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Quadratic Time-Frequency Representations with Scale Covariance and Generalized Time-Shift Covariance: A Unified Framework for the Affine, Hyperbolic, and Power Classes*

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Abstract—We propose the generalized class of quadratic time-frequency representations (QTFRs) that satisfy the scale covariance property, which is important in multiresolution analysis, and the generalized time-shift covariance property, which is important in the analysis of signals propagating through systems with specific dispersive characteristics. We discuss a formulation of the generalized class QTFRs in terms of two-dimensional kernel functions, a generalized signal expansion related to the generalized class time-frequency geometry, an important member of the generalized class, a set of desirable QTFR properties and their corresponding kernel constraints, and a "localized-kernel" generalized subclass that is characterized by one-dimensional kernels.

Special cases of the generalized QTFR class include the *affine class* and the new *hyperbolic class* and *power classes*. All these QTFR classes satisfy the scale covariance property. In addition, the affine QTFRs are covariant to constant time shifts, the hyperbolic QTFRs are covariant to hyperbolic time shifts, and the power QTFRs are covariant to power time shifts. We present a detailed study of these classes that includes their definition and formulation, an associated generalized signal expansion, important class members, desirable QTFR properties and corresponding kernel constraints, and localized-kernel subclasses. Also, we investigate the subclasses formed by the intersection between the affine and hyperbolic classes, the affine and power classes, and the hyperbolic and power classes. These subclasses are important since their members satisfy additional desirable properties.

We show that the hyperbolic class is obtained from Cohen's QTFR class using a "hyperbolic time-frequency warping", and that the power classes are obtained similarly by applying a "power time-frequency warping" to the affine class. The affine class is a special case of the power classes. Furthermore, we generalize the time-frequency warping so that when applied either to Cohen's class or to the affine class, it yields QTFRs that are always generalized time-shift covariant but not necessarily scale covariant.

1 INTRODUCTION

The Fourier transform has been used extensively for signal analysis [1]; however, it does not provide easily accessible information about the time localization of a given frequency component in a signal. Thus, it is of limited use for the analysis of nonstationary signals, that is, signals whose frequency content changes with time. Quadratic time-frequency representations (QTFRs), on the other hand, are functions of both time and frequency. They map a one-dimensional (1-D) signal of time, $x(t)$, with Fourier transform, $X(f)$, into a two-dimensional (2-D) function of time and frequency, $T_X(t, f)$. As a result, QTFRs are potentially capable of displaying the temporal localization of the signal's spectral components [2–6].

QTFRs have been used successfully for the analysis and synthesis of nonstationary or transient signals [4]. They are found in applications such as speech processing, signal detection and parameter estimation, the analysis of acoustical and biological signals, and the analysis of linear systems. Some well-known QTFRs include the Wigner distribution [2, 7, 8], the smoothed pseudo-Wigner distribution [3–5, 8], the Choi-Williams (exponential) distribution [3–5, 9–11], the generalized exponential distribution [12], the spectrogram (squared magnitude of the short-time Fourier transform) [13–15], the scalogram (squared magnitude of the wavelet transform) [4, 16, 17], the Altes-Marinovich Q-distribution [18, 19], and the Bertrand P_κ-distributions [20–22].

Since no one QTFR exists that can be used effectively in all possible applications, different QTFRs are best suited for analyzing signals with different types of properties. Thus, the choice of a QTFR depends on the specific application at hand and the QTFR properties that are desirable for this application. As a result, QTFRs can be classified based on the properties they satisfy. A classification that has been found useful is based on elementary "covariance" properties (i.e. properties that result in a QTFR preserving various time-frequency (TF) translations to the signal):

- *Cohen's class* of time-shift and frequency-shift covariant QTFRs [3–5, 23–25] comprises many important QTFRs such as the Wigner distribution, the smoothed pseudo-Wigner distribution, the Choi-Williams distribution, the generalized exponential distribution, and the spectrogram. Cohen's class QTFRs feature a TF analysis resolution independent of the analysis time and frequency ("constant-bandwidth analysis").

- The *affine class* [4, 5, 10, 17, 20, 26–28], as an alternative, has been proposed as a framework for "constant-Q" or "proportional-bandwidth" TF analysis whose time resolution (frequency resolution) is proportional (inversely proportional) to the analysis frequency. The QTFRs of the affine class are covariant to time shifts and TF scalings (expansions/compressions). Some members include the Wigner distribution, the Choi-Williams distribution, the scalogram, and the Bertrand P_κ-distributions.

- The *hyperbolic class* [27, 29–34] provides an alternative framework for constant-Q analysis. It includes QTFRs that are covariant to hyperbolic time shifts and TF scalings such as the Altes-Marinovich Q-distribution and the Bertrand P_0-distribution. The hyperbolic QTFRs exhibit constant-Q TF characteristics since the hyperbolic class is obtained by mapping Cohen's class of constant-bandwidth QTFRs using a "constant-Q

warping" [29, 31, 33–37].

- The κth *power class* [33, 34, 38–40], $\kappa \neq 0$, consists of power time-shift and scale covariant QTFRs such as the κth power Wigner distribution and the Bertrand P_κ-distributions. The power classes can be obtained from the affine class through a "power warping", and the affine class is the power class with $\kappa = 1$.

Cohen's class, the affine class, the hyperbolic class, and the power classes of QTFRs satisfy different covariance properties and have different TF resolution characteristics, and, as a consequence, they are best used for different types of applications.

In this paper, we propose a generalized QTFR class that provides a unified framework for the affine class, the hyperbolic class, and the power classes; this generalized class consists of QTFRs that are covariant to generalized time shifts and TF scalings. We consider some important QTFRs pertaining to this generalized framework. We formulate desirable QTFR properties and derive constraints on the QTFRs' kernel functions. We investigate "localized-kernel" subclasses. We also consider the intersection subclasses between the various special cases of the generalized class, such as the affine-hyperbolic intersection, the affine-power intersection, and the hyperbolic-power intersection. Finally, we provide examples illustrating the application of the proposed QTFRs.

Figures 1 and 2 provide a pictorial summary of the different classes to be discussed in the paper together with some of their important QTFR members and subclasses. Cohen's class, the affine class, the hyperbolic class, and the power classes are shown in Figure 1 together with their intersection subclasses, while the localized-kernel subclasses of the affine, hyperbolic, and power classes are added in Figure 2.

1.1 Cohen's QTFR Class

Prior to discussing QTFR classes based on generalized time-shift covariance and scale covariance, we briefly review the traditional framework for quadratic TF analysis. *Cohen's class with signal-independent kernels* (hereafter referred to as Cohen's class for simplicity) consists of all QTFRs that satisfy the time-shift and frequency-shift covariance properties (cf. Table 1) [3–5, 23–25]. In particular, let the time-shift operator, $\mathcal{S}_\tau$, and the frequency-shift operator, $\mathcal{M}_\nu$, be defined as

$$(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f}\, X(f)\,, \qquad (\mathcal{M}_\nu X)(f) = X(f-\nu)\,. \tag{1}$$

Any QTFR, $T_X^{(C)}(t,f)$, in Cohen's class satisfies

$$T_{\mathcal{S}_\tau X}^{(C)}(t,f) = T_X^{(C)}(t-\tau,\, f) \tag{2}$$

$$T_{\mathcal{M}_\nu X}^{(C)}(t,f) = T_X^{(C)}(t, f-\nu) \tag{3}$$

where, for example, $T_{\mathcal{S}_\tau X}^{(C)}(t,f)$ stands for $T_Y^{(C)}(t,f)$ with $Y(f) = (\mathcal{S}_\tau X)(f)$. Both the time and frequency shift covariance properties are important in applications where the signal needs to be analyzed at various TF points using the same time resolution and the same frequency resolution. Indeed, the TF analysis characteristics of

Cohen's class QTFRs do not change with time and frequency, and the QTFRs maintain the signal's time and frequency shifts. This is important in a broad range of applications involving the analysis of many natural or man-made signals. More generally, such a uniform TF analysis characteristic defaults reasonably whenever a specific TF signal structure (with actually a nonuniform analysis characteristic) is not known to exist.

Any QTFR of Cohen's class can be expressed in terms of the signal's Fourier transform, $X(f)$, as

$$T_X^{(C)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma_T^{(C)}(f-f_1, f-f_2)\, e^{j2\pi t(f_1-f_2)} X(f_1)\, X^*(f_2)\, df_1 df_2 \tag{4}$$

where $\Gamma_T^{(C)}(f_1,f_2)$ is a 2-D kernel uniquely characterizing the Cohen's class QTFR, $T^{(C)}$. Alternatively, any Cohen's class QTFR can be written in terms of the following four "normal forms":

$$\begin{aligned}
T_X^{(C)}(t,f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T^{(C)}(t-\hat{t},\tau)\, u_X(\hat{t},\tau)\, e^{-j2\pi f\tau}\, d\hat{t}\, d\tau && (5)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_T^{(C)}(f-\hat{f},\nu)\, U_X(\hat{f},\nu)\, e^{j2\pi t\nu}\, d\hat{f}\, d\nu && (6)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T^{(C)}(t-\hat{t}, f-\hat{f})\, W_X(\hat{t},\hat{f})\, d\hat{t}\, d\hat{f} && (7)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_T^{(C)}(\tau,\nu)\, A_X(\tau,\nu)\, e^{j2\pi(t\nu - f\tau)}\, d\tau\, d\nu\,, && (8)
\end{aligned}$$

where the signal products, $u_X(t,\tau)$ and $U_X(f,\nu)$, are defined as

$$u_X(t,\tau) \triangleq x\Big(t+\frac{\tau}{2}\Big)\, x^*\Big(t-\frac{\tau}{2}\Big)\,, \qquad U_X(f,\nu) \triangleq X\Big(f+\frac{\nu}{2}\Big)\, X^*\Big(f-\frac{\nu}{2}\Big)\,. \tag{9}$$

In (7), Cohen's class QTFRs are written in terms of the Wigner distribution, $W_X(t,f)$, using a 2-D convolution, and in (8) they are written in terms of the narrowband ambiguity function, $A_X(\tau,\nu)$, as a product plus a 2-D Fourier transform. The Wigner distribution and the ambiguity function are defined in terms of the signal products as

$$W_X(t,f) = \int_{-\infty}^{\infty} u_X(t,\tau)\, e^{-j2\pi f\tau}\, d\tau = \int_{-\infty}^{\infty} U_X(f,\nu)\, e^{j2\pi t\nu}\, d\nu \tag{10}$$

$$A_X(\tau,\nu) = \int_{-\infty}^{\infty} u_X(t,\tau)\, e^{-j2\pi\nu t}\, dt = \int_{-\infty}^{\infty} U_X(f,\nu)\, e^{j2\pi\tau f}\, df\,. \tag{11}$$

The 2-D kernel functions $\phi_T^{(C)}(t,\tau)$, $\Phi_T^{(C)}(f,\nu)$, $\psi_T^{(C)}(t,f)$, and $\Psi_T^{(C)}(\tau,\nu)$ are interrelated by Fourier transforms,

$$\begin{aligned}
\psi_T^{(C)}(t,f) &= \int_{-\infty}^{\infty} \phi_T^{(C)}(t,\tau)\, e^{-j2\pi f\tau}\, d\tau = \int_{-\infty}^{\infty} \Phi_T^{(C)}(f,\nu)\, e^{j2\pi t\nu}\, d\nu\,, && (12)\\
\Psi_T^{(C)}(\tau,\nu) &= \int_{-\infty}^{\infty} \phi_T^{(C)}(t,\tau)\, e^{-j2\pi\nu t}\, dt = \int_{-\infty}^{\infty} \Phi_T^{(C)}(f,\nu)\, e^{j2\pi\tau f}\, df\,, && (13)
\end{aligned}$$

and the kernels $\Gamma_T^{(C)}(f_1,f_2)$ in (4) and $\Phi_T^{(C)}(f,\nu)$ in (6) are related as $\Gamma_T^{(C)}(f_1,f_2) = \Phi_T^{(C)}(\frac{f_1+f_2}{2}, f_2-f_1)$. Any one of these kernels uniquely characterizes the QTFR $T^{(C)}$. Note that the kernels are not allowed to depend on the signal $X(f)$ since such a dependence would be incompatible with the quadratic dependence of $T_X^{(C)}(t,f)$ on $X(f)$. A table with numerous QTFR members of Cohen's class developed in the literature is provided in [4], together with a list of constraints on the kernels of Cohen's class corresponding to desirable QTFR properties.

1.2 Paper Survey

The important properties of generalized (possibly dispersive) time-shift covariance and scale covariance are combined together to form the foundation of the generalized QTFR class proposed in this paper. This QTFR class is particularly suited to the multiresolution analysis of signals propagating through systems with specific dispersive characteristics. Depending on the particular form of the dispersive characteristic, some special cases of the generalized time-shift covariance property include the constant (nondispersive) time-shift covariance of the affine class, the dispersive hyperbolic time-shift covariance of the hyperbolic class, and the dispersive power time-shift covariance of the power classes.

The paper is organized as follows. Sections 2-4 consider three special cases of the generalized class, namely, the affine, the hyperbolic, and the power classes. In Section 2, we discuss the affine class that is defined as the class of QTFRs that satisfy the nondispersive time-shift covariance and the scale covariance properties. Section 3 presents the recently proposed hyperbolic class that is defined based on the hyperbolic time-shift covariance and the scale covariance properties. Alternatively, but equivalently, the hyperbolic class is obtained by applying a "constant-Q warping" transformation to Cohen's class QTFRs[1]. Section 4 considers the power classes of QTFRs. The κth power class is defined based on the κth power time-shift covariance and the scale covariance properties. The power classes are also obtained from the affine class using a "power warping" transformation. The affine class is the special case of the power classes with $\kappa = 1$.

For each of these three QTFR classes, we discuss the formulation of any class member in terms of 2-D kernels, some important specific members, some desirable QTFR properties and the corresponding kernel constraints, and a "localized-kernel" subclass in which the QTFR formulation and the kernel constraints are simplified.

The various QTFR classes considered have some QTFRs in common. The resulting *intersection subclasses* are considered in Section 5. In these subclasses, a third covariance property is added to the two covariance properties of each class; as a result, these QTFRs are useful in applications that require all three properties.

In Section 6, we define the generalized time-shift covariance property and propose the generalized class formulation. The QTFRs of the generalized class are defined based on a specific dispersive time-shift covariance property and the scale covariance property. We formulate a condition on the phase function (i.e., the dispersive characteristic) that is necessary and sufficient for the existence of the generalized QTFRs. We show how the generalized class can be simplified to the affine class, the hyperbolic class, and the power classes by special choices of the phase function or its derivative, the group delay function, and we discuss the special role played by these particular classes within the general framework. Furthermore, we develop the generalized class formulation in terms of a 2-D kernel, a generalized signal expansion, some desirable QTFR properties and the associated kernel constraints, and a "localized-kernel" generalized subclass.

In Section 7, we propose a generalized "warping" transformation and its application to Cohen's class and the affine class. The resulting QTFRs satisfy the generalized time-shift covariance property, and an additional covariance property which, in general, is not the scale covariance.

Finally, in Section 8 we provide simulation results that illustrate the superior performance of the proposed QTFRs for various types of signals.

2 THE AFFINE QTFR CLASS

The affine class [4, 5, 10, 17, 20, 26–28] contains all QTFRs that satisfy the time-shift covariance property and the scale covariance property (cf. Table 1). Thus, any QTFR, $T_X^{(A)}(t,f)$, in the affine class satisfies

$$T_{\mathcal{S}_\tau X}^{(A)}(t,f) = T_X^{(A)}(t-\tau,\, f)\,, \tag{14}$$

$$T_{\mathcal{C}_a X}^{(A)}(t,f) = T_X^{(A)}\left(at, \frac{f}{a}\right)\,, \tag{15}$$

where the scaling operator $\mathcal{C}_a$ causes the signal to be expanded or compressed, and it is defined as

$$(\mathcal{C}_a X)(f) = \frac{1}{\sqrt{|a|}}\, X\left(\frac{f}{a}\right)\,. \tag{16}$$

The scale covariance property in (15) is important for multiscale or multiresolution analysis (cf. the wavelet transform [4, 16]), for self-similar signals [42], scale-covariant systems [43], and in the context of the wideband Doppler effect [44]. It is desirable to preserve scale changes on the signal in applications such as image enlargement or compression. Many affine class QTFRs (like the scalogram, i.e. the squared magnitude of the wavelet transform) have a constant-Q TF analysis characteristic where the analysis bandwidth is proportional to the analysis frequency. This constant-Q analysis offers an alternative to the constant-bandwidth analysis achieved by QTFRs of Cohen's class.

