# Design of 3-D FIR digital filters using integral squared error criterion and transformation method 

GUERGANA MOLLOVA WOLFGANG F.G. MECKLENBRÄUKER<br>Institute of Communications and Radio-Frequency Engineering<br>Vienna University of Technology<br>Gusshausstrasse 25/389, A-1040 Vienna<br>AUSTRIA

g.mollova@gmx.net http://www.nt.tuwien.ac.at/


#### Abstract

In this paper, the design of three-dimensional (3-D) FIR filters using McClellan transform method and integral squared error (ISE) criterion is presented. The 3-D filters with cone-shaped characteristics are considered in the approach. By minimizing the ISE function and double integration, a novel closed-form solution for the transform parameters is developed. The new relations for the transform coefficients are expressed in terms of the desired angle of inclination of the cone. The scaling problem of transformation function is also under discussion, as well as the choice of an appropriate cut-off frequency of the 1-D filter prototype. Several design examples are demonstrated to illustrate the merits of the proposed new technique.


Key-Words: - Three-dimensional filters, FIR digital filters, McClellan transformation, Integral squared error criterion, Transform parameters, Isopotential contours, Scaled transform function

## 1 Introduction

It is known, that McClellan transform is an efficient tool in designing multidimensional (M-D) digital filters. Especially methods for two-dimensional (2D) filters are very well developed in the literature. This transform is easy to perform and gives efficient 2-D realization structure [1]. Different optimality criteria have been applied in realization of these methods, e.g. the integral squared error (ISE) criterion for 2-D zero-phase FIR fan filters [2,3]. A method for 2-D least-squares (LS) contour mapping is shown in [4] with a simultaneous determination of the optimal 1-D cut-off frequency. Other LS 2-D approaches using McClellan transform are discussed in [5-7]. The coefficients of the M-D transform are obtained in [8] using LS optimization along to a series of contour points. A 3-D ellipsoidal frequency response is constructed as an example.

In this work we propose a new method for design of 3-D FIR filters with cone-shaped characteristics using the ISE criterion and McClellan transform method. As a result, the new analytical formulas for the transform parameters will be derived which lead to an effective computation procedure.

## 2 Design approach for 3-D FIR filters

### 2.1 The McClellan transform

The essence of the M-D McClellan transformation is given with the following expression:

$$
\begin{equation*}
\cos (\omega)=F\left(\omega_{1}, \omega_{2}, \ldots, \omega_{\mathrm{M}}\right), \tag{1}
\end{equation*}
$$

where $F\left(\omega_{1}, \omega_{2}, \ldots, \omega_{\mathrm{M}}\right)$ is M-D transform function and $\omega$ is the $1-\mathrm{D}$ frequency variable. Let we define our 3-D transform function as:

$$
\begin{align*}
& F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)= \\
& =t_{000}+t_{001} \cos \left(\omega_{3}\right)+t_{010} \cos \left(\omega_{2}\right)+ \\
& +t_{011} \cos \left(\omega_{2}\right) \cos \left(\omega_{3}\right)+t_{100} \cos \left(\omega_{1}\right)+  \tag{2}\\
& +t_{101} \cos \left(\omega_{1}\right) \cos \left(\omega_{3}\right)+t_{110} \cos \left(\omega_{1}\right) \cos \left(\omega_{2}\right)+ \\
& +t_{111} \cos \left(\omega_{1}\right) \cos \left(\omega_{2}\right) \cos \left(\omega_{3}\right)
\end{align*}
$$

which is one choice from the original McClellan transform from $(I, J, K)$-order:

$$
\begin{aligned}
& F_{\text {OM }}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)= \\
& =\sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} t_{i j k} \cos \left(i \omega_{1}\right) \cos \left(j \omega_{2}\right) \cos \left(k \omega_{3}\right) .
\end{aligned}
$$

