity of Moyal's formula [1], [6]. Note that this condition cannot be satisfied if $T^{(E)}(t, f)$ is a smoothed WD version (i.e., if $|\Psi_T(\tau, v)|$ decays for increasing $|\tau|$ and/or $|v|$ as discussed in Section V-C). Comparing (5.13), (5.14) with (5.9), (5.10), the inverse kernels of unitary BTFR's $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ are seen to be

$$\varphi_T^{-1}(t, \tau) = \varphi_T^*(-t, \tau), \quad \Psi_T^{-1}(\tau, v) = \Psi_T^*(\tau, v),$$

$$\psi_T^{-1}(t, f) = \psi_T^*(-t, -f).$$

G. Strict Unitarity

The equivalence of regularity and strict regularity in the case of shift invariance directly entails an equivalence of unitarity and strict unitarity: if a shift-invariant BTFR is unitary, it is also strictly unitary.

VI. THE SHIFT-SCALE-INVARIANT CLASS

A. Shift-Scale-Invariant BTFR's

Among the shift-invariant E-domain and C-domain BTFR's that have been considered in the literature, there are several which also satisfy a *scale-invariance* property. Examples are WD, AF, generalized WD (including the Rihaczek distribution) [3], [6], [36], generalized AF [9], and the Choi-Williams distribution [37]. In this section, we study the subclass

of shift-invariant BTFR's that are also scale-invariant. It will be seen that, inside this subclass, the description of BTFR's by means of kernels and inverse kernels is once again simplified.

An E-domain BTFR $T^{(E)}(t, f)$ is said to be *shift-scale-invariant* [9], [12] if it is shift-invariant and, in addition, satisfies the scale-invariance property

$$T_{h_x, h_y}^{(E)}(t, f) = T_{x, y}^{(E)}(at, f/a)$$

with the time-frequency scaling operator h defined by

$$(hx)(t) = \sqrt{|a|} x(at), \quad a \in \mathbb{R}, \quad a \neq 0.$$

If an E-domain BTFR $T^{(E)}(t, f)$ is shift-scale-invariant, then the dual C-domain BTFR $T^{(C)}(\tau, v)$ is shift-scale-invariant as well, i.e., it is C-domain shift-invariant and satisfies the scale-invariance property

$$T_{h_x, h_y}^{(C)}(\tau, v) = T_{x, y}^{(C)}(a\tau, v/a).$$

Shift-scale-invariant BTFR's are seen to form a subclass of the broader class of shift-invariant BTFR's. The kernel functions $\varphi_T(t, \tau)$, $\Psi_T(\tau, v)$, and $\psi_T(t, f)$ of dual, shift-scale-invariant BTFR's $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ can be written in terms of one-dimensional functions $g_T(\alpha)$, $G_T(\xi)$ as [9], [12]

$$\varphi_T(t, \tau) = \frac{1}{|\tau|} g_T\left(\frac{t}{\tau}\right), \quad \Psi_T(\tau, v) = G_T(\tau v),$$

$$\psi_T(t, f) = \int_{\alpha} g_T(\alpha) \frac{1}{|\alpha|} e^{-j2\pi(tf/\alpha)} d\alpha$$

where

$$G_T(\xi) = \mathcal{F}g_T(\alpha). \quad (6.1)$$

Note that $\Psi_T(\tau, v)$ and $\psi_T(t, f)$ depend only on the product of τ, v and t, f , respectively.

B. Regularity

The inverse kernels $\varphi_T^{-1}(t, \tau)$, $\Psi_T^{-1}(\tau, v)$, and $\psi_T^{-1}(t, f)$ of regular, shift-scale-invariant BTFR's $T^{(E)}(t, f)$ and $T^{(C)}(\tau, v)$ can be written as [12]

$$\varphi_T^{-1}(t, \tau) = \frac{1}{|\tau|} g_T^{-1}\left(\frac{t}{\tau}\right), \quad \Psi_T^{-1}(\tau, v) = G_T^{-1}(\tau v),$$

$$\psi_T^{-1}(t, f) = \int_{\alpha} g_T^{-1}(\alpha) \frac{1}{|\alpha|} e^{-j2\pi(tf/\alpha)} d\alpha,$$