2.1 Affine Class Formulation

Any affine class QTFR, $T_X^{(A)}(t,f)$, can be expressed in terms of $X(f)$ as

$$T_X^{(A)}(t,f) = \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma_T^{(A)}\Big(\frac{f_1}{f}, \frac{f_2}{f}\Big)\, e^{j2\pi t(f_1-f_2)}\, X(f_1)\, X^*(f_2)\, df_1 df_2 \tag{17}$$

where $\Gamma_T^{(A)}(b_1,b_2)$ is a 2-D kernel uniquely characterizing the affine QTFR, $T^{(A)}$. The affine class QTFRs can also be written in terms of 2-D kernels in a similar way as Cohen's class QTFRs were expressed in (5)-(8):

$$T_X^{(A)}(t,f) = |f| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T^{(A)}\Big(f(t-\hat{t}), f\tau\Big)\, u_X(\hat{t},\tau)\, d\hat{t}\, d\tau \tag{18}$$

$$= \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_T^{(A)}\Big(-\frac{\hat{f}}{f}, \frac{\nu}{f}\Big)\, U_X(\hat{f},\nu)\, e^{j2\pi t\nu} d\hat{f}\, d\nu \tag{19}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T^{(A)}\Big(f(t-\hat{t}), -\frac{\hat{f}}{f}\Big)\, W_X(\hat{t},\hat{f})\, d\hat{t}\, d\hat{f} \tag{20}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_T^{(A)}\Big(f\tau, \frac{\nu}{f}\Big)\, A_X(\tau,\nu)\, e^{j2\pi t\nu}\, d\tau\, d\nu \tag{21}$$

where the signal products, $u_X(t,\tau)$ and $U_X(f,\nu)$, are defined in (9), and the Wigner distribution, $W_X(t,f)$, and ambiguity function, $A_X(\tau,\nu)$, are defined in (10) and (11), respectively. The kernel functions $\phi_T^{(A)}(c,\zeta)$, $\Phi_T^{(A)}(b,\beta)$, $\psi_T^{(A)}(c,b)$, and $\Psi_T^{(A)}(\zeta,\beta)$ are all interrelated by Fourier transforms,

$$\psi_T^{(A)}(c,b) = \int_{-\infty}^{\infty} \phi_T^{(A)}(c,\zeta)\, e^{-j2\pi b\zeta}\, d\zeta = \int_{-\infty}^{\infty} \Phi_T^{(A)}(b,\beta)\, e^{j2\pi c\beta}\, d\beta\,, \tag{22}$$

$$\Psi_T^{(A)}(\zeta,\beta) = \int_{-\infty}^{\infty} \phi_T^{(A)}(c,\zeta)\, e^{-j2\pi\beta c}\, dc = \int_{-\infty}^{\infty} \Phi_T^{(A)}(b,\beta)\, e^{j2\pi\zeta b}\, db\,. \tag{23}$$

The kernel $\Phi_T^{(A)}(b,\beta)$ in (19) is related to the kernel $\Gamma_T^{(A)}(b_1,b_2)$ in (17) according to the relationship $\Gamma_T^{(A)}(b_1,b_2) = \Phi_T^{(A)}(-\frac{b_1+b_2}{2}, b_1-b_2)$. Note that the Wigner distribution is a member of both the affine class (with affine class kernel $\Phi_W^{(A)}(b,\beta) = \delta(b+1)$ in (19)), and Cohen's class (with Cohen's class kernel $\Phi_W^{(C)}(f,\nu) = \delta(f)$ in (6)). Thus, it is a member of the intersection between the two classes [10], and as such it is TF shift covariant as well as scale covariant.

2.2 Affine Signal Expansion

The affine class is intimately related to the Fourier transform. The brief discussion of the Fourier transform given in the following may appear trivial, but it will serve as a useful point of reference and comparison for more complex signal expansions, to be discussed in later sections.

Let us consider the family of Dirac impulses $i_c^{(A)}(t) = \frac{1}{\sqrt{f_r}}\,\delta(t-\frac{c}{f_r})$, where $c \in \mathbb{R}$ and f_r is a positive, fixed reference frequency. The Fourier transform of this impulse is

$$I_c^{(A)}(f) = \frac{1}{\sqrt{f_r}}\, e^{-j2\pi c\frac{f}{f_r}}\,; \tag{24}$$

it has constant spectral energy density $|I_c^{(A)}(f)|^2 = \frac{1}{f_r}$ and constant group delay $\frac{c}{f_r}$. A TF scaling of the impulse results in a scaling of the parameter c, $(\mathcal{C}_a I_c^{(A)})(f) = \frac{1}{\sqrt{|a|}}\, I_{c/a}^{(\kappa)}(f)$, whereas time-shifting the impulse by a constant amount $\frac{c_0}{f_r}$ results in a shift of the parameter c, $(\mathcal{S}_{c_0/f_r} I_c^{(A)})(f) = I_{c+c_0}^{(A)}(f)$.

The "affine signal expansion" related to the affine class is simply the Fourier transform

$$X(f) = \int_{-\infty}^{\infty} \rho_X^{(A)}(c)\, I_c^{(A)}(f)\, dc = \frac{1}{\sqrt{f_r}} \int_{-\infty}^{\infty} \rho_X^{(A)}(c)\; e^{-j2\pi c\frac{f}{f_r}}\, dc\,, \tag{25}$$

where the coefficient function, $\rho_X^{(A)}(c)$, is the inner product of the signal $X(f)$ with the impulse $I_c^{(A)}(f)$ in (24), which is essentially the time-domain signal, $x(t)$:

$$\rho_X^{(A)}(c) = \left\langle X, I_c^{(A)} \right\rangle = \int_{-\infty}^{\infty} X(f)\, I_c^{(A)*}(f)\, df = \frac{1}{\sqrt{f_r}} \int_{-\infty}^{\infty} X(f)\, e^{j2\pi c\frac{f}{f_r}}\, df = \frac{1}{\sqrt{f_r}}\, x\!\left(\frac{c}{f_r}\right)\,. \tag{26}$$

The validity of the Fourier transform signal expansion follows from the completeness relation

$$\int_{-\infty}^{\infty} I_c^{(A)}(f_1)\, I_c^{(A)*}(f_2)\, dc = \delta(f_1-f_2)\,.$$

Another important property of the impulses $I_c^{(A)}(f)$ is the orthogonality property

$$\left\langle I_{c_1}^{(A)}, I_{c_2}^{(A)} \right\rangle = \int_{-\infty}^{\infty} I_{c_1}^{(A)}(f)\, I_{c_2}^{(A)*}(f)\, df = \delta(c_1 - c_2)\,.$$

Some basic properties of the Fourier transform signal expansion are [1]:

- Unitarity (cf. Parseval's theorem), i.e., the preservation of inner products

$$\int_{-\infty}^{\infty} \rho_{X_1}^{(A)}(c)\, \rho_{X_2}^{(A)*}(c)\, dc = \int_{-\infty}^{\infty} X_1(f)\, X_2^*(f)\, df\,.$$

- Scaling the Fourier transform of the signal scales the coefficient function, $\rho_{\mathcal{C}_a X}^{(A)}(c) = \sqrt{|a|}\, \rho_X^{(A)}(ac)$.
- Time-shifting the signal by time lag $\tau = \frac{c_0}{f_r}$ shifts the coefficient function by c_0, $\rho_{\mathcal{S}_{c_0/f_r} X}^{(A)}(c) = \rho_X^{(A)}(c - c_0)$.
- The coefficient function of an impulse, $I_{c_0}^{(A)}(f)$, is a Dirac impulse centered at $c = c_0$, $\rho_{I_{c_0}^{(A)}}^{(A)}(c) = \delta(c - c_0)$.

2.3 Important Members, Properties, and Kernel Constraints

Some members of the affine class are listed below:

- *Wigner distribution*, $W_X(t, f)$, defined in (10).
- *Generalized Wigner distribution*, $W_X^{(\alpha)}(t, f)$ [2, 10]:

$$W_X^{(\alpha)}(t, f) = \int_{-\infty}^{\infty} X\Big(f + \big(\frac{1}{2} - \alpha\big)\nu\Big)\; X^*\Big(f - \big(\frac{1}{2} + \alpha\big)\nu\Big)\; e^{j2\pi t\nu}\, d\nu\,. \tag{27}$$

  Note that the generalized Wigner distribution equals the Wigner distribution for $\alpha = 0$, and the Rihaczek distribution [2, 4, 10] for $\alpha = \frac{1}{2}$.

- *Scalogram*, $\mathrm{SCAL}_X(t, f)$:

$$\mathrm{SCAL}_X(t, f) = \frac{f_r}{|f|} \left| \int_{-\infty}^{\infty} X(\hat{f})\, \Theta^*\Big(f_r \frac{\hat{f}}{f}\Big)\, e^{j2\pi t\hat{f}}\, d\hat{f} \right|^2. \tag{28}$$

  Here, $\Theta(f)$ is the Fourier transform of a wavelet function. The scalogram is the squared magnitude of the wavelet transform [16, 17].

- *Affine smoothed pseudo Wigner distribution*, $\mathrm{ASPWD}_X(t, f)$:

$$\mathrm{ASPW}_X(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(f(t - \hat{t}))\; H\Big(-\frac{\hat{f}}{f}\Big)\; W_X(\hat{t}, \hat{f})\; d\hat{t}\; d\hat{f} \tag{29}$$

  where $s(c)$ and $H(b)$ are two independent windows controlling the smoothing characteristics.

- *Bertrand unitary P_0-distribution*, $P_{0_X}(t, f)$ [20–22, 31, 45, 46]:

$$\begin{aligned} P_{0_X}(t, f) &= f \int_{-\infty}^{\infty} X\Big(f \frac{\beta/2}{\sinh(\beta/2)} e^{\frac{\beta}{2}}\Big)\, X^*\Big(f \frac{\beta/2}{\sinh(\beta/2)} e^{-\frac{\beta}{2}}\Big)\, \frac{\beta/2}{\sinh(\beta/2)}\, e^{j2\pi t f\beta}\, d\beta\,, \quad f > 0 \\ &= f \int_{-\infty}^{\infty} X\Big(f\big(\frac{\beta}{2} \coth \frac{\beta}{2} + \frac{\beta}{2}\big)\Big)\; X^*\Big(f\big(\frac{\beta}{2} \coth \frac{\beta}{2} - \frac{\beta}{2}\big)\Big)\, \frac{\beta/2}{\sinh(\beta/2)}\, e^{j2\pi t f\beta}\, d\beta\,, \quad f > 0\,. \end{aligned} \tag{30}$$

- *Flandrin D-distribution*, $D_X(t,f)$ [45]:

$$D_X(t,f) = |f| \int_{-\infty}^{\infty} X\Big(f\Big(1+\frac{\beta}{4}\Big)^2\Big)\, X^*\Big(f\Big(1-\frac{\beta}{4}\Big)^2\Big)\Big[1-\Big(\frac{\beta}{4}\Big)^2\Big]\, e^{j2\pi t f\beta}\, d\beta\,. \tag{31}$$

- *Passive Unterberger distribution*, $\mathrm{PUD}_X(t,f)$ [20, 45]:

$$\mathrm{PUD}_X(t,f) = |f| \int_{-\infty}^{\infty} X\left(f\left(\sqrt{1+\Big(\frac{\beta}{2}\Big)^2}+\frac{\beta}{2}\right)\right) X^*\left(f\left(\sqrt{1+\Big(\frac{\beta}{2}\Big)^2}-\frac{\beta}{2}\right)\right) \frac{1}{\sqrt{1+(\frac{\beta}{2})^2}}\, e^{j2\pi t f\beta}\, d\beta\,.$$

- *Active Unterberger distribution*, $\mathrm{AUD}_X(t,f)$ [20, 45]:

$$\mathrm{AUD}_X(t,f) = |f| \int_{-\infty}^{\infty} X\left(f\left(\sqrt{1+\Big(\frac{\beta}{2}\Big)^2}+\frac{\beta}{2}\right)\right) X^*\left(f\left(\sqrt{1+\Big(\frac{\beta}{2}\Big)^2}-\frac{\beta}{2}\right)\right) e^{j2\pi t f\beta}\, d\beta\,. \tag{32}$$

- An important QTFR family within the affine class is formed by the *Bertrand P_κ-distributions* $P_{\kappa_X}(t,f)$ [20, 21] defined as

$$P_{\kappa_X}(t,f) = |f| \int_{-\infty}^{\infty} X(f\lambda_\kappa(u))\; X^*(f\lambda_\kappa(-u))\; e^{j2\pi t f\,[\lambda_\kappa(u)-\lambda_\kappa(-u)]}\; \mu(u)\, du \tag{33}$$

where $\mu(u)$ is a real and even weighting function, and

$$\begin{aligned} \lambda_\kappa(u) &= \Big(\kappa\frac{e^{-u}-1}{e^{-\kappa u}-1}\Big)^{\frac{1}{\kappa-1}}, \quad \kappa \neq 0,1 \\ \lambda_0(u) &= \frac{u}{1-e^{-u}}, \qquad \lambda_1(u) = \exp\left(1+\frac{u\,e^{-u}}{e^{-u}-1}\right). \end{aligned} \tag{34}$$

The kernel in (17) of the Bertrand P_κ-distributions is $\Gamma^{(A)}_{P_\kappa}(b_1,b_2) = \int_{-\infty}^{\infty} \delta(b_1-\lambda_\kappa(u))\delta(b_2-\lambda_\kappa(-u))\, \mu(u) du$. In addition to being scale covariant and time-shift covariant, the Bertrand P_κ-distributions are also covariant to a power-law time-shift covariance property [21, 38] (see Sections 4 and 5.2). The following important affine QTFRs are special cases of the Bertrand P_κ-distributions in (33):

  - The Bertrand unitary P_0-distribution in (30) is obtained for κ=0 and $\mu(u)=\frac{u/2}{\sinh(u/2)}$.
  - The active Unterberger distribution in (32) is obtained for $\kappa=-1$ and $\mu(u)=\cosh(\frac{u}{2})$.
  - The Flandrin D-distribution in (31) is obtained for $\kappa=\frac{1}{2}$ and $\mu(u)=\lambda_{1/2}(u)\lambda_{1/2}(-u)$.
  - The Wigner distribution in (10) (restricted to analytic signals only) is obtained when κ=2 and $\mu(u)=\sqrt{\lambda_2(u)\lambda_2(-u)\,\frac{d}{du}(\lambda_2(u)-\lambda_2(-u))}$ [20, 28].

All affine class QTFRs satisfy the time-shift covariance property (14) and the scale covariance property (15). Additional desirable properties will be satisfied if the QTFR kernels satisfy certain constraints associated with these properties. A list of QTFR properties and the associated kernel constraints is provided in Table 2. In Table 3, we summarize the above affine QTFRs along with their kernels and the properties they satisfy.

2.4 Localized-Kernel Affine Subclass

"Localized-kernel" affine QTFRs have a kernel $\Phi_T^{(A)}(b,\beta)$ that is perfectly localized along a curve in the (b,β)-plane defined by $b = F_T^{(A)}(\beta)$, where $F_T^{(A)}(\beta)$ is a real-valued, 1-D kernel function [17, 26, 28, 45, 47]. The 2-D kernels for the localized-kernel affine QTFR subclass simplify to

$$\phi_T^{(A)}(c,\zeta) = \int_{-\infty}^{\infty} G_T^{(A)}(\beta)\, e^{j2\pi[c\beta+\zeta F_T^{(A)}(\beta)]}\, d\beta\,, \tag{35}$$

$$\Phi_T^{(A)}(b,\beta) = G_T^{(A)}(\beta)\, \delta\Big(b - F_T^{(A)}(\beta)\Big)\,, \tag{36}$$

$$\psi_T^{(A)}(c,b) = \int_{-\infty}^{\infty} G_T^{(A)}(\beta)\, \delta\Big(b - F_T^{(A)}(\beta)\Big)\, e^{j2\pi c\beta}\, d\beta\,, \tag{37}$$

$$\Psi_T^{(A)}(\zeta,\beta) = G_T^{(A)}(\beta)\, e^{j2\pi\zeta F_T^{(A)}(\beta)}\,, \tag{38}$$

where the 1-D kernels $F_T^{(A)}(\beta) \in \mathbb{R}$ and $G_T^{(A)}(\beta)$ characterize the affine QTFR, $T^{(A)}$. We see that the 2-D kernels in (18-21) are parameterized in terms of two 1-D kernels in (35-38). This results in simpler formulations for the localized-kernel affine QTFRs, and it simplifies the kernel constraints corresponding to desirable QTFR properties. Under certain assumptions, the above structure of the 2-D kernels is necessary for perfect localization along group delay laws [47].