Consider a 1-D zero-phase FIR filter of odd length $2 N+1$ with a frequency response:

$$
H(\omega)=\sum_{n=0}^{N} a(n) T_{n}[\cos (\omega)],
$$

where $T_{n}[\cdot]$ is the $n$-th order Chebyshev polynomial [9] and $a(n)$ can be expressed in terms of the impulse response coefficients. The 3-D frequency response is obtained using the substitution (1):

$$
\begin{aligned}
H\left(\omega_{1}, \omega_{2}, \omega_{3}\right) & =H(\omega)_{\mid \cos (\omega)=F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)}= \\
& =\sum_{n=0}^{N} a(n) T_{n}\left[F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)\right] .
\end{aligned}
$$

Since $|\cos (\omega)| \leq 1$, the function $F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ should satisfy the condition $\left|F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)\right| \leq 1$ for all frequencies $\omega_{1}, \omega_{2}$, and $\omega_{3}$ in the interval $[-\pi, \pi]$.

This transform is known as scaled or "wellbehaved" transform. The 3-D surfaces created by transformation (1) will be called isopotential surfaces. It is known that: (i) the function $F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ will determine the shapes of isopotential surfaces of a designed 3-D filter, and (ii) the values along these surfaces are fixed by means of the 1-D prototype frequency response.

The problem now is how to calculate the transform coefficients leading to the 3-D frequency response which approximates the ideal one. The ISE criterion will be used as an optimality criterion.

### 2.2 Determination of transform parameters and 1-D cut-off frequency

We focus our attention on 3-D filters with a particular symmetry class, namely with a conical type of the magnitude response. The ideal double cone filter oriented in $\omega_{3}$-direction is shown in Fig.1. On the surface of the cone:

$$
\begin{equation*}
\omega_{1}^{2}+\omega_{2}^{2}-\frac{\omega_{3}^{2}}{\gamma^{2}}=0, \tag{3}
\end{equation*}
$$

where $\gamma$ is the slope of the cone. Below, we denote the angle of inclination of the cone with $\theta$ :

$$
\theta=\arctan (\gamma)
$$

We impose the following conditions on the function $F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ :

$$
\begin{align*}
& F_{3}(0,0, \pi)=\cos (0)=1 \\
& F_{3}(\pi, \pi, \pi)=\cos (\pi)=-1  \tag{4}\\
& F_{3}(\pi, 0,0)=\cos (\pi)=-1
\end{align*}
$$

These conditions are necessary to have a true mapping from 1-D lowpass to 3-D lowpass filter. With the first condition we transform 1-D point $\omega=0$ into the point from passband of the cone $(0,0, \pi)$. By analogy, the $\pi$-point ( 1 -D plane) is mapped into the stopband point $(\pi, \pi, \pi)$ from 3-D plane. As the point $(\pi, 0,0)$ is outside of the cone, we assume satisfaction of the third condition from (4). Finally, the following system of equations is obtained having 8 unknown parameters:

$$
\left\lvert\, \begin{aligned}
& t_{000}-t_{001}+t_{010}-t_{011}+t_{100}-t_{101}+t_{110}-t_{111}=1 \\
& t_{000}+t_{001}+t_{010}+t_{011}-t_{100}-t_{101}-t_{110}-t_{111}=-1 \\
& t_{000}+t_{001}-t_{010}-t_{011}+t_{100}+t_{101}-t_{110}-t_{111}=-1
\end{aligned}\right.
$$

Therefore, we could express three of the parameters as a function of the rest five:

$$
\left\lvert\, \begin{align*}
& t_{010}=t_{101}+t_{100}-t_{011}  \tag{5}\\
& t_{000}=t_{011}+t_{111}-t_{100} \\
& t_{001}=-1+t_{100}-t_{011}+t_{110}
\end{align*}\right.
$$