where $g_T^{-1}(\alpha)$ and $G_T^{-1}(\xi)$ are again a Fourier transform pair according to (6.1), and

$$\int_{\alpha'} g_T^{-1}(\alpha - \alpha') g_T(\alpha') d\alpha' = \delta(\alpha),$$

$$G_T^{-1}(\xi) G_T(\xi) \equiv 1, \quad (6.2)$$

hence, in particular,

$$G_T^{-1}(\xi) = \frac{1}{G_T(\xi)}.$$

Example: In the case of WD/AF, the kernels $g_T(\alpha)$, $G_T(\xi)$ and inverse kernels $g_T^{-1}(\alpha)$, $G_T^{-1}(\xi)$ are obtained as

$$g_{W/A}(\alpha) = g_{W/A}^{-1}(\alpha) = \delta(\alpha),$$

$$G_{W/A}(\xi) = G_{W/A}^{-1}(\xi) \equiv 1.$$

C. Relations of Shift-Scale-Invariant BTFR's

If shift-scale-invariant BTFR's are linearly related, then this relation is the usual convolution (E-domain) or multiplication (C-domain) as in (5.11), (5.12). However, owing to the scale invariance of the BTFR's involved, the transformation kernels $\lambda(t, f)$, $\Lambda(\tau, \nu)$ can now be written in terms of one-dimensional functions $\omega(\alpha)$ and $\Omega(\xi)$ as [9]

$$\lambda(t, f) = \int_{\alpha} \omega(\alpha) \frac{1}{|\alpha|} e^{-j2\pi(tf/\alpha)} d\alpha,$$

$$\Lambda(\tau, \nu) = \Omega(\tau\nu),$$

where

$$\Omega(\xi) = \mathcal{F}\omega(\alpha).$$

If $T^{(E)}(t, f)$ and $T^{(C)}(\tau, \nu)$ are regular, then $\omega(\alpha)$ and $\Omega(\xi)$ can be constructed as

$$\omega(\alpha) = \int_{\alpha'} g_T(\alpha - \alpha') g_T^{-1}(\alpha') d\alpha',$$

$$\Omega(\xi) = G_T(\xi) G_T^{-1}(\xi) = \frac{G_T(\xi)}{G_T(\xi)}.$$

D. Unitarity

For shift-scale-invariant BTFR's $T^{(E)}(t, f)$ and $T^{(C)}(\tau, \nu)$, the unitarity conditions (5.13), (5.14) reduce to

$$\int_{\alpha'} g_T^*(\alpha' - \alpha) g_T(\alpha') d\alpha' = \delta(\alpha),$$

$$G_T^*(\xi) G_T(\xi) \equiv 1,$$

hence, in particular,

$$|G_T(\xi)| \equiv 1.$$

Comparing with (6.2), it follows that the inverse kernels are here given by

$$g_T^{-1}(\alpha) = g_T^*(-\alpha), \quad G_T^{-1}(\xi) = G_T^*(\xi).$$

VII. CONCLUSION

The definitions of the two BTFR properties considered in this paper, termed regularity and unitarity, are both based on the observation that a BTFR outcome $T_{x,y}(t, f)$ or $T_{x,y}(\tau, \nu)$ can be viewed as the result of a linear transformation of the outer signal product $q_{x,y}(t_1, t_2) = x(t_1)y^*(t_2)$, i.e., $T_{x,y} = \mathbf{u}_T q_{x,y}$, where the linear operator $\mathbf{u}_T$ provides a complete and compact characterization of the BTFR T . In this framework, regularity and unitarity are indeed quite natural properties since they essentially express (respectively) the reversibility and unitarity of the "BTFR operator" $\mathbf{u}_T$.