For a localized-kernel affine QTFR, the general form in (19) simplifies to

$$T_X^{(A)}(t,f) = |f| \int_{-\infty}^{\infty} U_X\Big(-f F_T^{(A)}(\beta), f\beta\Big)\, G_T^{(A)}(\beta)\, e^{j2\pi t f\beta}\, d\beta \tag{39}$$

$$= |f| \int_{-\infty}^{\infty} X\Big(f\Big(-F_T^{(A)}(\beta) + \frac{\beta}{2}\Big)\Big)\, X^*\Big(f\Big(-F_T^{(A)}(\beta) - \frac{\beta}{2}\Big)\Big)\, G_T^{(A)}(\beta)\, e^{j2\pi t f\beta}\, d\beta\,. \tag{40}$$

The Wigner distribution, $W_X(t,f)$, the generalized Wigner distribution, $W_X^{(\alpha)}(t,f)$, the Bertrand unitary P_0-distribution, $P_{0_X}(t,f)$, the Flandrin D-distribution, $D_X(t,f)$, the passive Unterberger distribution, $\mathrm{PUD}_X(t,f)$, and the active Unterberger distribution, $\mathrm{AUD}_X(t,f)$ are members of the localized-kernel affine subclass. Their 1-D kernels[2] are given as

$$\begin{array}{ll}
F_W^{(A)}(\beta) = -1\,, & G_W^{(A)}(\beta) = 1\,; \\
F_{W^{(\alpha)}}^{(A)}(\beta) = \alpha\beta - 1\,, & G_{W^{(\alpha)}}^{(A)}(\beta) = 1\,; \\
F_{P_0}^{(A)}(\beta) = -\frac{\beta}{2}\coth(\frac{\beta}{2})\,, & G_{P_0}^{(A)}(\beta) = \frac{\beta/2}{\sinh(\beta/2)}\,; \\
F_D^{(A)}(\beta) = -1 - (\frac{\beta}{4})^2\,, & G_D^{(A)}(\beta) = 1 - (\frac{\beta}{4})^2\,; \\
F_{\mathrm{PUD}}^{(A)}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}\,, & G_{\mathrm{PUD}}^{(A)}(\beta) = \frac{1}{\sqrt{1+(\frac{\beta}{2})^2}}\,; \\
F_{\mathrm{AUD}}^{(A)}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}\,, & G_{\mathrm{AUD}}^{(A)}(\beta) = 1\,.
\end{array}$$

The simplified kernel constraints for localized-kernel affine QTFRs are given in the third column of Table 2, where the simplified form of the kernel $\Phi_T^{(A)}(b,\beta)$ in (36) is exploited.

3 THE HYPERBOLIC QTFR CLASS

The recently proposed hyperbolic class [27, 29–34] is defined based on the hyperbolic time-shift covariance property and the scale covariance property that all its QTFR members satisfy. That is, if $T_X^{(H)}(t,f)$ is a QTFR in the hyperbolic class, where $X(f)$ is the Fourier transform of an analytic signal (i.e., $X(f)=0$ for $f<0$), then it satisfies the hyperbolic time-shift and scale covariance properties (cf. Table 1) defined respectively as

$$T_{\mathcal{H}_c X}^{(H)}(t,f) = T_X^{(H)}\left(t-\frac{c}{f},f\right), \tag{41}$$

$$T_{\mathcal{C}_a X}^{(H)}(t,f) = T_X^{(H)}\left(at,\frac{f}{a}\right). \tag{42}$$

Here, the hyperbolic time-shift operator $\mathcal{H}_c$ is defined as

$$(\mathcal{H}_c X)(f) = e^{-j2\pi c \ln\frac{f}{f_r}}\, X(f), \qquad f>0, \tag{43}$$

where f_r is a positive reference frequency. The hyperbolic time-shift operator $\mathcal{H}_c$ can be interpreted as an allpass, linear, time-invariant system with frequency response, $\exp(-j2\pi c \ln\frac{f}{f_r})$, and hyperbolic group delay, $\tau(f)=c/f$. For example, if $c>0$, then signal components at low frequencies are delayed more than high-frequency components. The two properties (41)-(42) defining the hyperbolic class are important for the analysis of Doppler-invariant signals similar to the signals used by bats and dolphins for echolocation [18, 48], and for the analysis of self-similar random processes [42].

By constraining any QTFR to satisfy the two covariance properties in (41)-(42), any hyperbolic QTFR, $T_X^{(H)}(t,f)$, can be expressed in terms of the signal $X(f)$ as

$$T_X^{(H)}(t,f) = \frac{1}{f}\int_0^\infty\int_0^\infty \Gamma_T^{(H)}\left(\frac{f_1}{f},\frac{f_2}{f}\right) e^{j2\pi t f \ln\frac{f_1}{f_2}}\, X(f_1)\, X^*(f_2)\, df_1 df_2, \qquad f>0, \tag{44}$$

where the 2-D kernel $\Gamma_T^{(H)}(b_1,b_2)$ uniquely characterizes the hyperbolic QTFR, $T^{(H)}$.

3.1 Constant-Q Warping

The hyperbolic class can alternatively be defined by applying a "constant-Q warping" transformation to Cohen's class QTFRs [29, 31]. This warping converts the constant-bandwidth QTFRs of Cohen's class into constant-Q (or proportional-bandwidth) QTFRs. As a result, the new hyperbolic QTFRs can now be used in applications where a constant-Q TF analysis is more appropriate than the constant-bandwidth analysis of Cohen's class QTFRs. The constant-Q warping is defined as

$$T_X^{(H)}(t,f) \triangleq T_{\mathcal{W}_H X}^{(C)}\left(\frac{tf}{f_r}, f_r \ln\frac{f}{f_r}\right), \qquad f>0, \tag{45}$$

where the warped signal $(\mathcal{W}_H X)(f)$ is defined as

$$(\mathcal{W}_H X)(f) \triangleq \sqrt{e^{f/f_r}}\, X\left(f_r\, e^{f/f_r}\right), \qquad -\infty<f<\infty. \tag{46}$$

The geometry of this warping is discussed in [29]. Note that $T_X^{(H)}(t,f)$ is only defined for positive frequencies, which corresponds to the frequency support of the analytic signal $X(f)$. Thus, a hyperbolic QTFR can be obtained by computing the corresponding Cohen's class QTFR $T_Y^{(C)}(t,f)$ for the warped signal, $Y(f) = (\mathcal{W}_H X)(f)$ in (46), and then transforming its time and frequency axes as in (45) for correct TF localization. We note that a generalization of the constant-Q warping of Cohen's class has been proposed in [33–37, 49, 50].

Conversely, any member of Cohen's class can be obtained from the corresponding member of the hyperbolic class using an inverse logarithmic warping

$$T_X^{(C)}(t,f) = T^{(H)}_{\mathcal{W}_H^{-1}X}\Big(t\, e^{-f/f_r},\, f_r\, e^{f/f_r}\Big)$$

where the operator $\mathcal{W}_H^{-1}$ is given by

$$(\mathcal{W}_H^{-1}X)(f) = \sqrt{\frac{f_r}{f}}\; X\Big(f_r \ln\frac{f}{f_r}\Big)\,, \qquad f>0\,.$$

This is the inverse of the operator $\mathcal{W}_H$ in (46), i.e. $(\mathcal{W}_H^{-1}\mathcal{W}_H X)(f) = X(f)$.

3.2 Hyperbolic Class Formulation

Any hyperbolic class QTFR, $T_X^{(H)}(t,f)$, can be formulated as in (44). The hyperbolic class QTFRs can also be formulated in four "normal forms", which are obtained from the normal forms of Cohen's class in (5)-(8) via the constant-Q warping in (45):

$$\begin{aligned}
T_X^{(H)}(t,f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T^{(H)}(tf-c,\,\zeta)\; v_X^{(H)}(c,\zeta)\; e^{-j2\pi\zeta\ln\frac{f}{f_r}}\; dc\, d\zeta && (47)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_T^{(H)}\Big(\ln\frac{f}{f_r}-b,\,\beta\Big)\; V_X^{(H)}(b,\beta)\; e^{j2\pi t f\beta}\; db\, d\beta && (48)\\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} \psi_T^{(H)}\Big(tf-\hat{t}\hat{f},\,\ln\frac{f}{\hat{f}}\Big)\; Q_X(\hat{t},\hat{f})\; d\hat{t}\, d\hat{f} && (49)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_T^{(H)}(\zeta,\beta)\; A_X^{(H)}(\zeta,\beta)\; e^{j2\pi\left(tf\beta-\zeta\ln\frac{f}{f_r}\right)}\; d\zeta\, d\beta\,. && (50)
\end{aligned}$$

Here, the "hyperbolic signal products" are scaled versions of the signal products (9) of the warped signal,

$$\begin{aligned}
v_X^{(H)}(c,\zeta) &\triangleq \frac{1}{f_r}\, u_{\mathcal{W}_H X}\Big(\frac{c}{f_r},\frac{\zeta}{f_r}\Big) \;=\; \rho_X^{(H)}\Big(c+\frac{\zeta}{2}\Big)\; \rho_X^{(H)*}\Big(c-\frac{\zeta}{2}\Big)\,, && (51)\\
V_X^{(H)}(b,\beta) &\triangleq f_r\, U_{\mathcal{W}_H X}(f_r b, f_r\beta) \;=\; f_r e^{b}\; X\Big(f_r\, e^{b+\beta/2}\Big)\, X^*\Big(f_r\, e^{b-\beta/2}\Big)\,, && (52)
\end{aligned}$$

and $\rho_X^{(H)}(c)$ is the hyperbolic coefficient function (see Section 3.3),

$$\rho_X^{(H)}(c) = \int_0^{\infty} X(f)\,\frac{1}{\sqrt{f}}\, e^{j2\pi c\ln\frac{f}{f_r}}\, df\,. \qquad (53)$$

Furthermore, the Altes-Marinovich Q-distribution [18, 19, 29, 31],

$$\begin{aligned} Q_X(t,f) &= W_{\mathcal{W}_H X}\Big(\frac{tf}{f_r},\, f_r \ln\frac{f}{f_r}\Big) \\ &= \int_{-\infty}^{\infty} V_X^{(H)}\Big(\ln\frac{f}{f_r},\beta\Big)\, e^{j2\pi t f\beta} d\beta = \int_{-\infty}^{\infty} v_X^{(H)}(tf,\zeta)\, e^{-j2\pi\zeta \ln\frac{f}{f_r}}\, d\zeta \\ &= f\int_{-\infty}^{\infty} X\Big(fe^{\beta/2}\Big)\; X^*\Big(fe^{-\beta/2}\Big)\; e^{j2\pi t f\beta}\, d\beta\,, \quad f>0\,, \end{aligned} \tag{54}$$

is a warped version of the Wigner distribution in (10), and the "hyperbolic ambiguity function" [19, 29, 42],

$$\begin{aligned} A_X^{(H)}(\zeta,\beta) &= A_{\mathcal{W}_H X}\Big(\frac{\zeta}{f_r}, f_r\beta\Big) = \int_{-\infty}^{\infty} V_X^{(H)}(b,\beta)\; e^{j2\pi\zeta b}\, db = \int_{-\infty}^{\infty} v_X^{(H)}(c,\zeta)\; e^{-j2\pi\beta c}\, dc \\ &= \int_0^{\infty} X\Big(fe^{\beta/2}\Big)\; X^*\Big(fe^{-\beta/2}\Big)\; e^{j2\pi\zeta\ln\frac{f}{f_r}}\; df \end{aligned} \tag{55}$$

is a scaled version of the narrowband ambiguity function in (11) applied to the warped signal. According to the four normal forms (47)-(50), the hyperbolic QTFRs are formulated in terms of the kernels $\phi_T^{(H)}(c,\zeta)$, $\Phi_T^{(H)}(b,\beta)$, $\psi_T^{(H)}(c,b)$, and $\Psi_T^{(H)}(\zeta,\beta)$, each of which uniquely characterizes the hyperbolic QTFR, $T^{(H)}$. The hyperbolic class kernels are simply scaled versions of the respective kernels in Cohen's class,

$$\phi_T^{(H)}(c,\zeta) = \frac{1}{f_r}\,\phi_T^{(C)}\Big(\frac{c}{f_r},\frac{\zeta}{f_r}\Big)\,, \qquad \Phi_T^{(H)}(b,\beta) = f_r\,\Phi_T^{(C)}(f_r b, f_r\beta)\,,$$
$$\psi_T^{(H)}(c,b) = \psi_T^{(C)}\Big(\frac{c}{f_r}, f_r b\Big)\,, \qquad \Psi_T^{(H)}(\zeta,\beta) = \Psi_T^{(C)}\Big(\frac{\zeta}{f_r}, f_r\beta\Big)\,,$$

and are interrelated by Fourier transforms in the same way as the Cohen's class kernels are in (12)-(13),

$$\begin{aligned} \psi_T^{(H)}(c,b) &= \int_{-\infty}^{\infty} \phi_T^{(H)}(c,\zeta)\, e^{-j2\pi b\zeta}\, d\zeta \;= \int_{-\infty}^{\infty} \Phi_T^{(H)}(b,\beta)\, e^{j2\pi c\beta}\, d\beta\,, \\ \Psi_T^{(H)}(\zeta,\beta) &= \int_{-\infty}^{\infty} \phi_T^{(H)}(c,\zeta)\, e^{-j2\pi\beta c}\, dc \;= \int_{-\infty}^{\infty} \Phi_T^{(H)}(b,\beta)\, e^{j2\pi\zeta b}\, db\,. \end{aligned}$$

Note that the kernel $\Phi_T^{(H)}(b,\beta)$ in (48) is related to the kernel $\Gamma_T^{(H)}(b_1,b_2)$ in (44) according to the relationship $\Gamma_T^{(H)}(b_1,b_2) = \frac{1}{\sqrt{|b_1 b_2|}}\,\Phi_T^{(H)}(-\ln\sqrt{b_1 b_2}, \ln\frac{b_1}{b_2})$. Also, $\Gamma_T^{(H)}(b_1,b_2) = \frac{f_r}{\sqrt{|b_1 b_2|}}\,\Gamma_T^{(C)}(f_r\ln b_1, f_r\ln b_2)$ where $\Gamma_T^{(C)}(f_1,f_2)$ is the Cohen's class kernel in (4).

3.3 Hyperbolic Signal Expansion

The hyperbolic class is closely related to a specific signal expansion. The *hyperbolic impulse* [51] is an analytic signal defined as

$$H_c(f) \;\triangleq\; \frac{1}{\sqrt{f}}\, e^{-j2\pi c\ln\frac{f}{f_r}} = \frac{1}{\sqrt{f_r}}\,\Big(\frac{f}{f_r}\Big)^{-j2\pi c-\frac{1}{2}}\,, \qquad f>0\,. \tag{56}$$

The group delay of $H_c(f)$ corresponds to a hyperbola $t = c/f$ in the TF plane where the parameter $c \in \mathbb{R}$ determines the shape of the hyperbola. The hyperbolic impulse is invariant to scalings up to a phase factor, i.e. $(\mathcal{C}_a H_c)(f) = e^{j2\pi c\ln a}\, H_c(f)$. Thus, the hyperbolic impulse is a Doppler-invariant signal, similar to the signals

used by bats and dolphins for echolocation [48]. Furthermore, hyperbolically time-shifting the hyperbolic impulse results in another hyperbolic impulse with shifted parameter c, i.e. $(\mathcal{H}_{c_0} H_c)(f) = H_{c+c_0}(f)$. The hyperbolic impulse satisfies the completeness relation

$$\int_{-\infty}^{\infty} H_c(f_1)\, H_c^*(f_2)\, dc = \delta(f_1 - f_2)\,, \quad f_1, f_2 > 0 \tag{57}$$

and the orthogonality property

$$\langle H_{c_1}, H_{c_2} \rangle = \int_0^{\infty} H_{c_1}(f)\, H_{c_2}^*(f)\, df = \delta(c_1 - c_2)\,.$$

From the completeness relation (57), it follows that any finite-energy, analytic signal $X(f)$ can be represented as a superposition of hyperbolic impulses [29]

$$X(f) = \int_{-\infty}^{\infty} \rho_X^{(H)}(c)\, H_c(f)\, dc = \int_{-\infty}^{\infty} \rho_X^{(H)}(c)\, \frac{1}{\sqrt{f}}\, e^{-j2\pi c \ln \frac{f}{f_r}}\, dc\,, \tag{58}$$

where $\rho_X^{(H)}(c)$, the "hyperbolic coefficient function", is the inner product of the analytic signal $X(f)$ with the hyperbolic impulse $H_c(f)$ (cf. (53)):

$$\rho_X^{(H)}(c) = \langle X, H_c \rangle = \int_0^{\infty} X(f)\, H_c^*(f)\, df = \int_0^{\infty} X(f)\, \frac{1}{\sqrt{f}}\, e^{j2\pi c \ln \frac{f}{f_r}}\, df = \frac{1}{\sqrt{f_r}} \int_0^{\infty} X(f)\, \left(\frac{f}{f_r}\right)^{j2\pi c - \frac{1}{2}}\, df\,. \tag{59}$$

Some important properties of the hyperbolic signal expansion include the following:

- Unitarity (cf. Parseval's theorem) states that the inner product of the hyperbolic coefficients of two signals is the inner product of the corresponding signals,

$$\int_{-\infty}^{\infty} \rho_{X_1}^{(H)}(c)\, \rho_{X_2}^{(H)*}(c)\, dc = \int_0^{\infty} X_1(f)\, X_2^*(f)\, df\,.$$

- A TF scaling of the signal leaves the magnitude of the hyperbolic coefficient function invariant,

$$\rho_{\mathcal{C}_a X}^{(H)}(c) = e^{j2\pi(\ln a)c}\, \rho_X^{(H)}(c).$$

- Hyperbolically time-shifting the signal (with hyperbolic parameter c_0) shifts the hyperbolic coefficient function by c_0, $\rho_{\mathcal{H}_{c_0} X}^{(H)}(c) = \rho_X^{(H)}(c - c_0)$.