After replacement in (2), we rewrite transform function as a function of 5 parameters $\left(t_{011}, t_{111}, t_{100}\right.$, $t_{110}$, and $t_{101}$ ):

$$
\begin{align*}
& F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)= \\
& =t_{011}\left(1-\cos \left(\omega_{3}\right)-\cos \left(\omega_{2}\right)+\cos \left(\omega_{2}\right) \cos \left(\omega_{3}\right)\right)+ \\
& +t_{111}\left(1+\cos \left(\omega_{1}\right) \cos \left(\omega_{2}\right) \cos \left(\omega_{3}\right)\right)+  \tag{6}\\
& +t_{100}\left(-1+\cos \left(\omega_{3}\right)+\cos \left(\omega_{2}\right)+\cos \left(\omega_{1}\right)\right)+ \\
& +t_{110}\left(\cos \left(\omega_{3}\right)+\cos \left(\omega_{1}\right) \cos \left(\omega_{2}\right)\right)+ \\
& +t_{101}\left(\cos \left(\omega_{2}\right)+\cos \left(\omega_{1}\right) \cos \left(\omega_{3}\right)\right)-\cos \left(\omega_{3}\right)
\end{align*}
$$

Let the cut-off frequency of 1-D prototype is denoted with $\omega_{0}$. We formulate our approximation problem as follows: to determine transform parameters from (6) and optimal $\omega_{0}$ under given angle of the cone $\theta \in(0, \pi / 2)$, such that the isopotential surface corresponding to the $\omega_{0}$ approximates the cone surface defined by (3). The same design problem was considered in [2], but applied for 2-D FIR fan filters.

To solve the above problem, at first we define the following deviation function on the cut-off isopotential surface:

$$
\begin{equation*}
E\left(\omega_{1}, \omega_{2}, \omega_{3}\right)=F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)-\cos \left(\omega_{0}\right) \tag{7}
\end{equation*}
$$

considering only the small values of cosine arguments:

$$
\begin{equation*}
\cos (x) \approx 1-\frac{x^{2}}{2} \tag{8}
\end{equation*}
$$

Thus, using (6), (7), and (8), we get the following relation from the equation $E\left(\omega_{1}, \omega_{2}, \omega_{3}\right)=0$ :

$$
\begin{equation*}
\omega_{1}^{2}+\omega_{2}^{2}-\frac{\omega_{3}^{2}}{p /(1-p)}=\frac{2}{p}[(2 p-1-\cos (\omega))+H] \tag{9}
\end{equation*}
$$

where the values of $p$ and $H$ have been calculated as:

$$
\begin{aligned}
& p=t_{111}+t_{110}+t_{100}+t_{101}, \\
& H=\frac{\omega_{1}^{2} \omega_{2}^{2}}{4}\left(t_{111}+t_{110}\right)+\frac{\omega_{2}^{2} \omega_{3}^{2}}{4}\left(t_{011}+t_{111}\right)+ \\
& +\frac{\omega_{1}^{2} \omega_{3}^{2}}{4}\left(t_{111}+t_{101}\right)-t_{111} \frac{\omega_{1}^{2} \omega_{2}^{2} \omega_{3}^{2}}{8} .
\end{aligned}
$$

As we stated before, we would like the cut-off isopotential surface (i.e. the surface for $\omega=\omega_{0}$ ) to be


Fig.1. Ideal frequency response of a cone filter (with $\gamma=1$ )
as close as possible to the equation describing the cone (3). Assuming this equivalence between (3) and (9), we get:

$$
2 p-1-\cos \left(\omega_{0}\right)=0 \quad \gamma^{2}=\frac{p}{1-p}
$$

and the cut-off frequency is finally determined as:

$$
\omega_{0}=\pi-2 \theta
$$

Also, in order to reduce the approximation error (considering the expression for $H$ ), we found that:

$$
\begin{align*}
& t_{110}=t_{011}=t_{101}=-t_{111} \\
& t_{100}=t_{111}+\frac{1-\cos (2 \theta)}{2} \tag{10}
\end{align*}
$$