The property of *regularity* expresses the existence of a (left) inverse BTFR operator $\mathbf{u}_T^{-1}$ such that $T_{x,y} = \mathbf{u}_T q_{x,y}$

can be inverted, $q_{x,y} = \mathbf{u}_T^{-1} T_{x,y}$. It has been shown in Section III that this simple property has far-reaching consequences. If a BTFR is regular, then 1) a signal can be recovered from its BTFR outcome up to a constant phase factor; 2) any BSR can be derived from the BTFR via a linear transform; 3) any pair of linear signal transforms induces an associated BTFR transform; and 4) a basis of a linear signal space induces a basis of the induced T -domain space. Finally, if a BTFR satisfies a stronger version of regularity, then the induced T -domain space associated with $L_2(\mathbb{R})$ is $L_2(\mathbb{R}^2)$.

The property of *unitarity*, formulated as $\mathbf{u}_T^+ \mathbf{u}_T = \mathbf{I}$, expresses the fact that the BTFR operator $\mathbf{u}_T$ preserves inner products and norms. Unitarity of a BTFR was shown in Section IV to be a necessary and sufficient condition for validity of Moyal's formula. Unitarity is of fundamental importance for BTFR-based signal synthesis and detection/estimation methods. Apart from this, unitarity also permits the systematic derivation of "product relations" of BTFR's, as well as the construction of induced orthogonal projection operators and induced orthonormal bases.

While the properties of regularity and unitarity are best introduced and understood in the general linear-operator framework of BTFR's, practical calculations using kernel functions and inverse kernel functions are significantly simplified if the BTFR's satisfy a natural shift-invariance property. Therefore, the class of *shift-invariant* BTFR's (which, in the E-domain, equals the well-known Cohen class) was considered in Section V. In Section VI, it was shown that a further simplification occurs if, in addition to shift-invariance, the BTFR's satisfy a scale-invariance property.

It is to be noted that most of the BTFR's proposed on theoretical grounds (in particular, WD, Rihaczek distribution, Page and Levin distribution, and AF) are regular and also unitary while those BTFR's specifically adapted to practical exigencies (smoothed WD versions like pseudo WD, spectrogram, and the Choi-Williams distribution) are non-regular and thus also nonunitary. BTFR's of the latter type generally have the advantage of meeting two practical requirements: firstly, the possibility of calculating BTFR's on a short-time basis (using a sliding analysis window of finite length) [13], [32], and secondly, the reduction of cross or interference terms [30], [38], [39]. Unfortunately, *both goals are incompatible with the properties of regularity and unitarity*. Thus, while smoothed WD versions are better suited than WD itself in a number of applications, their inherent nonunitarity causes problems in other applications. Specifically, optimal detection/estimation cannot be performed by means of smoothed WD versions. (Optimal detection using spectrograms has been considered in [40]; however, this approach uses a problematic deconvolution step to transform the spectrogram into the Rihaczek distribution, i.e., into a unitary BTFR.) Also, owing to the nonunitarity of smoothed WD versions, no closed-form solution can be given to the problem of optimal signal synthesis (here, only iterative methods have been proposed so far [33], [41], [42]).

From this, it is apparent that BTFR's can be divided into a group of regular/unitary BTFR's and a group of nonregu-

lar/nonunitary BTFR's. Clearly, this distinction is of great importance for deciding which BTFR is suited for a given application.

APPENDIX A PROOF OF THEOREM 1

1) Let $\mathcal{R}$ denote the range of the BTFR operator u_T . We first show that any $\tilde{T}(t, f) \in \mathcal{R}$ is an element of the induced T -domain space $[L_2(\mathbb{R})]_T$.

Let $\tilde{T}(t, f) \in \mathcal{R}$. Then there exists at least one $\tilde{q}(t_1, t_2) \in L_2(\mathbb{R}^2)$ (recall that $L_2(\mathbb{R}^2)$ is the domain of u_T) such that $\tilde{T} = u_T \tilde{q}$.

If $\{e_k(t)\}$ is an orthonormal basis of $L_2(\mathbb{R})$, then it can be shown [43, p. 56] that $\{q_{e_k, e_l}(t_1, t_2)\}$ is an orthonormal basis of $L_2(\mathbb{R}^2)$; hence $\tilde{q}(t_1, t_2) \in L_2(\mathbb{R}^2)$ can be expanded as

$$\tilde{q}(t_1, t_2) = \sum_k \sum_l c_{kl} q_{e_k, e_l}(t_1, t_2) \text{ with } \sum_k \sum_l |c_{kl}|^2 < \infty, \quad (\text{A.1})$$

hence,

$$\tilde{T} = u_T \tilde{q} = \sum_k \sum_l c_{kl} u_T q_{e_k, e_l} = \sum_k \sum_l c_{kl} T_{e_k, e_l}.$$

From Definition 1, it then follows that $\tilde{T}(t, f) \in [L_2(\mathbb{R})]_T$.