- The hyperbolic coefficient function of a hyperbolic impulse $H_{c_0}(f)$ is a Dirac impulse centered at the hyperbolic parameter value $c = c_0$, i.e. $\rho_{H_{c_0}}^{(H)}(c) = \delta(c - c_0)$.

The mapping $X(f) \to \rho_X^{(H)}(c)$ defined in (59) is a version of the *Mellin transform* in the form used in [46, 51–53]. Note also that the inverse Fourier transform of the warped signal $(\mathcal{W}_H X)(f)$ in (46) is proportional to a scaled version of the hyperbolic coefficient function $\rho_X^{(H)}(c)$:

$$\int_0^{\infty} (\mathcal{W}_H X)(f)\, e^{j2\pi t f}\, df = \sqrt{f_r}\, \rho_X^{(H)}(f_r t)\,.$$

3.4 Important Members, Properties, and Kernel Constraints

Any hyperbolic QTFR can be obtained from a corresponding QTFR of Cohen's class using the constant-Q warping in (45). In particular, the following QTFRs can be obtained from well-known Cohen's class QTFRs such as the Wigner distribution, the generalized Wigner distribution, the spectrogram, the pseudo Wigner distribution, and the smoothed pseudo Wigner distribution [2–4]:

- *Altes-Marinovich Q-distribution*, $Q_X(t,f)$ [18, 19, 29] (warped version of the Wigner distribution). The Q-distribution has been defined in (54); it can alternatively be written as

$$Q_X(t,f) = \int_{-\infty}^{\infty} \rho_X^{(H)}\Big(tf + \frac{\zeta}{2}\Big)\, \rho_X^{(H)*}\Big(tf - \frac{\zeta}{2}\Big)\, e^{-j2\pi\zeta \ln\frac{f}{f_r}}\, d\zeta\,.$$

- *Generalized Q-distribution*, $Q_X^{(\alpha)}(t,f)$ (warped version of the generalized Wigner distribution (27)):

$$Q_X^{(\alpha)}(t,f) = f \int_{-\infty}^{\infty} e^{-\alpha\beta}\, X\Big(fe^{(\frac{1}{2}-\alpha)\beta}\Big)\; X^*\Big(fe^{-(\frac{1}{2}+\alpha)\beta}\Big)\; e^{j2\pi tf\beta}\, d\beta\,.$$

  When α=0, this simplifies to the Q-distribution, $Q_X^{(0)}(t,f) = Q_X(t,f)$.

- *Bertrand P_0-distribution*, $P_{0_X}(t,f)$, defined in (30).

- *κth power form of the Bertrand P_0-distribution*, $P_{0_X}^{(\kappa)}(t,f)$, where $\kappa \neq 0$ [30, 33, 34]:

$$P_{0_X}^{(\kappa)}(t,f) = f\int_{-\infty}^{\infty} X\Big(f\Big(\frac{\kappa\beta/2}{\sinh(\kappa\beta/2)}\Big)^{1/\kappa} e^{\frac{\beta}{2}}\Big)\, X^*\Big(f\Big(\frac{\kappa\beta/2}{\sinh(\kappa\beta/2)}\Big)^{1/\kappa} e^{-\frac{\beta}{2}}\Big)\, \Big(\frac{\beta/2}{\sinh(\beta/2)}\Big)^{1/\kappa} e^{j2\pi tf\beta}\, d\beta\,. \quad (60)$$

  It will be shown in Section 4 that this is also a member of the κth power class. Note that for $\kappa = 1$, (60) simplifies to the Bertrand P_0-distribution in (30), i.e. $P_{0_X}^{(1)}(t,f) = P_{0_X}(t,f)$.

- *Hyperbologram*, $Y_X(t,f)$ [29] (warped version of the spectrogram [13–15]):

$$Y_X(t,f) = \frac{f_r}{f}\left|\int_0^{\infty} X(\hat{f})\, \Theta^*\Big(\frac{f_r}{f}\hat{f}\Big)\, e^{j2\pi tf \ln\frac{\hat{f}}{f_r}}\, d\hat{f}\right|^2$$

  where $\Theta(f)$ is the Fourier transform of an analytic analysis wavelet concentrated about $f = f_r$. The hyperbologram is obtained from the spectrogram, the squared magnitude of the short-time Fourier transform, using the constant-Q warping in (45). The hyperbologram is the squared magnitude of the "hyperbolic wavelet transform"

$$\mathrm{HWT}_X(t,f) = \int_0^{\infty} X(\hat{f})\, \Big[(\mathcal{H}_{tf}\, \mathcal{C}_{f/f_r}\Theta)(\hat{f})\Big]^*\, d\hat{f} = \sqrt{\frac{f_r}{f}}\int_0^{\infty} X(\hat{f})\, \Theta^*\Big(f_r\frac{\hat{f}}{f}\Big)\, e^{j2\pi tf \ln\frac{\hat{f}}{f_r}}\, d\hat{f}$$

  which is the inner product of the signal $X(f)$ and the transformed version of $\Theta(f)$, $(\mathcal{H}_c\, \mathcal{C}_a\Theta)(f)$, where $c = tf$ and $a = f/f_r$.

- *Pseudo Q-distribution*, $\mathrm{PAD}_X(t,f)$ [29] (warped version of the pseudo Wigner distribution [7, 8]):

$$\mathrm{PAD}_X(t,f) = f_r \int_0^{\infty} Q_\Theta\Big(0, f_r\frac{f}{\hat{f}}\Big)\, Q_X\Big(\frac{tf}{\hat{f}}, \hat{f}\Big)\, \frac{d\hat{f}}{\hat{f}}$$

  where $\Theta(f)$ is the Fourier transform of an analysis wavelet.

- *Smoothed pseudo Q-distribution*, $\mathrm{SPAD}_X(t,f)$ [29] (warped version of the smoothed pseudo Wigner distribution [4, 5, 8]):

$$\mathrm{SPAD}_X(t,f) = \int_{-\infty}^{\infty} s(tf-c)\, \mathrm{PAD}_X\Big(\frac{c}{f}, f\Big)\; dc \tag{61}$$

where $s(c)$ is a smoothing function.

All hyperbolic QTFRs satisfy the hyperbolic time-shift covariance property (41) and the scale covariance property (42). In addition, they will also satisfy other desirable properties provided that the QTFR kernels satisfy associated constraints. A number of desirable QTFR properties in the hyperbolic class, together with their corresponding kernel constraints, are given in Table 4. For example, the hyperbolic marginal property P_6 in Table 4 states that if we integrate a hyperbolic QTFR over all hyperbolae in the TF plane, we obtain the squared magnitude of the hyperbolic coefficient in (59). The specific hyperbolic QTFRs listed above are summarized in Table 5 together with their kernels and the properties they satisfy. For example, the Q-distribution, $Q_X(t,f)$, satisfies properties P_1-P_{12} in Table 4, since its kernel, $\Phi_Q^{(H)}(b,\beta) = \delta(b)$ (see Table 5), satisfies the corresponding constraints. All properties in Table 4 correspond to well-known properties in Cohen's class via the constant-Q warping. An extensive list of corresponding properties is provided in [29].

3.5 Localized-Kernel Hyperbolic Subclass

An important subclass of the hyperbolic class is the localized-kernel hyperbolic class [30, 33]. This subclass is important as it allows perfect localization along given group delay laws [30]. For a localized-kernel hyperbolic QTFR, the 2-D kernel $\Phi_T^{(H)}(b,\beta)$ in (48) is perfectly localized along a curve in the (b,β)-plane defined by $b = F_T^{(H)}(\beta)$ where $F_T^{(H)}(\beta)$ is a 1-D kernel. The 2-D kernels are parameterized in terms of two 1-D kernels, which results in a simplification of QTFR expressions and kernel constraints. The four kernels used in the normal forms in (47)-(50) simplify to

$$\begin{aligned}
\phi_T^{(H)}(c,\zeta) &= \int_{-\infty}^{\infty} G_T^{(H)}(\beta)\, e^{j2\pi[c\beta+\zeta F_T^{(H)}(\beta)]}\, d\beta\,,\\
\Phi_T^{(H)}(b,\beta) &= G_T^{(H)}(\beta)\, \delta\Big(b - F_T^{(H)}(\beta)\Big)\,,\\
\psi_T^{(H)}(c,b) &= \int_{-\infty}^{\infty} G_T^{(H)}(\beta)\, \delta\Big(b - F_T^{(H)}(\beta)\Big)\, e^{j2\pi c\beta}\, d\beta\,,\\
\Psi_T^{(H)}(\zeta,\beta) &= G_T^{(H)}(\beta)\, e^{j2\pi\zeta F_T^{(H)}(\beta)}\,,
\end{aligned} \tag{62}$$

where the 1-D kernels $F_T^{(H)}(\beta) \in \mathbb{R}$ and $G_T^{(H)}(\beta)$ characterize the hyperbolic QTFR, $T^{(H)}$. Any localized-kernel hyperbolic QTFR can be written as

$$T_X^{(H)}(t,f) = \int_{-\infty}^{\infty} V_X^{(H)}\Big(\ln\frac{f}{f_r} - F_T^{(H)}(\beta),\, \beta\Big)\; G_T^{(H)}(\beta)\, e^{j2\pi t f\beta}\, d\beta \tag{63}$$

$$= f\int_{-\infty}^{\infty} X\Big(f\, e^{-F_T^{(H)}(\beta)+\frac{\beta}{2}}\Big)\; X^*\Big(f\, e^{-F_T^{(H)}(\beta)-\frac{\beta}{2}}\Big)\; e^{-F_T^{(H)}(\beta)}\; G_T^{(H)}(\beta)\, e^{j2\pi t f\beta}\, d\beta\,. \tag{64}$$

Some important QTFRs of the localized-kernel subclass include the Q-distribution, $Q_X(t,f)$, the generalized Q-distribution, $Q_X^{(\alpha)}(t,f)$, the P_0-distribution, $P_{0_X}(t,f)$, and the κth power form of the P_0-distribution, $P_{0_X}^{(\kappa)}(t,f)$, with kernels

$$\begin{array}{ll} F_Q^{(H)}(\beta) = 0\,, & G_Q^{(H)}(\beta) = 1\,; \\ F_{Q^{(\alpha)}}^{(H)}(\beta) = \alpha\beta\,, & G_{Q^{(\alpha)}}^{(H)}(\beta) = 1\,; \\ F_{P_0}^{(H)}(\beta) = \ln\left(\frac{\sinh(\beta/2)}{\beta/2}\right)\,, & G_{P_0}^{(H)}(\beta) = 1\,; \\ F_{P_0^{(\kappa)}}^{(H)}(\beta) = \frac{1}{\kappa}\ln\left(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\right)\,, & G_{P_0^{(\kappa)}}^{(H)}(\beta) = 1\,. \end{array}$$

The simplified kernel constraints corresponding to desirable QTFR properties are given in the third column of Table 4.

4 THE POWER QTFR CLASSES

The *power classes* [33, 34, 38–40] have recently been defined as the QTFR classes containing all QTFRs, $T_X^{(\kappa)}(t,f)$, that satisfy the power time-shift covariance and the scale covariance properties (cf. Table 1)

$$T_{\mathcal{D}_c^{(\kappa)}X}^{(\kappa)}(t,f) \;=\; T_X^{(\kappa)}(t - c\tau_\kappa(f), f) \;=\; T_X^{(\kappa)}\left(t - \frac{c\kappa}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}, f\right) \tag{65}$$

$$T_{\mathcal{C}_a X}^{(\kappa)}(t,f) \;=\; T_X^{(\kappa)}\left(at, \frac{f}{a}\right)\,, \tag{66}$$

where $\kappa \in \mathbb{R}$, $\kappa \neq 0$ is the power parameter associated with each power class, and the power time-shift operator $\mathcal{D}_c^{(\kappa)}$ (cf. Table 1) is defined as [38, 54]

$$(\mathcal{D}_c^{(\kappa)}X)(f) = e^{-j2\pi c\,\xi_\kappa\left(\frac{f}{f_r}\right)}\,X(f) = e^{-j2\pi c\,\mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^\kappa}\,X(f)\,. \tag{67}$$

Here, the κth power phase function is defined as

$$\xi_\kappa(b) \;\triangleq\; \mathrm{sgn}(b)\,|b|^\kappa, \qquad b \in \mathbb{R} \tag{68}$$

where $\mathrm{sgn}(b)$ is -1 for $b < 0$ and 1 for $b > 0$. Note that $\xi_\kappa(b)$ is a power function extended to $b < 0$ such that $\xi_\kappa(b)$ is an odd, strictly monotonic function constituting a one-to-one mapping from $\mathbb{R}$ to $\mathbb{R}$. The group delay function associated with the phase function $\xi_\kappa(b)$ is given as

$$\tau_\kappa(f) = \frac{d}{df}\,\xi_\kappa\!\left(\frac{f}{f_r}\right) = \frac{\kappa}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}, \qquad f \in \mathbb{R}\,. \tag{69}$$

Note that the inverse power phase function (defined by $\xi_\kappa^{-1}(\xi_\kappa(b)) = b$) is given by $\xi_\kappa^{-1}(b) = \xi_{1/\kappa}(b)$.

The κth power class QTFRs are obtained when QTFRs are constrained to satisfy both of the covariance properties in (65) and (66), respectively. As a result, it follows that any κth power class QTFR, $T_X^{(\kappa)}(t,f)$, can be expressed in terms of the signal $X(f)$ as

$$T_X^{(\kappa)}(t,f) \;=\; \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_T^{(\kappa)}\!\left(\frac{f_1}{f}, \frac{f_2}{f}\right) e^{j2\pi\frac{t}{\tau_\kappa(f)}\left[\xi_\kappa\left(\frac{f_1}{f_r}\right) - \xi_\kappa\left(\frac{f_2}{f_r}\right)\right]}\,X(f_1)\,X^*(f_2)\,df_1df_2 \tag{70}$$

where the 2-D kernel $\Gamma_T^{(\kappa)}(b_1, b_2)$ uniquely characterizes the power class QTFR, $T^{(\kappa)}$.

The family of power classes generalizes the affine class in Section 2. In particular, the affine class is a special case of the power classes obtained when the power parameter $\kappa = 1$. Thus, the affine class is characterized by the phase function $\xi_1(b) = b$ in (68), and the group delay function $\tau_1(f) = \frac{d}{df}\,\xi_1\Big(\frac{f}{f_r}\Big) = \frac{1}{f_r}$ in (69). Indeed, when $\kappa = 1$ the power time-shift operator $\mathcal{D}_c^{(\kappa)}$ in (67) reduces to the nondispersive time-shift operator, $\mathcal{D}_{f_r\tau}^{(1)} = \mathcal{S}_\tau$, and the power time-shift covariance property in (65) simplifies to the constant time-shift covariance property in (14). Furthermore, when $\kappa = 1$, the κth power class QTFR formulation in (70) simplifies to the affine class formulation in (17) and thus $\Gamma_T^{(A)}(b_1, b_2) = \Gamma_T^{(1)}(b_1, b_2)$. Accordingly, an affine class QTFR can be equally denoted as $T_X^{(A)}(t, f)$ as in Section 2, or as $T_X^{(1)}(t, f)$.

The power time-shift covariance property in (65) makes the power classes QTFRs useful in the analysis of signals whose TF localization is related to a power-law TF geometry. In addition, power QTFRs can be used in multiresolution analysis applications since they are scale covariant.