Substituting the above obtained values (10) in (6), allow us to write the transform function in terms of only one parameter $\left(t_{111}\right)$ :

$$
\begin{align*}
& F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)=\left(t_{111}+r\right)\left(\cos \left(\omega_{1}\right)+\cos \left(\omega_{2}\right)-1\right)+ \\
& +\left(t_{111}-1+r\right) \cos \left(\omega_{3}\right)-t_{111} \cos \left(\omega_{1}\right) \cos \left(\omega_{3}\right)+  \tag{11}\\
& +t_{111} \cos \left(\omega_{2}\right) \cos \left(\omega_{3}\right)\left(\cos \left(\omega_{1}\right)-1\right)-t_{111} \cos \left(\omega_{1}\right) \cos \left(\omega_{2}\right), \\
& \text { where: } \quad r=\frac{1-\cos (2 \theta)}{2} . \tag{12}
\end{align*}
$$

The equation for isopotential surfaces can be obtained by solving $F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)=\cos (\omega)$ for frequency $\omega_{3}$ as a function of $\omega_{1}$ and $\omega_{2}$. For the cutoff frequency $\omega=\omega_{0}$ this equation is:
$\omega_{3}=\arccos \left(\frac{\cos \left(\omega_{0}\right)+\left(t_{111}+r\right)(1-A)+t_{111} B}{t_{111}(1-A+B)+r-1}\right)$,
where:

$$
\begin{align*}
& A=\cos \left(\omega_{1}\right)+\cos \left(\omega_{2}\right)  \tag{13}\\
& B=\cos \left(\omega_{1}\right) \cos \left(\omega_{2}\right)
\end{align*}
$$

Using the fact that $\cos \left(\omega_{0}\right)=-\cos (2 \theta)=2 r-1$, the expression (13) can be further optimized as:

$$
\begin{equation*}
\omega_{3}=\arccos \left(\frac{t_{111}(1-A+B)+r(3-A)-1}{t_{111}(1-A+B)+r-1}\right) \tag{14}
\end{equation*}
$$

As we proved, the following relation holds:

$$
F_{3}(0,0,0)=\cos \left(\omega_{0}\right) .
$$

This may be explained by the fact that the isopotential cut-off surface passes through the origin of a 3-D plane.

### 2.3 An application of the ISE criterion

According to the results obtained in the previous section, the transform function (11) is expressed in terms of the parameter $t_{111}$ (for small values of the cosine arguments). In order to find an analytical solution for this coefficient, we define the ISE function $M\left(t_{111}\right)$ as:

$$
\begin{equation*}
M\left(t_{111}\right)=\int_{0}^{\pi} \int_{0}^{\pi} E^{2}\left(\omega_{1}, \omega_{2}=f\left(\omega_{1}, \omega_{3}\right), \omega_{3}\right) d \omega_{1} d \omega_{3} \tag{15}
\end{equation*}
$$

where $E\left(\omega_{1}, \omega_{2}=f\left(\omega_{1}, \omega_{3}\right), \omega_{3}\right)$ is calculated using definition (7) and:

$$
\begin{equation*}
\cos \left(\omega_{2}\right) \approx 1+\omega_{1}^{2}-\frac{(1-r) \omega_{3}^{2}}{r} \tag{16}
\end{equation*}
$$

and has the following value:

$$
\begin{aligned}
& E\left(\omega_{1}, \omega_{2}=f\left(\omega_{1}, \omega_{3}\right), \omega_{3}\right)=1-2 r-\cos \left(\omega_{3}\right)+ \\
& +\left(\cos \left(\omega_{1}\right)+\cos \left(\omega_{3}\right)\right)\left(r-t_{111} \omega_{1}^{2}+t_{111} \frac{1-r}{r} \omega_{3}^{2}\right)+ \\
& +\left(r+t_{111}+t_{111} \cos \left(\omega_{1}\right) \cos \left(\omega_{3}\right)\right)\left(\omega_{1}^{2}-\frac{1-r}{r} \omega_{3}^{2}\right)
\end{aligned}
$$

In above considerations, we assume satisfying (16) for $\cos \left(\omega_{2}\right)$ as a result of eqs. (3) and (8).