2) We next show that, conversely, any $\tilde{T}(t, f) \in [L_2(\mathbb{R})]_T$ is an element of $\mathcal{R}$.

Let $\tilde{T}(t, f) \in [L_2(\mathbb{R})]_T$. According to Definition 1,

$$\tilde{T}(t, f) = \sum_k \sum_l c_{kl} T_{e_k, e_l}(t, f) \text{ with } \sum_k \sum_l |c_{kl}|^2 < \infty,$$

where $\{e_k(t)\}$ is an orthonormal basis of $L_2(\mathbb{R})$. Obviously, $\tilde{T}(t, f)$ can be written as

$$\tilde{T} = u_T \tilde{q} \text{ with } \tilde{q}(t_1, t_2) = \sum_k \sum_l c_{kl} q_{e_k, e_l}(t_1, t_2).$$

Since $\sum_k \sum_l |c_{kl}|^2 < \infty$, it follows that $\tilde{q}(t_1, t_2) \in L_2(\mathbb{R}^2)$, i.e., $\tilde{q}(t_1, t_2)$ is an element of the domain of u_T . Hence $\tilde{T} = u_T \tilde{q}$ must be an element of the range $\mathcal{R}$.

APPENDIX B PROOF OF THEOREM 6

Given a basis $\{s_k(t)\}$ of the linear signal space S (i.e., the signals $s_k(t)$ are linearly independent and complete in S) and a regular BTFR $T(t, f)$, we have to show that the functions $T_{s_k, s_l}(t, f)$ are 1) linearly independent and 2) complete in the induced T -domain space S_T .

1) The functions $T_{s_k, s_l}(t, f)$ are *linearly independent*, if and only if

$$\sum_k \sum_l a_{kl} T_{s_k, s_l}(t, f) \equiv 0 \quad (\text{B.1})$$

implies $a_{kl} = 0$ for all k, l . Since the BTFR T is regular, we can apply the inverse BSR operator u_T^{-1} to (B.1) (note that 0 is an element of $[L_2(\mathbb{R})]_T$ and u_T^{-1} is defined on $[L_2(\mathbb{R})]_T$) and obtain

$$\sum_k \sum_l a_{kl} q_{s_k, s_l}(t_1, t_2) \equiv 0,$$

which can be rewritten as

$$\sum_l b_l^*(t_1) s_l(t_2) \equiv 0 \quad (\text{B.2})$$

with

$$b_l(t_1) = \sum_k a_{kl} s_k(t_1). \quad (\text{B.3})$$

Now, it follows from (B.2) and the linear independence of the $s_l(t)$ that $b_l(t_1) \equiv 0$ for all l ; this, together with (B.3), finally shows that $a_{kl} = 0$ for all k, l .

2) The functions $T_{s_k, s_l}(t, f)$ are *complete* in S_T , if and only if any $\tilde{T}(t, f) \in S_T$ can be represented as

$$\tilde{T}(t, f) = \sum_k \sum_l a_{kl} T_{s_k, s_l}(t, f). \quad (\text{B.4})$$

We first recall (see Definition 1) that any $\tilde{T}(t, f) \in S_T$ can be written as

$$\tilde{T}(t, f) = \sum_m \sum_n c_{mn} T_{e_m, e_n}(t, f), \quad (\text{B.5})$$

where $\{e_m(t)\}$ is an orthonormal basis of S . Since the $s_k(t)$ are complete in S , the $e_m(t)$ can be expanded as

$$e_m(t) = \sum_k \alpha_{mk} s_k(t),$$

hence, with (2.1),

$$T_{e_m, e_n}(t, f) = \sum_k \sum_l \alpha_{mk} \alpha_{nl}^* T_{s_k, s_l}(t, f).$$

Inserting this into (B.5) yields the desired representation (B.4) with

$$a_{kl} = \sum_m \sum_n c_{mn} \alpha_{mk} \alpha_{nl}^*.$$

Note that the regularity of T is not necessary for the functions $T_{s_k, s_l}(t, f)$ to be complete in S_T .