4.1 Power Warping

The power classes have been defined based on the power time-shift covariance and the scale covariance properties they satisfy. Equivalently, they can also be defined by applying a unitary mapping to the affine class [38], similar to the way in which the hyperbolic class was obtained from Cohen's class in Section 3.1. This unitary mapping (called "power warping" in what follows) is defined as[3]

$$T_X^{(\kappa)}(t, f) \;=\; T_{\mathcal{W}_\kappa X}^{(1)}\Big(\frac{t}{f_r \tau_\kappa(f)},\, f_r\, \xi_\kappa\Big(\frac{f}{f_r}\Big)\Big) \;=\; T_{\mathcal{W}_\kappa X}^{(A)}\left(\frac{t}{\kappa\, \big|\frac{f}{f_r}\big|^{\kappa-1}},\, f_r\, \mathrm{sgn}(f) \left|\frac{f}{f_r}\right|^{\kappa}\right), \tag{71}$$

with the power warping operator $\mathcal{W}_\kappa$ defined as

$$(\mathcal{W}_\kappa X)(f) \;=\; \frac{1}{\sqrt{f_r \left|\tau_\kappa\Big(f_r\, \xi_\kappa^{-1}\Big(\frac{f}{f_r}\Big)\Big)\right|}}\; X\Big(f_r\, \xi_\kappa^{-1}\Big(\frac{f}{f_r}\Big)\Big) \;=\; \frac{1}{\sqrt{|\kappa|}\, \left|\frac{f}{f_r}\right|^{\frac{\kappa-1}{2\kappa}}}\; X\left(f_r\, \mathrm{sgn}(f) \left|\frac{f}{f_r}\right|^{1/\kappa}\right). \tag{72}$$

As a result, any member of the κth power class, $T_X^{(\kappa)}(t, f)$, can be obtained from the corresponding member of the affine class, $T_X^{(1)}(t, f) = T_X^{(A)}(t, f)$, since the power warping in (71) establishes a one-to-one mapping between the two classes. According to the power warping, a power class QTFR is obtained by first computing the corresponding affine class QTFR of the frequency-warped signal in (72), and then performing an area-preserving[4], nonlinear TF coordinate transform $(t, f) \to \Big(\frac{t}{f_r \tau_\kappa(f)},\, f_r\, \xi_\kappa\Big(\frac{f}{f_r}\Big)\Big)$. For $\kappa = 1$, the warping operator simplifies to the identity operator, $(\mathcal{W}_1 X)(f) = (\mathcal{I} X)(f) = X(f)$, and the coordinate transform simplifies to the identity transform $(t, f) \to (t, f)$.

Conversely, any member of the affine class can be obtained from the corresponding member of the κth power class using an inverse power warping

$$T_X^{(A)}(t, f) \;=\; T_X^{(1)}(t, f) \;=\; T_{\mathcal{W}_\kappa^{-1} X}^{(\kappa)}\Big(\frac{t}{f_r \tau_{1/\kappa}(f)},\, f_r\, \xi_{1/\kappa}\Big(\frac{f}{f_r}\Big)\Big) \;=\; T_{\mathcal{W}_\kappa^{-1} X}^{(\kappa)}\left(\kappa t \left|\frac{f}{f_r}\right|^{\frac{\kappa-1}{\kappa}}, f_r\, \mathrm{sgn}(f) \left|\frac{f}{f_r}\right|^{\frac{1}{\kappa}}\right) \tag{73}$$

where $\mathcal{W}_\kappa^{-1}$, the inverse signal warping operator defined by $\mathcal{W}_\kappa^{-1}\mathcal{W}_\kappa = \mathcal{I}$, is given by $\mathcal{W}_\kappa^{-1} = \mathcal{W}_{1/\kappa}$. As a result, if an algorithm exists that implements the power warping in (71)-(72), then the same algorithm can be used to implement the inverse power warping in (73) simply by replacing κ with $1/\kappa$ in the functions $(\mathcal{W}_\kappa X)(f)$, $\xi_\kappa(b)$, and $\tau_\kappa(f)$ in (71)-(72).

Composing the power warping in (71) and the inverse power warping in (73), it is also possible to show that any QTFR member of a power class with power parameter m, $T_X^{(m)}(t,f)$, can be obtained from the corresponding QTFR member of a power class with power parameter κ, $T_X^{(\kappa)}(t,f)$, using a similar unitary warping. In particular,

$$T_X^{(m)}(t,f) \;=\; T^{(\kappa)}_{\mathcal{W}_{m/\kappa}X}\left(\frac{t}{f_r\tau_{m/\kappa}(f)},\; f_r\,\xi_{m/\kappa}\Big(\frac{f}{f_r}\Big)\right) \;=\; T^{(\kappa)}_{\mathcal{W}_{m/\kappa}X}\left(\frac{\kappa t}{m}\Big|\frac{f}{f_r}\Big|^{\frac{\kappa-m}{\kappa}}, f_r\,\mathrm{sgn}(f)\Big|\frac{f}{f_r}\Big|^{\frac{m}{\kappa}}\right).$$

Thus, there is a one-to-one correspondence between any two power classes.

4.2 Power Classes Formulation

Any QTFR of the κth power class, $T_X^{(\kappa)}(t,f)$, is expressed as in (70). The power class QTFRs can also be expressed in the following four normal forms, which are obtained from the normal forms of the affine class QTFRs, (18)-(21), through the power warping in (71):[5]

$$\begin{aligned}
T_X^{(\kappa)}(t,f) &= \Big|\xi_\kappa\Big(\frac{f}{f_r}\Big)\Big| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_T^{(A)}\Big(\xi_\kappa\Big(\frac{f}{f_r}\Big)\Big[\frac{t}{\tau_\kappa(f)} - c\Big], \xi_\kappa\Big(\frac{f}{f_r}\Big)\zeta\Big)\, v_X^{(\kappa)}(c,\zeta)\, dc\, d\zeta && (74)\\
&= \frac{1}{\Big|\xi_\kappa\Big(\frac{f}{f_r}\Big)\Big|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_T^{(A)}\left(\frac{-b}{\xi_\kappa\Big(\frac{f}{f_r}\Big)}, \frac{\beta}{\xi_\kappa\Big(\frac{f}{f_r}\Big)}\right) V_X^{(\kappa)}(b,\beta)\, e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\, db\, d\beta && (75)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T^{(A)}\left(\xi_\kappa\Big(\frac{f}{f_r}\Big)\Big[\frac{t}{\tau_\kappa(f)} - \frac{\hat{t}}{\tau_\kappa(\hat{f})}\Big], -\xi_\kappa\Big(\frac{\hat{f}}{f}\Big)\right) W_X^{(\kappa)}(\hat{t},\hat{f})\, d\hat{t}\, d\hat{f} && (76)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_T^{(A)}\left(\xi_\kappa\Big(\frac{f}{f_r}\Big)\zeta, \frac{\beta}{\xi_\kappa\Big(\frac{f}{f_r}\Big)}\right) A_X^{(\kappa)}(\zeta,\beta)\, e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\, d\zeta\, d\beta. && (77)
\end{aligned}$$

Here, the κth "power signal products" are scaled versions of the signal products in (9) using the warped signal $(\mathcal{W}_\kappa X)(f)$ in (72),

$$\begin{aligned}
v_X^{(\kappa)}(c,\zeta) &= \frac{1}{f_r}\, u_{\mathcal{W}_\kappa X}\Big(\frac{c}{f_r}, \frac{\zeta}{f_r}\Big) \;=\; \rho_X^{(\kappa)}\Big(c + \frac{\zeta}{2}\Big)\, \rho_X^{(\kappa)*}\Big(c - \frac{\zeta}{2}\Big) && (78)\\
V_X^{(\kappa)}(b,\beta) &= f_r\, U_{\mathcal{W}_\kappa X}(f_r b, f_r\beta) && (79)\\
&= \frac{f_r}{|\kappa|\,\Big|b^2 - \frac{\beta^2}{4}\Big|^{\frac{\kappa-1}{2\kappa}}}\, X\Big(f_r\,\xi_\kappa^{-1}\Big(b + \frac{\beta}{2}\Big)\Big)\; X^*\Big(f_r\,\xi_\kappa^{-1}\Big(b - \frac{\beta}{2}\Big)\Big)\;,
\end{aligned}$$

where $\rho_X^{(\kappa)}(c)$ is the κth power coefficient function (defined in (84) in Section 4.3). Furthermore, $W_X^{(\kappa)}(t,f)$, the κth power form of the Wigner distribution [38], is the power warped version of the Wigner distribution in (10),

$$W_X^{(\kappa)}(t,f) \;=\; W_{\mathcal{W}_\kappa X}\left(\frac{t}{f_r\tau_\kappa(f)},\; f_r\,\xi_\kappa\Big(\frac{f}{f_r}\Big)\right)$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} V_X^{(\kappa)}\Big(\xi_\kappa\Big(\frac{f}{f_r}\Big),\beta\Big)\, e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\, d\beta = \int_{-\infty}^{\infty} v_X^{(\kappa)}\Big(\frac{t}{\tau_\kappa(f)},\zeta\Big)\, e^{-j2\pi\zeta\xi_\kappa(\frac{f}{f_r})}\, d\zeta \qquad (80)\\
&= \Big|\frac{f}{\kappa}\Big| \int_{-\infty}^{\infty} X\Big(f\,\xi_\kappa^{-1}\Big(1+\frac{\beta}{2}\Big)\Big)\, X^*\Big(f\,\xi_\kappa^{-1}\Big(1-\frac{\beta}{2}\Big)\Big)\, e^{j2\pi\frac{tf}{\kappa}\beta}\, \frac{1}{\big|1-\frac{\beta^2}{4}\big|^{\frac{\kappa-1}{2\kappa}}}\, d\beta,
\end{aligned}
$$

and $A_X^{(\kappa)}(\zeta,\beta)$, the κth "power ambiguity function", is a scaled narrowband ambiguity function (see (11)) of the warped signal,

$$
\begin{aligned}
A_X^{(\kappa)}(\zeta,\beta) &= A_{\mathcal{W}_\kappa X}\Big(\frac{\zeta}{f_r}, f_r\beta\Big) = \int_{-\infty}^{\infty} V_X^{(\kappa)}(b,\beta)\, e^{j2\pi\zeta b}\, db = \int_{-\infty}^{\infty} v_X^{(\kappa)}(c,\zeta)\, e^{-j2\pi\beta c}\, dc \qquad (81)\\
&= \frac{f_r}{|\kappa|}\int_{-\infty}^{\infty} X\Big(f_r\,\xi_\kappa^{-1}\Big(b+\frac{\beta}{2}\Big)\Big)\; X^*\Big(f_r\,\xi_\kappa^{-1}\Big(b-\frac{\beta}{2}\Big)\Big)\, e^{j2\pi\zeta b}\, \frac{1}{\big|b^2-\frac{\beta^2}{4}\big|^{\frac{\kappa-1}{2\kappa}}}\, db\,.
\end{aligned}
$$

Note that the 2-D kernel functions $\phi_T^{(A)}(c,\zeta)$, $\Phi_T^{(A)}(b,\beta)$, $\psi_T^{(A)}(c,b)$, and $\Psi_T^{(A)}(\zeta,\beta)$ that uniquely characterize each member of the power classes are interrelated by Fourier transforms as in (22)-(23) as they are the *same* kernel functions as the ones in the affine class normal forms in (18)-(21) that characterize their affine class counterparts. For example, the kernel of the κth power Wigner distribution is $\Phi_{W^{(\kappa)}}^{(A)}(b,\beta) = \delta(b+1)$, and the affine class kernel for the Wigner distribution (the affine class counterpart QTFR of the κth power Wigner distribution) is $\Phi_W^{(A)}(b,\beta) = \delta(b+1)$. As a result, the functional form of the power classes kernels is independent of the power parameter κ; only the arguments of the kernels in (74)-(77) depend upon κ. Consequently, corresponding QTFRs of two different power classes are characterized by the same kernels. Note, however, that the kernel $\Gamma_T^{(\kappa)}(b_1,b_2)$ in (70) depends on κ; it is related to the kernel $\Phi_T^{(A)}(b,\beta)$ in (75) using the relationship $\Gamma_T^{(\kappa)}(b_1,b_2) = \Big|\xi_\kappa'\Big(\sqrt{|b_1 b_2|}\Big)\Big|\; \Phi_T^{(A)}\Big(-\frac{\xi_\kappa(b_1)+\xi_\kappa(b_2)}{2}, \xi_\kappa(b_1)-\xi_\kappa(b_2)\Big)$ where $\xi_\kappa'(b) = \frac{d}{db}\xi_\kappa(b)$. The kernel $\Gamma_T^{(\kappa)}(b_1,b_2)$ in (70) of the power class QTFR $T_X^{(\kappa)}(t,f)$ is also related to the kernel $\Gamma_T^{(A)}(b_1,b_2)$ in (17) of the affine class QTFR $T_X^{(A)}(t,f)$ as $\Gamma_T^{(\kappa)}(b_1,b_2) = \Big|\xi_\kappa'\Big(\sqrt{|b_1 b_2|}\Big)\Big|\; \Gamma_T^{(A)}(\xi_\kappa(b_1),\xi_\kappa(b_2))$ provided that $T_X^{(A)}(t,f)$ and $T_X^{(\kappa)}(t,f)$ are QTFR counterparts (i.e. provided they are related as in (71)).

4.3 Power Signal Expansion

An important signal that is matched to the power class TF geometry is the power impulse, $I_c^{(\kappa)}(f)$, defined as

$$
I_c^{(\kappa)}(f) \triangleq \sqrt{|\tau_\kappa(f)|}\, e^{-j2\pi c\,\xi_\kappa(\frac{f}{f_r})} = \sqrt{\frac{|\kappa|}{f_r}\Big|\frac{f}{f_r}\Big|^{\kappa-1}}\, e^{-j2\pi c\,\mathrm{sgn}(f)\big|\frac{f}{f_r}\big|^{\kappa}}\,. \qquad (82)
$$

It has spectral energy density $|I_c^{(\kappa)}(f)|^2 = |\tau_\kappa(f)|$ and group delay $c\tau_\kappa(f)$ where $\tau_\kappa(f)$ is given in (69). The parameter $c \in \mathbb{R}$ influences the shape of the group delay. Note that when $\kappa = 1$ (affine class), the corresponding power impulse, $I_c^{(1)}(f) = I_c^{(A)}(f)$, is the signal in (24). Scaling the power impulse results in scaling the parameter c, $(\mathcal{C}_a I_c^{(\kappa)})(f) = \frac{1}{\sqrt{|\xi_\kappa(a)|}}\, I_{c/[\xi_\kappa(a)]}^{(\kappa)}(f)$, whereas power time-shifting the power impulse results in another power impulse with a shifted parameter, $(\mathcal{D}_{c_0}^{(\kappa)} I_c^{(\kappa)})(f) = I_{c+c_0}^{(\kappa)}(f)$. The power impulse satisfies the completeness relation

$$
\int_{-\infty}^{\infty} I_c^{(\kappa)}(f_1)\, I_c^{(\kappa)*}(f_2)\, dc = \delta(f_1 - f_2)
$$

and the orthogonality property

$$\left\langle I_{c_1}^{(\kappa)}, I_{c_2}^{(\kappa)} \right\rangle = \int_{-\infty}^{\infty} I_{c_1}^{(\kappa)}(f)\, I_{c_2}^{(\kappa)*}(f)\, df = \delta(c_1 - c_2)\,.$$

Any finite-energy signal, $X(f)$, can be expanded into power impulses $I_c^{(\kappa)}(f)$ as

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} \rho_X^{(\kappa)}(c)\, I_c^{(\kappa)}(f)\, dc = \sqrt{|\tau_\kappa(f)|} \int_{-\infty}^{\infty} \rho_X^{(\kappa)}(c)\, e^{-j2\pi c\, \xi_\kappa\left(\frac{f}{f_r}\right)}\, dc \\ &= \sqrt{\frac{|\kappa|}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}} \int_{-\infty}^{\infty} \rho_X^{(\kappa)}(c)\, e^{-j2\pi c\, \mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^{\kappa}}\, dc, \end{aligned} \tag{83}$$

where the power coefficient function, $\rho_X^{(\kappa)}(c)$, is the inner product of the signal $X(f)$ with the power impulse $I_c^{(\kappa)}(f)$,

$$\begin{aligned} \rho_X^{(\kappa)}(c) &= \left\langle X, I_c^{(\kappa)} \right\rangle = \int_{-\infty}^{\infty} X(f)\, I_c^{(\kappa)*}(f)\, df \\ &= \int_{-\infty}^{\infty} X(f) \sqrt{|\tau_\kappa(f)|}\, e^{j2\pi c\, \xi_\kappa\left(\frac{f}{f_r}\right)}\, df = \int_{-\infty}^{\infty} X(f) \sqrt{\frac{|\kappa|}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}}\, e^{j2\pi c\, \mathrm{sgn}(f)\left|\frac{f}{f_r}\right|^{\kappa}}\, df\,. \end{aligned} \tag{84}$$

The basic properties of the power signal expansion are similar to the ones for the affine signal expansion in Section 2.2 and the hyperbolic signal expansion in Section 3.3:

- Unitarity (cf. Parseval's theorem) states that the inner product of the power coefficient functions of two signals is equal to the inner product of the signals,

$$\int_{-\infty}^{\infty} \rho_{X_1}^{(\kappa)}(c)\, \rho_{X_2}^{(\kappa)*}(c)\, dc = \int_{-\infty}^{\infty} X_1(f)\, X_2^*(f)\, df\,.$$

- Scaling the signal scales the power coefficient function,

$$\rho_{\mathcal{C}_a X}^{(\kappa)}(c) = \sqrt{|\xi_\kappa(a)|}\, \rho_X^{(\kappa)}(\xi_\kappa(a)\, c) = \sqrt{|a|^\kappa}\, \rho_X^{(\kappa)}(\mathrm{sgn}(a)\, |a|^\kappa c)\,.$$

- Power time-shifting the signal (with parameter c_0) as defined in (67) shifts the power coefficient function by c_0, $\rho_{\mathcal{D}_{c_0}^{(\kappa)} X}^{(\kappa)}(c) = \rho_X^{(\kappa)}(c - c_0)$ with $\mathcal{D}_c^{(\kappa)}$ defined in (67).