In other words, the 3-D cone filter design problem can be reformulated as follows: to find value of $t_{111}$ that minimizes the ISE function defined by (15) under given angle $\theta$ of the cone. The minimization is with respect to one parameter and we can find an analytical solution. In order to determine the double integral in (15), we use the following statement:

$$
\begin{equation*}
M\left(t_{111}\right)=\int_{0}^{\pi}\left[\int_{0}^{\pi} E^{2}\left(\omega_{1}, \omega_{2}=f\left(\omega_{1}, \omega_{3}\right), \omega_{3}\right) d \omega_{3}\right] d \omega_{1} \tag{17}
\end{equation*}
$$

which allow us to decompose the above task into solving a set of single integrals (with respect to $\omega_{3}$ and $\omega_{1}$ ). After analytical evaluation of all elementary integrals included in (17), we derived the following final result:

$$
\begin{equation*}
M\left(t_{111}\right)=P_{1}+P_{2}+P_{3}+2\left(P_{4}+P_{5}+P_{6}\right) \tag{18}
\end{equation*}
$$

where:

$$
\begin{aligned}
& P_{1}= \pi^{2}+r\left(\frac{2 \pi^{4}}{3}-4 \pi^{2}\right)+r^{2}\left(\frac{\pi^{6}}{5}-\frac{4 \pi^{4}}{3}+\frac{\pi^{2}}{2}\right)+ \\
&+t_{111}\left[\begin{array}{l}
\frac{2 \pi^{4}}{3}+4 \pi^{2}+ \\
\left.+r\left(\frac{2 \pi^{6}}{5}+\frac{19 \pi^{4}}{3}-\frac{121 \pi^{2}}{2}\right)\right]+t_{111}^{2}\left(\frac{3 \pi^{6}}{10}+\frac{17 \pi^{4}}{2}-\frac{195 \pi^{2}}{4}\right) \\
P_{2}
\end{array}=(r-1)^{2}\left(\frac{\pi^{6}}{5}-\frac{7 \pi^{2}}{2}\right)+t_{111} \frac{(r-1)^{2}}{r}\left(\frac{2 \pi^{6}}{5}-4 \pi^{2}\right)+\right. \\
&+{t_{111}}_{2}^{(r-1)^{2}} \frac{3 \pi^{6}}{r^{2}} \\
& P_{3}= t_{t_{111}}^{2}\left[\frac{3 \pi^{6}}{20}+\frac{17 \pi^{4}}{4}-\frac{195 \pi^{2}}{8}+\frac{(r-1)}{r}\left(\frac{(1-r)^{2}}{r^{2}}\left(\frac{3 \pi^{6}}{20}+\frac{3 \pi^{4}}{4}-\frac{9 \pi^{4}}{3}+\frac{17 \pi^{2}}{8}\right)+\right]\right. \\
& P_{4}= t_{111}\left[(r-1)\left(3 \pi^{2}+\frac{\pi^{4}}{2}\right)+\frac{(r-1)^{2}}{r}\left(\frac{23 \pi^{4}}{6}-\frac{97 \pi^{2}}{4}\right)\right]+ \\
&+{ }_{t_{111}}^{2}\left[\frac{r-1}{r}\left(\pi^{4}+\frac{17 \pi^{2}}{2}\right)+\frac{(r-1)^{2}}{r^{2}}\left(6 \pi^{4}-36 \pi^{2}\right)\right]
\end{aligned}
$$