APPENDIX C PROOF OF THEOREM 13

Using the unitarity of T , it is fairly easy to show that the operator $P_{S_T} = u_T z_P u_T^+$ of (4.5) is both self-adjoint and idempotent and therefore a valid orthogonal projection operator. To check that P_{S_T} is indeed the projection operator of S_T , we have to show that 1) $P_{S_T} \tilde{T} \in S_T$ for any $\tilde{T} \in L_2(\mathbb{R}^2)$, and 2) $P_{S_T} \tilde{T} = \tilde{T}$ for any $\tilde{T} \in S_T$.

1) Let $\tilde{T}(t, f) \in L_2(\mathbb{R}^2)$ and consider

$$P_{S_T} \tilde{T} = (u_T z_P u_T^+) \tilde{T} = u_T z_P \tilde{q}, \quad (\text{C.1})$$

where $\tilde{q} = u_T^+ \tilde{T} \in L_2(\mathbb{R}^2)$ since u_T —and thus also u_T^+ —is a bounded operator. Hence, $\tilde{q}$ can be expanded as (cf. (A.1))

$$\tilde{q}(t_1, t_2) = \sum_k \sum_l c_{kl} q_{e_k, e_l}(t_1, t_2) \text{ with } \sum_k \sum_l |c_{kl}|^2 < \infty, \quad (\text{C.2})$$

where $\{e_k(t)\}$ is an orthonormal basis of $L_2(\mathbb{R})$. Insertion of (C.2) into (C.1) yields

$$P_{S_T} \tilde{T} = u_T z_P \left\{ \sum_k \sum_l c_{kl} q_{e_k, e_l} \right\} = \sum_k \sum_l c_{kl} u_T z_P q_{e_k, e_l}. \quad (\text{C.3})$$

With P_S denoting the orthogonal projection operator of S , it is straightforward to show

$$z_P q_{e_k, e_l} = q_{x_k, x_l} \text{ with } x_k(t) = (P_S e_k)(t) \in S. \quad (\text{C.4})$$

The projections $x_k(t) \in S$ can be expanded as

$$x_k(t) = \sum_m \alpha_{km} s_m(t), \quad (\text{C.5})$$

where $\{s_m(t)\}$ is an orthonormal basis of S . Inserting (C.4) and (C.5) into (C.3), we obtain

$$P_{S_T} \tilde{T} = \sum_m \sum_n d_{mn} T_{s_m, s_n} \text{ with } d_{mn} = \sum_k \sum_l c_{kl} \alpha_{km} \alpha_{ln}^*. \quad (\text{C.6})$$

There must be $\sum_m \sum_n |d_{mn}|^2 < \infty$ since $P_{S_T} \tilde{T} \in L_2(\mathbb{R}^2)$ and $\{T_{s_k, s_l}(t, f)\}$ is an orthonormal basis. From (C.6) and Definition 1, it then follows that $P_{S_T} \tilde{T} \in S_T$.

2) Let $\tilde{T} \in S_T$, hence, $\tilde{T}$ can be expanded as

$$\tilde{T}(t, f) = \sum_k \sum_l c_{kl} T_{s_k, s_l}(t, f),$$

with $\{s_k(t)\}$ again being an orthonormal basis of S . Proceeding in analogy to Part 1), we have

$$\begin{aligned} P_{S_T} \tilde{T} &= \sum_k \sum_l c_{kl} \mathbf{u}_T \mathbf{z}_P q_{s_k, s_l} = \sum_k \sum_l c_{kl} \mathbf{u}_T q_{s_k, s_l} \\ &= \sum_k \sum_l c_{kl} T_{s_k, s_l} = \tilde{T}, \end{aligned}$$

where we have used that $\mathbf{z}_P q_{s_k, s_l} = q_{s_k, s_l}$ due to (C.4) and the fact that $s_k(t) \in S$.

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