- The power coefficient function of a power impulse, $I_{c_0}^{(\kappa)}(f)$, is a Dirac impulse centered at $c = c_0$, $\rho_{I_{c_0}^{(\kappa)}}^{(\kappa)}(c) = \delta(c - c_0)$.

The power signal expansion in (83)-(84) constitutes a unitary, linear signal transformation $X(f) \longleftrightarrow \rho_X^{(\kappa)}(c)$ that simplifies to the Fourier transform in (25) when $\kappa = 1$ (corresponding to the affine class). Thus, the power signal expansion generalizes the Fourier transform. Note also that the power signal expansion is closely related to the power frequency warping, $\mathcal{W}_\kappa$, in (72). In particular, the inverse Fourier transform of $(\mathcal{W}_\kappa X)(f)$ is a scaled version of the power coefficient function $\rho_X^{(\kappa)}(c)$:

$$\int_{-\infty}^{\infty} (\mathcal{W}_\kappa X)(f)\, e^{j2\pi t f}\, df = \sqrt{f_r}\, \rho_X^{(\kappa)}(f_r t)\,.$$

4.4 Important Members, Properties, and Kernel Constraints

Any member of the power classes can be obtained from its affine class counterpart using the power warping in (71). In particular, the following QTFRs can be obtained from well-known members of the affine class such as the Wigner distribution, the generalized Wigner distribution, the scalogram, the affine smoothed pseudo Wigner distribution, the Bertrand P_0-distribution, the Flandrin D-distribution, and the passive and active Unterberger distributions (see Section 2.3):

- The *Power Wigner distribution*, $W_X^{(\kappa)}(t,f)$ (the κth power warped version of the Wigner distribution), has been defined in (80). It can also be written in terms of $\rho_X^{(\kappa)}$ as

$$W_X^{(\kappa)}(t,f) \;=\; \int_{-\infty}^{\infty} \rho_X^{(\kappa)}\Big(\frac{t}{\tau_\kappa(f)}+\frac{\zeta}{2}\Big)\,\rho_X^{(\kappa)*}\Big(\frac{t}{\tau_\kappa(f)}-\frac{\zeta}{2}\Big)\,e^{-j2\pi\xi_\kappa(\frac{f}{f_r})\zeta}\,d\zeta\,.$$

- *Generalized power Wigner distribution*, $W_X^{(\kappa)(\alpha)}(t,f)$ (the κth power warped version of the generalized Wigner distribution):

$$W_X^{(\kappa)(\alpha)}(t,f) = \int_{-\infty}^{\infty} V_X^{(\kappa)}\Big(\xi_\kappa\Big(\frac{f}{f_r}\Big)-\alpha\beta,\beta\Big)\;e^{j2\pi\frac{t}{\tau_\kappa(f)}\beta}\,d\beta\,.$$

  Note that when $\alpha=0$, this simplifies to the power Wigner distribution, i.e. $W_X^{(\kappa)(0)}(t,f)=W_X^{(\kappa)}(t,f)$.

- *Powergram*, $Y_X^{(\kappa)}(t,f)$ (the κth power warped version of the scalogram):

$$Y_X^{(\kappa)}(t,f) = \frac{f_r}{|f|}\left|\int_{-\infty}^{\infty} X(\hat{f})\,\tilde{\Theta}^*\Big(f_r\frac{\hat{f}}{f}\Big)\,e^{j2\pi\frac{t}{\tau_\kappa(f)}\xi_\kappa\left(\frac{\hat{f}}{f_r}\right)}d\hat{f}\right|^2, \tag{85}$$

  where $\tilde{\Theta}(f)$[6] is the Fourier transform of an analysis wavelet concentrated about $f=f_r$. The powergram is the squared magnitude of the "power wavelet transform" defined as the inner product of the signal $X(f)$ and the transformed version of $\tilde{\Theta}(f)$, $(\mathcal{D}_c^{(\kappa)}\mathcal{C}_a\tilde{\Theta})(f)$, where $c=t/\tau_\kappa(f)$ and $a=f/f_r$:

$$\begin{aligned}\mathrm{PWT}_X^{(\kappa)}(t,f) \;&=\; \int_{-\infty}^{\infty} X(\hat{f})\,\Big[\Big(\mathcal{D}^{(\kappa)}_{t/\tau_\kappa(f)}\,\mathcal{C}_{f/f_r}\tilde{\Theta}\Big)(\hat{f})\Big]^*\,d\hat{f}\\ &=\; \sqrt{\frac{f_r}{|f|}}\int_{-\infty}^{\infty} X(\hat{f})\,\tilde{\Theta}^*\Big(f_r\frac{\hat{f}}{f}\Big)\,e^{j2\pi\frac{t}{\tau_\kappa(f)}\xi_\kappa\left(\frac{\hat{f}}{f_r}\right)}d\hat{f}\,.\end{aligned}$$

- *Smoothed pseudo power Wigner distribution*, $\mathrm{ASPW}_X^{(\kappa)}$ (the κth power warped version of the affine smoothed pseudo Wigner distribution):

$$\mathrm{ASPW}_X^{(\kappa)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s\left(\xi_\kappa\Big(\frac{f}{f_r}\Big)\Big[\frac{t}{\tau_\kappa(f)}-\frac{\hat{t}}{\tau_\kappa(\hat{f})}\Big]\right)\,H\left(-\xi_\kappa\left(\frac{\hat{f}}{f}\right)\right)\,W_X^{(\kappa)}(\hat{t},\hat{f})\,d\hat{t}\,d\hat{f} \tag{86}$$

  where $s(c)$ and $H(b)$ are two independent windows controlling the smoothing characteristics.

- *Power Bertrand P_0-distribution*, $P_{0X}^{(\kappa)}(t,f)$ (the κth power warped version of the Bertrand P_0-distribution):

$$P_{0X}^{(\kappa)}(t,f) = \left|\xi_\kappa\Big(\frac{f}{f_r}\Big)\right|\int_{-\infty}^{\infty} V_X^{(\kappa)}\Big(-\xi_\kappa\Big(\frac{f}{f_r}\Big)\,F^{(A)}_{P_0^{(\kappa)}}(\beta),\,\xi_\kappa\Big(\frac{f}{f_r}\Big)\beta\Big)\;G^{(A)}_{P_0^{(\kappa)}}(\beta)\;e^{j2\pi\frac{tf}{\kappa}\beta}\,d\beta \tag{87}$$

  where $F^{(A)}_{P_0^{(\kappa)}}(\beta)=-\frac{\beta}{2}\coth(\frac{\beta}{2})$ and $G^{(A)}_{P_0^{(\kappa)}}(\beta)=\frac{\beta/2}{\sinh(\beta/2)}$. An equivalent expression is given in (60).

- *Power Flandrin D-distribution*, $D_X^{(\kappa)}(t,f)$, defined as in (87) with $F_{D^{(\kappa)}}^{(A)}(\beta) = -1 - (\frac{\beta}{4})^2$ and $G_{D^{(\kappa)}}^{(A)}(\beta) = 1 - (\frac{\beta}{4})^2$ (the κth power warped version of the Flandrin D-distribution).

- *Power passive Unterberger distribution*, $\mathrm{PUD}_X^{(\kappa)}(t,f)$, defined as in (87) with $F_{\mathrm{PUD}^{(\kappa)}}^{(A)}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}$ and $G_{\mathrm{PUD}^{(\kappa)}}^{(A)}(\beta) = \frac{1}{\sqrt{1+(\frac{\beta}{2})^2}}$ (the κth power warped version of the passive Unterberger distribution).

- *Power active Unterberger distribution*, $\mathrm{AUD}_X^{(\kappa)}(t,f)$, defined as in (87) with $F_{\mathrm{AUD}^{(\kappa)}}^{(A)}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}$ and $G_{\mathrm{AUD}^{(\kappa)}}^{(A)}(\beta) = 1$ (the κth power warped version of the active Unterberger distribution).

Note that when $\kappa = 1$, all these QTFRs reduce to the corresponding affine QTFRs in Section 2.3.

- Another important family of power class QTFRs are the *Bertrand P_κ-distributions*, $P_{\kappa_X}(t,f)$, defined in (33) [20, 21] with power parameter $\kappa \neq 0$. Their kernel in the κth power class in (70) is given by $\Gamma_{P_\kappa}^{(\kappa)}(b_1,b_2) = \int_{-\infty}^{\infty} \delta(b_1 - \lambda_\kappa(u))\delta(b_2 - \lambda_\kappa(-u))\, \mu(u)du$. The Bertrand P_κ-distributions satisfy the constant time-shift covariance property in addition to the power time-shift and scale covariance properties inherent to the power classes [21]. As a result, they are also members of the affine class, and thus they belong to the intersection subclass of the affine class and the power classes that is considered in detail in Section 5.2.

  As the Bertrand P_κ-distributions are also members of the affine class (see Section 2.3), they can be warped using (71) to obtain a new family of power class QTFRs. The resulting mth power warped version[7] of the Bertrand P_κ-distributions, $P_{\kappa_X}^{(m)}(t,f)$, is defined as

$$P_{\kappa_X}^{(m)}(t,f) = \frac{f}{|m|}\int_{-\infty}^{\infty} X(f[\lambda_\kappa(u)]^{\frac{1}{m}})X^*(f[\lambda_\kappa(-u)]^{\frac{1}{m}})e^{j2\pi\frac{tf}{m}[\lambda_\kappa(u)-\lambda_\kappa(-u)]}[\lambda_\kappa(u)\lambda_\kappa(-u)]^{\frac{1-m}{2m}}\mu(u)du,\ f>0, \quad (88)$$

  where $\lambda_\kappa(u)$ is defined in (33). The kernel in (70) of the mth power class QTFRs in (88) is given by $\Gamma_{P_\kappa^{(m)}}^{(m)}(b_1,b_2) = |\xi_m'(\sqrt{|b_1 b_2|})| \int_{-\infty}^{\infty} \delta(\xi_m(b_1) - \lambda_\kappa(u))\delta(\xi_m(b_2) - \lambda_\kappa(-u))\, \mu(u)du$. Note that the mth power warped versions of the P_κ-distributions in (88) are covariant not only to scale changes and mth power time shifts according to $(\mathcal{D}_c^{(m)}X)(f) = e^{-j2\pi c\xi_m(\frac{f}{f_r})}X(f)$, but they are also covariant to pth power time shifts (where $p = m\kappa$) according to $(\mathcal{D}_c^{(m\kappa)}X)(f) = e^{-j2\pi c\xi_{m\kappa}(\frac{f}{f_r})}X(f)$, on the analysis signal. In particular, if the power used in the warping is given by $m = 1/\kappa$, then the resulting QTFRs in (88) are covariant to constant time shifts according to $(\mathcal{D}_c^{(\frac{\kappa}{\kappa})}X)(f) = (\mathcal{D}_c^{(1)}X)(f) = (\mathcal{S}_{c/f_r}X)(f) = e^{-j2\pi c\frac{f}{f_r}}X(f)$ (cf. (14)).

In Table 6, we list some desirable properties that one might want the power QTFRs to satisfy along with their corresponding kernel constraints. The kernels of the power QTFRs mentioned above and the properties that they satisfy are given in Table 7. A power class QTFR satisfies a property from Table 6 if its kernel satisfies the associated constraint. For example, the power Wigner distribution, $W_X^{(\kappa)}(t,f)$, satisfies properties P_1-P_{12} in Table 6 since its kernel in Table 7, $\Phi_{W^{(\kappa)}}^{(A)}(b,\beta) = \delta(b+1)$, satisfies the associated constraints. Note that if a QTFR in the κth power class, $T_X^{(\kappa)}(t,f)$, satisfies a property $\mathrm{P_i}$ with the parameter κ in Table 6, then the corresponding QTFR of the mth power class, $T_X^{(m)}(t,f)$, will satisfy the corresponding property $\mathrm{P_i}$ with the

parameter m since the kernel constraints in Table 6 are independent of the power. This fact is especially useful when constructing QTFRs of the κth power class by applying the power warping mapping to well-known QTFRs of the affine class (where $\kappa = 1$) since all affine class properties map into corresponding properties of the κth power class QTFR. Note also that the kernel constraints for the κth power class equal the corresponding kernel constraints for the affine class in Table 2. For example, the Wigner distribution, $W_X(t,f)$, which is a member of the affine class, satisfies the temporal marginal property, $\int_{-\infty}^{\infty} W_X(t,f)\,df = |x(t)|^2$ (property P_6 in Table 2). In Table 6, this property is recognized as the "power marginal property" (P_6) when $\kappa = 1$. Said differently, the power marginal property in the κth power class in Table 6 corresponds to the time marginal property in the affine class in Table 2. Thus, since $W_X^{(\kappa)}(t,f)$ in (80) is the κth power class QTFR corresponding to $W_X(t,f)$, it follows that $W_X^{(\kappa)}(t,f)$ must satisfy the power marginal property P_6 in Table 6. Similarly, $W_X^{(\kappa)}(t,f)$ will satisfy Moyal's formula P_9 in Table 6 for all values of $\kappa \neq 0$ since this property is the same for all power classes and it is satisfied by the Wigner distribution $W_X(t,f) = W_X^{(1)}(t,f)$.

4.5 Localized-Kernel Power Subclasses

An important subclass of the power classes that allows perfect localization along given group delay laws [40] is the "localized kernel" power subclass. For the localized-kernel power QTFRs, the 2-D kernels $\phi_T^{(A)}(c,\zeta)$, $\Phi_T^{(A)}(b,\beta)$, $\psi_T^{(A)}(c,b)$, and $\Psi_T^{(A)}(\zeta,\beta)$ occurring in the normal forms (74)–(77) are parameterized by two 1-D kernels $F_T^{(A)}(\beta) \in \mathbb{R}$ and $G_T^{(A)}(\beta)$ as defined in (35)–(38) [40, 55]. In particular, the kernel $\Phi_T^{(A)}(b,\beta)$ is perfectly localized along a curve in the (b,β)-plane defined by $b = F_T^{(A)}(\beta)$,

$$\Phi_T^{(A)}(b,\beta) = G_T^{(A)}(\beta)\;\delta\Big(b - F_T^{(A)}(\beta)\Big). \tag{89}$$

The 1-D kernels $F_T^{(A)}(\beta) \in \mathbb{R}$ and $G_T^{(A)}(\beta)$ are arbitrary functions that characterize the κth power QTFR, $T^{(\kappa)}$. Just as in the localized-kernel affine class in Section 2.4 and in the localized-kernel hyperbolic class in Section 3.5, the kernel structure in (89) is necessary (under certain assumptions) for localization along group delay laws [40, 55]. Also, it results in simpler formulations for the localized-kernel power QTFRs, and it simplifies the property constraints for the power classes.

In the κth localized-kernel power subclass, the general form in (75) simplifies to

$$T_X^{(\kappa)}(t,f) = \left|\xi_\kappa\Big(\frac{f}{f_r}\Big)\right| \int_{-\infty}^{\infty} V_X^{(\kappa)}\Big(-\xi_\kappa\Big(\frac{f}{f_r}\Big)F_T^{(A)}(\beta),\, \xi_\kappa\Big(\frac{f}{f_r}\Big)\beta\Big)\; G_T^{(A)}(\beta)\; e^{j2\pi\frac{tf}{\kappa}\beta}\,d\beta \tag{90}$$

$$= \left|\frac{f}{\kappa}\right| \int_{-\infty}^{\infty} X\Big(f\,\xi_\kappa^{-1}\Big(-F_T^{(A)}(\beta) + \frac{\beta}{2}\Big)\Big)\, X^*\Big(f\,\xi_\kappa^{-1}\Big(-F_T^{(A)}(\beta) - \frac{\beta}{2}\Big)\Big)\, e^{j2\pi\frac{tf}{\kappa}\beta}\, \frac{G_T^{(A)}(\beta)}{\Big|F_T^{(A)2}(\beta) - \frac{\beta^2}{4}\Big|^{\frac{\kappa-1}{2\kappa}}}\, d\beta. \tag{91}$$

When $\kappa = 1$, (90)–(91) reduce to the localized-kernel affine subclass as given by (39)–(40). Note, also, that the κth localized-kernel power subclass is obtained when applying the power warping in (71) to the QTFRs of the localized-kernel affine subclass.