$P_{5}=\frac{(1-r) \pi^{3}}{3 r}\left[\begin{array}{l}2 r^{2} \pi-r \pi-\frac{r^{2} \pi^{3}}{3}- \\ -t_{111}\left(\frac{2 r \pi^{3}}{3}+\frac{3 r \pi}{2}+\pi\right)-t_{111}^{2}\left(\frac{\pi^{3}}{2}+\frac{17 \pi}{4}\right)\end{array}\right]$
$P_{6}=\frac{2 \pi(1-r)}{r}\left[t_{111}\left(\frac{r \pi}{2}-\frac{r \pi^{3}}{3}-\pi\right)-{ }_{t_{111}}^{2}\left(\frac{\pi^{3}}{2}+\frac{17 \pi}{4}\right)\right]$.
Using expression (18) for $M\left(t_{111}\right)$, we should further determine the value of the coefficient $t_{111}$ for which $\frac{\partial M\left(t_{111}\right)}{\partial t_{111}}=0$. Hence, we obtain the following closed-form relation for $t_{111}$ :
$t_{111}=\frac{\frac{4 \pi^{4}}{9}+\frac{8 \pi^{2}}{3}-r\left(\frac{38 \pi^{4}}{45}+\frac{29 \pi^{2}}{3}-\frac{113}{2}\right)-\frac{r-1}{r}\left(\frac{2 \pi^{2}}{3}+4\right)-\frac{(r-1)^{2}}{r} I_{1}}{I_{2}+\frac{r-1}{r}\left(\pi^{4}+17 \pi^{2}+\frac{289}{4}\right)+\frac{(r-1)^{2}}{r^{2}} I_{2}}$
where: $\quad I_{1}=\frac{2 \pi^{4}}{5}+\frac{23 \pi^{2}}{3}-\frac{105}{2}, \quad I_{2}=\frac{9 \pi^{4}}{10}+\frac{51 \pi^{2}}{2}-\frac{585}{4}$ and this value of $t_{111}$ is a solution of our optimization task as defined by (15). Applying above relation, the transform function from (11) is fully determined.

## 3 Scaled transform function

As we discussed in section 2.1, we would like the

transform function to be properly scaled. Our investigations have shown that $F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$ from (11) needs a special scaling procedure, because its extremal values are:

$$
\begin{equation*}
F_{3 \text { max }}=1 \quad F_{3 \text { min }}=-2+\cos (2 \theta) . \tag{20}
\end{equation*}
$$

That means that $\left|F_{3 \text { min }}\right|>1$ and the condition for scaled function $\left|F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)\right| \leq 1$ is not fulfilled. We apply below the scaling scheme proposed by Mersereau et al. [10] to get a scaled transform function:

$$
F_{3}^{S}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)=C_{1} F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)-C_{2},
$$

where:

$$
\begin{equation*}
C_{1}=2 /\left(F_{3 \max }-F_{3 \text { min }}\right) \quad C_{2}=C_{1} F_{3 \text { max }}-1 \tag{21}
\end{equation*}
$$

Using relations (20) and (21) in our case we determine:
$C_{1}=2 /(3-\cos (2 \theta)) \quad C_{2}=(\cos (2 \theta)-1) /(3-\cos (2 \theta))$ and the scaled transform function will be:

$$
\begin{align*}
& F_{3}^{S}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)= \\
& =\frac{1}{r+1}\left(\begin{array}{l}
-t_{111}+\left(t_{111}+r\right)\left(\cos \left(\omega_{1}\right)+\cos \left(\omega_{2}\right)\right)+ \\
+\left(t_{111}-1+r\right) \cos \left(\omega_{3}\right)+ \\
+t_{111} \cos \left(\omega_{2}\right) \cos \left(\omega_{3}\right)\left(\cos \left(\omega_{1}\right)-1\right)- \\
-t_{111} \cos \left(\omega_{1}\right) \cos \left(\omega_{3}\right)-t_{111} \cos \left(\omega_{1}\right) \cos \left(\omega_{2}\right)
\end{array}\right) \tag{22}
\end{align*}
$$



Fig.2. The isopotential cut-off surfaces plotted for different angles $\theta\left(20^{\circ}, 40^{\circ}, 55^{\circ}\right.$, and $\left.75^{\circ}\right)$
where $r$ and $t_{111}$ are given by (12) and (19), respectively.