Some important localized-kernel power subclass QTFRs include the power Wigner distribution, $W_X^{(\kappa)}(t,f)$, the generalized power Wigner distribution, $W_X^{(\kappa)(\alpha)}(t,f)$, the power Bertrand P_0-distribution, $P_{0X}^{(\kappa)}(t,f)$, the power Flandrin D-distribution, $D_X^{(\kappa)}(t,f)$, the power passive Unterberger distribution, $\mathrm{PUD}_X^{(\kappa)}(t,f)$, and the power active Unterberger distribution, $\mathrm{AUD}_X^{(\kappa)}(t,f)$. Their 1-D kernels are given as

$$
\begin{aligned}
&F^{(A)}_{Q^{(\kappa)}}(\beta) = -1\,, && G^{(A)}_{Q^{(\kappa)}}(\beta) = 1\,;\\
&F^{(A)}_{Q^{(\kappa)(\alpha)}}(\beta) = \alpha\beta - 1\,, && G^{(A)}_{Q^{(\kappa)(\alpha)}}(\beta) = 1\,;\\
&F^{(A)}_{P_0^{(\kappa)}}(\beta) = -\tfrac{\beta}{2}\coth(\tfrac{\beta}{2})\,, && G^{(A)}_{P_0^{(\kappa)}}(\beta) = \tfrac{\beta/2}{\sinh(\beta/2)}\,;\\
&F^{(A)}_{D^{(\kappa)}}(\beta) = -1 - (\tfrac{\beta}{4})^2\,, && G^{(A)}_{D^{(\kappa)}}(\beta) = 1 - (\tfrac{\beta}{4})^2\,;\\
&F^{(A)}_{\mathrm{PUD}^{(\kappa)}}(\beta) = -\sqrt{1+(\tfrac{\beta}{2})^2}\,, && G^{(A)}_{\mathrm{PUD}^{(\kappa)}}(\beta) = \tfrac{1}{\sqrt{1+(\frac{\beta}{2})^2}}\,;\\
&F^{(A)}_{\mathrm{AUD}^{(\kappa)}}(\beta) = -\sqrt{1+(\tfrac{\beta}{2})^2}\,, && G^{(A)}_{\mathrm{AUD}^{(\kappa)}}(\beta) = 1\,.
\end{aligned}
$$

Note that these kernels are identical to the corresponding ones listed in Section 2.4 (e.g., $F^{(A)}_{W^{(\kappa)}} = F^{(A)}_W$).

The simplified kernel constraints for various QTFR properties are given in the third column of Table 6, where the simplified form of the kernel $\Phi_T^{(A)}(b,\beta)$ in (89) is exploited.

5 INTERSECTION SUBCLASSES

The affine, hyperbolic, and power QTFR classes that we have considered in the previous sections all satisfy the scale covariance property. In addition, each class satisfies a specific group delay dependent time-shift covariance property. There exist QTFRs in these classes that satisfy more than one form of time-shift covariance, and thus are members of the intersections between the QTFR classes considered. These intersection subclasses and some of their members are discussed in this section, and are shown in Figure 1.

5.1 Affine-Hyperbolic Intersection Subclass

The affine-hyperbolic intersection subclass consists of all QTFRs satisfying the scale covariance in (15), the constant time-shift covariance in (14), and the hyperbolic time-shift covariance in (41) [30, 32, 33]. Thus, it contains QTFRs that are members of both the affine class and the hyperbolic class. Note that these QTFRs are only defined for analytic signals.

By imposing the constant time-shift covariance property in the hyperbolic class, the following condition on the hyperbolic class kernel $\Phi_T^{(H)}(b,\beta)$ is derived [30]:

$$\Phi_T^{(H)}(b,\beta) = G_T^{(H)}(\beta)\,\delta\Big(b - \ln\Big(\frac{\sinh(\beta/2)}{\beta/2}\Big)\Big)\,. \tag{92}$$

Here, $G_T^{(H)}(\beta)$ is a 1-D kernel that characterizes the QTFR, $T^{(H)}$. Thus, the 2-D kernel $\Phi_T^{(H)}(b,\beta)$ simplifies to only a 1-D kernel $G_T^{(H)}(\beta)$. This causes the kernel constraints for the hyperbolic class in Table 4 to simplify considerably for the affine-hyperbolic intersection [30]. In addition, the formulation of the time-shift covariant

hyperbolic QTFRs (which are simultaneously hyperbolic time-shift covariant affine QTFRs), $T_X^{(A\cap H)}(t,f)$, is also simplified and is given for $f>0$ as

$$T_X^{(A\cap H)}(t,f) = f\int_{-\infty}^{\infty} X\Big(f\,\frac{\beta/2}{\sinh(\beta/2)}\,e^{\beta/2}\Big)\,X^*\Big(f\,\frac{\beta/2}{\sinh(\beta/2)}\,e^{-\beta/2}\Big)\,\frac{\beta/2}{\sinh(\beta/2)}\,G_T^{(H)}(\beta)\,e^{j2\pi tf\beta}\,d\beta \quad (93)$$

$$= f\int_{-\infty}^{\infty} X\Big(f\Big(\frac{\beta}{2}\coth\frac{\beta}{2}+\frac{\beta}{2}\Big)\Big)\,X^*\Big(f\Big(\frac{\beta}{2}\coth\frac{\beta}{2}-\frac{\beta}{2}\Big)\Big)\,G_T^{(A)}(\beta)\,e^{j2\pi tf\beta}\,d\beta\,. \quad (94)$$

We note that (92) is a special case of the kernel for the localized-kernel hyperbolic class in (62), and that (93) is a special case of (64). This shows that the affine-hyperbolic intersection is a subclass of the localized-kernel hyperbolic class considered in Section 3.5, with $F_T^{(H)}(\beta)$ fixed since $F_T^{(H)}(\beta)=F_{P_0}^{(H)}(\beta)=\ln\Big(\frac{\sinh(\beta/2)}{\beta/2}\Big)$, and with $G_T^{(H)}(\beta)$ arbitrary. In addition, the affine-hyperbolic intersection in (94) is a subclass of the localized-kernel affine class in (40) with $F_T^{(A)}(\beta)=F_{P_0}^{(A)}(\beta)=-\frac{\beta}{2}\coth\frac{\beta}{2}$ and $G_T^{(A)}(\beta)$ arbitrary. Note that the equivalence of (93) with (94) follows from the fact that $\frac{\beta/2}{\sinh(\beta/2)}\,e^{\beta/2}=\frac{\beta}{2}\coth\frac{\beta}{2}+\frac{\beta}{2}$. An important member of the affine-hyperbolic intersection is the Bertrand unitary P_0-distribution [20, 21], $P_{0_X}(t,f)$, defined in (30). It has a very simple kernel $G_{P_0}^{(H)}(\beta)=1$ in the hyperbolic class and $G_{P_0}^{(A)}(\beta)=\frac{\beta/2}{\sinh(\beta/2)}$ in the affine class.

The desirable QTFR properties and corresponding kernel constraints for the affine-hyperbolic intersection subclass are listed in the second column of Table 8. The kernel constraints can be easily obtained from those of the localized-kernel hyperbolic class in the third column of Table 4 when $F_T^{(H)}(\beta)=F_{P_0}^{(H)}(\beta)=\ln\Big(\frac{\sinh(\beta/2)}{\beta/2}\Big)$. Thus, the kernel constraints simplify further since, for example, $F_{P_0}^{(H)}(0)=0$, $F_{P_0}^{(H)\prime}(0)=0$ (where $F_{P_0}^{(H)\prime}(\beta)=\frac{d}{d\beta}F_{P_0}^{(H)}(\beta)$), and $F_{P_0}^{(H)}(\beta)=F_{P_0}^{(H)}(-\beta)$. In particular, the finite frequency support property, P_{11}, is always satisfied for the affine-hyperbolic QTFRs, since the relationship $|F_T^{(H)}(\beta)|=|F_{P_0}^{(H)}(\beta)|\le|\beta/2|$ always holds, indicating that no constraints are necessary on the support of $G_T^{(H)}(\beta)$ (cf. Table 4).

The Bertrand P_0-distribution is a unitary [25, 27, 30] member of the affine-hyperbolic intersection since it satisfies Moyal's formula, property P_9, in Table 4. It follows from the unitarity property that any member of the affine-hyperbolic intersection can be derived from the Bertrand P_0-distribution, $P_{0_X}(t,f)$. It can be shown that this is accomplished through a convolution with respect to the time variable,

$$T_X^{(A\cap H)}(t,f)=f\int_{-\infty}^{\infty} g_T^{(H)}(f(t-\hat{t}))\,P_{0_X}(\hat{t},f)\,d\hat{t}\,,\quad f>0, \quad (95)$$

where $g_T^{(H)}(c)$ is the inverse Fourier transform of the kernel $G_T^{(H)}(\beta)$ in (92). For the Bertrand P_0-distribution, $g_{P_0}^{(H)}(c)=\delta(c)$ since $G_{P_0}^{(H)}(\beta)=1$.

5.2 Affine-Power Intersection Subclass

The intersection subclass between the affine class (the power class for $\kappa=1$) and the κth power class for $\kappa\neq 0,1$ consists of all QTFRs satisfying the scale covariance in (15), the constant time-shift covariance in (14), and the power time-shift covariance in (65). It has been shown in [21] that, with the restriction to analytic signals, the affine-power intersection subclass is given by the Bertrand P_κ-distributions in (33). Suitably extending the

derivation in [21] it can be shown that, for general (non-analytic) signals, the affine-power intersection QTFRs, $T_X^{(A\cap\kappa)}(t,f)$, are characterized as follows:

- For $\kappa > 0$ and $\kappa \neq 1$, $T_X^{(A\cap\kappa)}(t,f)$ can be written as

  $$T_X^{(A\cap\kappa)}(t,f) \;=\; T_X^{(A\cap\kappa)++}(t,f) \,+\, T_X^{(A\cap\kappa)--}(t,f) \,+\, T_X^{(A\cap\kappa)+-}(t,f) \,+\, T_X^{(A\cap\kappa)-+}(t,f)\,, \;\; -\infty < f < \infty\,.$$

  Here, the component QTFRs $T_X^{(A\cap\kappa)\,s\,s'}(t,f)$, where s and s' are $+$ or $-$, are given by

  $$T_X^{(A\cap\kappa)\,s\,s'}(t,f) = \begin{cases} |f| \int_{-\infty}^{\infty} X(sf\lambda_\kappa(u))\; X^*(sf\lambda_\kappa(-u))\; e^{j2\pi tfs\,[\lambda_\kappa(u)-\lambda_\kappa(-u)]}\; \mu_T^s(u)\; du\,, & \text{for } s = s' \\ |f| \int_{-\infty}^{\infty} X(sf\tilde{\lambda}_\kappa(u))\; X^*(-sf\tilde{\lambda}_\kappa(-u))\; e^{j2\pi tfs\,[\tilde{\lambda}_\kappa(u)+\tilde{\lambda}_\kappa(-u)]}\; \tilde{\mu}_T^s(u)\; du\,, & \text{for } s = -s' \end{cases}$$

  where (cf. (34))

  $$\lambda_\kappa(u) = \Big(\kappa\frac{e^{-u}-1}{e^{-\kappa u}-1}\Big)^{\frac{1}{\kappa-1}}\,, \qquad \tilde{\lambda}_\kappa(u) = \Big(\kappa\frac{e^{-u}+1}{e^{-\kappa u}+1}\Big)^{\frac{1}{\kappa-1}}\,, \qquad \kappa > 0\,,\;\; \kappa \neq 1.$$

  We emphasize that $T_X^{(A\cap\kappa)++}(t,f)$ is equal to the Bertrand P_κ-distributions, $P_{\kappa_X}(t,f)$, in (33). While $T_X^{(A\cap\kappa)++}(t,f) = P_{\kappa_X}(t,f)$ assigns positive and negative signal frequencies to positive and negative QTFR frequencies, respectively, the other QTFR components $T_X^{(A\cap\kappa)--}(t,f)$, $T_X^{(A\cap\kappa)+-}(t,f)$, and $T_X^{(A\cap\kappa)-+}(t,f)$ involve a crossover of positive and negative frequencies. The four-component QTFRs of the affine-power intersection are parameterized in terms of the four 1-D functions $\mu_T^+(u)$, $\mu_T^-(u)$, $\tilde{\mu}_T^+(u)$, and $\tilde{\mu}_T^-(u)$.

- For $\kappa < 0$, $T_X^{(A\cap\kappa)}(t,f)$ is given as

  $$T_X^{(A\cap\kappa)}(t,f) \;=\; T_X^{(A\cap\kappa)++}(t,f) \,+\, T_X^{(A\cap\kappa)--}(t,f)\,, \qquad -\infty < f < \infty\,,$$

  with $T_X^{(A\cap\kappa)++}(t,f)$ and $T_X^{(A\cap\kappa)--}(t,f)$ as defined above. Here, $T_X^{(A\cap\kappa)}(t,f)$ is parameterized in terms of the two 1-D kernel functions $\mu_T^+(u)$ and $\mu_T^-(u)$.

It can be shown that the component QTFRs $T_X^{(A\cap\kappa)++}(t,f)$ and $T_X^{(A\cap\kappa)--}(t,f)$ and, assuming invertibility of the even function $\tilde{\lambda}_\kappa(u) + \tilde{\lambda}_\kappa(-u)$ for $u > 0$, also $T_X^{(A\cap\kappa)+-}(t,f)$ and $T_X^{(A\cap\kappa)-+}(t,f)$ are members of the localized-kernel power subclass in Section 4.5. The QTFRs of the affine-power intersection, being the sum of all four component QTFRs, are however not, in general, members of the localized-kernel power subclass.

5.3 Hyperbolic-Power Intersection Subclass

The κth power class and the hyperbolic class have some common QTFR members that form the κth hyperbolic-power intersection subclass. QTFRs in this subclass satisfy the scale covariance property in (42), the hyperbolic time-shift covariance property in (41), and the power time-shift covariance property in (65). Note that since the hyperbolic class is only defined for positive frequencies, we are here considering the power classes for analytic signals only.

By imposing the hyperbolic time-shift covariance in (41) on the power classes, the associated condition on the power classes kernel $\Phi_T^{(A)}(b,\beta)$ in (75) is obtained as

$$\Phi_T^{(A)}(b,\beta) = G_T^{(A)}(\beta)\ \delta\Big(b + \frac{\beta}{2}\coth\Big(\frac{\beta}{2}\Big)\Big)\ , \tag{96}$$

where the arbitrary 1-D kernel $G_T^{(A)}(\beta)$ characterizes the QTFR $T^{(\kappa)}$. Similarly, by imposing the power time-shift covariance in (65) on the hyperbolic class, the associated condition on the hyperbolic class kernel $\Phi_T^{(H)}(b,\beta)$ in (48) is obtained as

$$\Phi_T^{(H)}(b,\beta) = G_T^{(H)}(\beta)\ \delta\Big(b - \frac{1}{\kappa}\ \ln\Big(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\Big)\Big)\ , \tag{97}$$

where the arbitrary 1-D kernel $G_T^{(H)}(\beta)$ characterizes the QTFR $T^{(H)}$. Thus, in the case of a hyperbolic time-shift covariant power QTFR, $T_X^{(H\cap\kappa)}(t,f)$, which is simultaneously a power time-shift covariant hyperbolic QTFR, (48) and (75) simplify, respectively, to

$$T_X^{(H\cap\kappa)}(t,f) = \int_{-\infty}^{\infty} V_X^{(H)}\Big(\ln\frac{f}{f_r} - \frac{1}{\kappa}\ln\Big(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\Big),\ \beta\Big)\ G_T^{(H)}(\beta)\ e^{j2\pi t f\beta}\, d\beta\,, \quad f>0, \tag{98}$$

$$= \Big|\xi_\kappa\Big(\frac{f}{f_r}\Big)\Big| \int_{-\infty}^{\infty} V_X^{(\kappa)}\Big(\xi_\kappa\Big(\frac{f}{f_r}\Big)\frac{\beta}{2}\coth\Big(\frac{\beta}{2}\Big),\ \xi_\kappa\Big(\frac{f}{f_r}\Big)\beta\Big)\ G_T^{(A)}(\beta)\ e^{j2\pi\frac{tf}{\kappa}\beta}\, d\beta. \tag{99}$$

We note that (96) is a special case of the localized-kernel power subclass in (89) with $F_T^{(A)}(\beta) = F_{P_0^{(\kappa)}}^{(A)}(\beta) = -\frac{\beta}{2}\coth(\frac{\beta}{2})$ and $G_T^{(A)}(\beta)$ arbitrary, and that (97) is a special case of the localized-kernel hyperbolic subclass in (62) with $F_T^{(H)}(\beta) = F_{P_0^{(\kappa)}}^{(H)}(\beta) = \frac{1}{\kappa}\ \ln\Big(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\Big)$ and $G_T^{(H)}(\beta)$ arbitrary. Thus, (99) is a special case of (90) and, equivalently, (98) is a special case of (63). This shows that the hyperbolic-power intersection is a subclass of both the localized-kernel power class (with the respective power κ) and the localized-kernel hyperbolic class. An important member of the hyperbolic-power intersection is the power Bertrand P_0-distribution, $P_{0_X}^{(\kappa)}(t,f)$, that has the simple kernels $G_{P_0^{(\kappa)}}^{(H)}(\beta) = 1$ in the hyperbolic class, and $G_{P_0^{(\kappa)}}^{(A)}(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$ in the κth power class. Note that for $\kappa = 1$, the hyperbolic-power intersection simplifies to the affine-hyperbolic intersection considered previously in Section 5.1.