The scaling method [10] is originally developed for 2-D digital filters, but our investigations proved, that it is also applicable for 3-D case. The shapes of the surfaces obtained with the scaled function (22) will be the same as these ones with non-scaled function $F_{3}\left(\omega_{1}, \omega_{2}, \omega_{3}\right)$. There will be only change of a 1-D frequency corresponding to each surface. The new 1-D cut-off frequency associated with the scaled function has to be determined as [10]:

$$
\begin{equation*}
\omega_{0}^{S}=\cos ^{-1}\left(C_{1} \cos \left(\omega_{0}\right)-C_{2}\right)=\cos ^{-1}\left(\frac{1-3 \cos (2 \theta)}{3-\cos (2 \theta)}\right) \tag{23}
\end{equation*}
$$

As concern the isopotential cut-off surface obtained with the function (22), we proved that its equation is exactly the same as this one with the non-scaled function (14).

We have to point out that the method presented in this work differs from the approach [11] in the following items: (i) three input conditions are imposed instead of four as in [11], (ii) the ISE criterion is applied, and (iii) the new relations for transform parameters are proposed.


## 4 Examples and discussion

All results given in this section are obtained using a Matlab simulation based on the derived expressions. The graphical view of the isopotential cut-off surfaces (for different angles $\theta$ ) is shown in Fig.2. The plots are generated using expression (14). The better accuracy we get for smaller values of frequencies (because of the applied approximation (8)) and for angles $\theta$ which are closer to 90 degrees.

The following methodology has been used for design of 3-D cone filters:

- Determination of the 1-D cut-off frequency using relation (23) under given angle $\theta$;
- Design a zero-phase 1-D FIR lowpass filter (we apply a McClellan-Parks algorithm in our examples);
- Calculation of transform parameter $t_{111}$ (see the expression (19)) in order to obtain scaled transform function (22);
- Using the presentation of a 1-D frequency response in terms of the Chebyshev polynomial (given in section 2.1) and relation (1), we design


Fig.3. The 3-D magnitude responses obtained for $\omega_{3}=\omega_{0}{ }^{\mathrm{s}}$ and different angles $\theta\left(58^{\circ}, 65^{\circ}, 75^{\circ}\right.$, and $\left.86^{\circ}\right)$
our 3-D cone filter response. The result for the transform function from the previous step is applied.
Fig. 3 presents several slices of the resulting 3-D magnitude responses plotted for a specific value of the frequency $\omega_{3}$, namely $\omega_{3}=\omega_{0}{ }^{\mathrm{s}}$. Different angles of the cone have been examined. The cut-off frequencies $\omega_{0}{ }^{\mathrm{S}}$ of corresponding 1-D filter prototypes have the following values: $\omega_{0}{ }^{\mathrm{S}}=0.2649 \pi$ $\left(\theta=58^{\circ}\right), 0.2028 \pi\left(\theta=65^{\circ}\right), 0.1192 \pi\left(\theta=75^{\circ}\right)$, and $0.0314 \pi\left(\theta=86^{\circ}\right)$. These results are obtained with prototypes of length 33 and transition band $0.1 \pi$. As we expected, the 3-D magnitude responses are equiripple (because 1-D filter prototypes determine the values along isopotential contours of the cone).

Our investigations have also shown that obtained graphical results (for both cut-off isopotential surfaces and 3-D magnitude responses) are close to those obtained with technique [11] for angles of the cone $\theta>45^{\circ}$. For angles below $45^{\circ}$, we get better accuracy with the proposed new method.

## 5 Conclusion

In this paper, the application of the ISE criterion for design of the McClellan based 3-D cone FIR filters is shown. The closed-form expressions for transform coefficients of 3-D FIR cone filters are derived.

The proposed method enjoys very short computation time without time-consuming iterative procedure. It could be also extended to design of other types of 3-D FIR filters (e.g. with spherical and elliptical characteristics).

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The first author is on leave from the University of Architecture, Civil Engineering and Geodesy, Dept. of Computer-Aided Engineering, 1 Hr . Smirnenski Blvd., 1046 Sofia, Bulgaria.

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