The desirable QTFR properties and associated kernel constraints for the hyperbolic-power intersection subclass are listed in the third column of Table 8. The kernel constraints are the same as those of the localized-kernel hyperbolic class in the third column of Table 4 with $F_T^{(H)}(\beta) = F_{P_0^{(\kappa)}}^{(H)}(\beta) = \frac{1}{\kappa}\ \ln\Big(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\Big)$, or as those of the localized-kernel power classes in the third column of Table 6 with $F_T^{(A)}(\beta) = F_{P_0^{(\kappa)}}^{(A)}(\beta) = -\frac{\beta}{2}\coth(\frac{\beta}{2})$. In particular, the finite frequency support property P_{11} is always satisfied by all hyperbolic-power (and thus all affine-hyperbolic) QTFRs. This follows since $|F_T^{(A)}(\beta)+1| = \Big|F_{P_0^{(\kappa)}}^{(A)}(\beta)+1\Big| = \Big|-\frac{\beta}{2}\coth(\frac{\beta}{2})+1\Big| \le |\beta/2|$ always holds or, equivalently, the corresponding relationship in the hyperbolic class $|F_T^{(H)}(\beta)| = \Big|F_{P_0^{(\kappa)}}^{(H)}(\beta)\Big| = \Big|\frac{1}{\kappa}\ \ln\Big(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\Big)\Big| \le |\beta/2|$ always holds.

The power Bertrand P_0-distribution is a unitary member of the hyperbolic-power intersection as it satisfies Moyal's formula, property P_9, in Table 6. Thus, any member of the hyperbolic-power intersection can be

derived from the power Bertrand P_0-distribution, $P_{0_X}^{(\kappa)}(t,f)$. It can be shown that this is accomplished through a convolution with respect to the time variable,

$$T_X^{(H\cap\kappa)}(t,f) = |f| \int_{-\infty}^{\infty} g_T^{(H)}(f(t-\hat{t}))\, P_{0_X}^{(\kappa)}(\hat{t},f)\, d\hat{t}\,, \quad f>0\,, \tag{100}$$

where $g_T^{(H)}(c)$ is the inverse Fourier transform of the kernel $G_T^{(H)}(\beta)$. For the power Bertrand P_0-distribution, $g_{P_0^{(\kappa)}}^{(H)}(c) = \delta(c)$ since $G_{P_0^{(\kappa)}}^{(H)}(\beta) = 1$. Note that (100) simplifies to (95) when $\kappa = 1$, as expected, since the power class with $\kappa = 1$ is equal to the affine class, and $P_{0_X}^{(1)}(t,f) = P_{0_X}(t,f)$.

6 THE GENERALIZED QTFR CLASS

The scale covariance property in (15) is very important in many applications such as multiscale and wavelet analysis. Another important property is the generalized time-shift covariance (see Table 1) that is related to the group delay or dispersion characteristic of a system under analysis. In particular, the generalized time-shift covariance is an important property when analyzing signals propagating through systems with specific dispersive characteristics such as constant, hyperbolic, and power.

We propose a generalized class of QTFRs, $T_X^{(G)}(t,f)$, that satisfy the generalized time-shift covariance and scale covariance properties defined, respectively, as

$$T_{\mathcal{D}_c X}^{(G)}(t,f) = T_X^{(G)}(t - c\tau(f), f) \tag{101}$$

$$T_{\mathcal{C}_a X}^{(G)}(t,f) = T_X^{(G)}\left(at, \frac{f}{a}\right). \tag{102}$$

The generalized time-shift operator $\mathcal{D}_c$ is defined as

$$(\mathcal{D}_c X)(f) = e^{-j2\pi c\,\xi(\frac{f}{f_r})}\, X(f)\,, \qquad c \in \mathbb{R}\,, \tag{103}$$

where the generalized time-shift $\tau(f) = \frac{d}{df}\xi(\frac{f}{f_r})$ is the derivative of the one-to-one phase function, $\xi(b)$. The generalized time-shift operator in (103) corresponds to an allpass, linear, time-invariant system producing a frequency-dependent time-shift defined by a group delay law, $c\tau(f)$ (cf. Figure 3). The following specific choices of the phase function and the group delay function have been considered in previous sections (cf. Table 9):

- When $\xi(b) = b$ and $\tau(f) = 1/f_r =$ constant, the generalized time-shift covariance for the generalized class in (101) simplifies to the nondispersive time-shift covariance of the affine class in (14), and (103) simplifies to the time-shift operator in (1), $\mathcal{D}_c = \mathcal{S}_{c/f_r}$.

- When $\xi(b) = \ln b$ and $\tau(f) = 1/f$, (101) simplifies to the hyperbolic time-shift covariance of the hyperbolic class in (41), and (103) simplifies to the hyperbolic time-shift operator in (43), $\mathcal{D}_c = \mathcal{H}_c$.

- When $\xi(b) = \xi_\kappa(b) = \mathrm{sgn}(b)\,|b|^\kappa$ (68) and $\tau(f) = \tau_\kappa(f) = \frac{\kappa}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1}$ (69), (101) simplifies to the power time-shift covariance of the κth power class in (65), and (103) simplifies to the power time-shift operator in (67), $\mathcal{D}_c = \mathcal{D}_c^{(\kappa)}$.

6.1 Generalized Class Formulation

We shall now investigate the conditions under which QTFRs satisfying the generalized time-shift covariance in (101) and the scale covariance in (102) exist, and we shall derive general formulations for such QTFRs.

Any QTFR, $T_X(t,f)$, can be written in terms of a four-dimensional (4-D) kernel function $K_T(t,f;f_1,f_2)$ [21, 25],

$$T_X(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_T(t,f;f_1,f_2)\, X(f_1)\, X^*(f_2)\, df_1 df_2\,. \tag{104}$$

For (101) and (102) to be satisfied, it is necessary and sufficient that the following two combined covariances are satisfied:

$$T^{(G)}_{\mathcal{D}_c\mathcal{C}_a X}(t,f) \;=\; T^{(G)}_X\Big(a(t-c\tau(f)), \frac{f}{a}\Big) \tag{105}$$

$$T^{(G)}_{\mathcal{C}_a\mathcal{D}_c X}(t,f) \;=\; T^{(G)}_X\Big(at-c\tau\Big(\frac{f}{a}\Big), \frac{f}{a}\Big)\,. \tag{106}$$

If we substitute (104) in both (105) and (106) and simplify, we obtain the result that the 4-D kernel must have the following two equivalent forms:

$$K_T(t,f;f_1,f_2) \;=\; \frac{1}{|f|}\,\Gamma^{(G)}_T\Big(\frac{f_1}{f},\frac{f_2}{f}\Big)\, e^{j2\pi\frac{t}{\tau(f)}\left[\xi\left(\frac{f_1}{f_r}\right)-\xi\left(\frac{f_2}{f_r}\right)\right]} \tag{107}$$

$$\;=\; \frac{1}{|f|}\,\Gamma^{(G)}_T\Big(\frac{f_1}{f},\frac{f_2}{f}\Big)\, e^{j2\pi\frac{tf}{f_r\tau(f_r)}\left[\xi\left(\frac{f_1}{f}\right)-\xi\left(\frac{f_2}{f}\right)\right]}\,. \tag{108}$$

By equating (107) with (108), we obtain a necessary condition in terms of the phase function $\xi(b)$ and the 2-D kernel $\Gamma^{(G)}_T(b_1,b_2)$ for any QTFR that satisfies the generalized time-shift covariance and the scale covariance properties. This condition is easily shown to be sufficient for (101) and (102) as well [38]. It is stated in the following theorem:

Theorem 1. A QTFR satisfying the generalized time-shift covariance (101) and the scale covariance (102) exists if and only if there exists a kernel function $\Gamma^{(G)}_T(b_1,b_2)$ such that

$$\Gamma^{(G)}_T(b_1,b_2)\,\frac{\xi(\alpha b_1)-\xi(\alpha b_2)}{\alpha\,\xi'(\alpha)} \;\equiv\; \Gamma^{(G)}_T(b_1,b_2)\,\frac{\xi(b_1)-\xi(b_2)}{\xi'(1)}\,, \qquad \forall\, b_1,b_2,\alpha, \tag{109}$$

where $\xi'(b) = \frac{d}{db}\xi(b)$.

If (109) is satisfied, then the QTFR can be expressed as[8]

$$T^{(G)}_X(t,f) \;=\; \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma^{(G)}_T\Big(\frac{f_1}{f},\frac{f_2}{f}\Big)\, e^{j2\pi\frac{t}{\tau(f)}\left[\xi\left(\frac{f_1}{f_r}\right)-\xi\left(\frac{f_2}{f_r}\right)\right]}\, X(f_1)\, X^*(f_2)\, df_1 df_2 \tag{110}$$

$$\;=\; \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma^{(G)}_T\Big(\frac{f_1}{f},\frac{f_2}{f}\Big)\, e^{j2\pi\frac{tf}{f_r\tau(f_r)}\left[\xi\left(\frac{f_1}{f}\right)-\xi\left(\frac{f_2}{f}\right)\right]}\, X(f_1)\, X^*(f_2)\, df_1 df_2\,. \tag{111}$$

Note that (110) is obtained by substituting (107) in the generalized QTFR form (104) while (111) is obtained by substituting the equivalent form (108) in (104). As a result, the QTFR constructed according to (110) or (111),

using a kernel $\Gamma_T^{(G)}(b_1,b_2)$, satisfies the combined covariances of generalized time-shift and scaling in (105) and (106) if and only if the kernel $\Gamma_T^{(G)}(b_1,b_2)$ satisfies (109) for the specified phase function $\xi(b)$. The generalized class of QTFRs satisfying generalized time-shift covariance and scale covariance is thus equivalently formulated by (110) or (111) with $\Gamma_T^{(G)}(b_1,b_2)$ constrained by (109). Three cases can now be distinguished:

1. A maximally wide QTFR class for a given phase function, $\xi(b)$, is obtained when (109) holds for *all* $\Gamma_T^{(G)}(b_1,b_2)$. This means that $\xi(b)$ must satisfy

$$\frac{\xi(\alpha b_1)-\xi(\alpha b_2)}{\alpha\,\xi'(\alpha)} \equiv \frac{\xi(b_1)-\xi(b_2)}{\xi'(1)}\,, \qquad \forall\, b_1,b_2,\alpha\,. \tag{112}$$

   It is shown in Appendix A that (112) is satisfied if and only if $\xi(b)$ is proportional to either a power function or a logarithm [58]; in both cases, the basic group delay $\tau(f)=\frac{d}{df}\,\xi(\frac{f}{f_r})$ will be a *power function*, which leads to the affine class, the hyperbolic class, and the power classes:

   - For the *affine class* with phase function $\xi(b)=b$ and group delay function $\tau(f)=1/f_r$ (see Table 9), condition (112) is satisfied. The generalized forms (110) and (111) simplify to (17), without any restrictions on the kernel function $\Gamma_T^{(A)}(b_1,b_2)$.
   - For the *hyperbolic class* with phase function $\xi(b)=\ln b$ and group delay function $\tau(f)=1/f$, condition (112) is satisfied since $\ln(\alpha b_1)-\ln(\alpha b_2)=\ln b_1/b_2$ and $\xi'(\alpha)=1/\alpha$. The generalized forms (110) and (111) simplify to (44), without any restrictions on the kernel function $\Gamma_T^{(H)}(b_1,b_2)$.
   - For the *power classes* with phase function $\xi(b)=\xi_\kappa(b)$ in (68) and group delay function $\tau(f)=\tau_\kappa(f)$ in (69), condition (112) is satisfied since $\xi_\kappa(\alpha b_1)-\xi_\kappa(\alpha b_2)=\xi_\kappa(\alpha)\big[\xi_\kappa(b_1)-\xi_\kappa(b_2)\big]$. The generalized forms in (110) and (111) simplify to (70), without any restrictions on the kernel function $\Gamma_T^{(\kappa)}(b_1,b_2)$.

2. The second case to be considered is when the phase function does *not* satisfy (112) but it satisfies (109) for some specific type of kernel $\Gamma_T^{(G)}(b_1,b_2)$. This results in a covariant QTFR class that is less wide than in Case 1, since in Case 1 the kernel $\Gamma_T^{(G)}(b_1,b_2)$ was unconstrained. An important example of this situation is given by the phase function $\xi(b)=b\ln b$ corresponding to the group delay function $\tau(f)=\frac{1}{f_r}(1+\ln\frac{f}{f_r})$, which only satisfies (109) for the special case

$$\Gamma_{P_1}(b_1,b_2)=\int_{-\infty}^{\infty}\delta(b_1-\lambda_1(u))\;\delta(b_2-\lambda_1(-u))\;\mu(u)\,du\,, \tag{113}$$

   where $\lambda_1(u)=\exp\big(1+\frac{u\,e^{-u}}{e^{-u}-1}\big)$ and $\mu(u)$ is a real and even function [20, 21]. This kernel is not arbitrary but has a specific form; it satisfies (109) with $\xi(b)=b\,\ln b$ since it is constrained to have a specific "delta function structure" which is non-zero only at certain combinations of b_1 and b_2 (namely, $b_1=\lambda_1(u)$, $b_2=\lambda_1(-u)$ where $\lambda_1(u)$ is matched to the given phase function $\xi(b)=b\ln b$). As a consequence, the choice of the 2-D kernel, $\Gamma_T(b_1,b_2)$, reduces to the choice of a 1-D weighting function $\mu(u)$. Substituting (113) into (110) or (111) and simplifying yields the class of *Bertrand P_1-distributions* defined as [20, 21]

$$P_{1_X}(t,f)=|f|\int_{-\infty}^{\infty}X(f\lambda_1(u))\;X^*(f\lambda_1(-u))\;e^{j2\pi tf[\lambda_1(u)-\lambda_1(-u)]}\,\mu(u)\,du\,, \tag{114}$$

which satisfies the generalized time-shift covariance in (101) with $\xi(b) = b \ln b$ and $\tau(f) = \frac{1}{f_r}(1 + \ln \frac{f}{f_r})$ and the scale covariance in (102). This class is less wide than a QTFR class where the kernel $\Gamma_T(b_1, b_2)$ is unconstrained. Note that the Bertrand P_1-distributions are also members of the affine class [21], and hence they also satisfy the conventional (nondispersive) time-shift covariance property.

3. Finally, the phase function $\xi(b)$ may be such that condition (109) is not satisfied for any kernel $\Gamma_T^{(G)}(b_1, b_2)$. Here, there does not exist any QTFR that satisfies the generalized time-shift covariance in (101) with the given phase function $\xi(b)$ and the scale covariance in (102).

Just as for the affine, hyperbolic, and power classes in the previous sections, a normal form can also be obtained for the generalized QTFR class. Any generalized class QTFR, $T_X(t, f)$, can be written as

$$\begin{aligned} T_X^{(G)}(t,f) &= \left|\frac{f_r}{f}\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_T^{(G)}\left(-\frac{\xi(\frac{f_r}{f}\, d(b,\beta)) + \xi(\frac{f_r}{f}\, d(b,-\beta))}{2}, \xi\Big(\frac{f_r}{f}\, d(b,\beta)\Big) - \xi\Big(\frac{f_r}{f}\, d(b,-\beta)\Big)\right) \\ &\quad \cdot V_X(b,\beta)\, \frac{\left|\xi'\Big(\frac{f_r}{f}\sqrt{|d(b,\beta)\, d(b,-\beta)|}\Big)\right|}{\sqrt{|\xi'(d(b,\beta))\, \xi'(d(b,-\beta))|}}\, e^{j2\pi \frac{t}{\tau(f)}\beta}\, db\, d\beta \end{aligned} \tag{115}$$

where $d(b, \beta) = \xi^{-1}(b + \frac{\beta}{2})$ and the signal product $V_X(b, \beta)$ is given as

$$V_X(b,\beta) = \frac{f_r}{\sqrt{|\xi'(d(b,\beta))\, \xi'(d(b,-\beta))|}}\, X(f_r\, d(b,\beta))\, X^*(f_r\, d(b,-\beta)). \tag{116}$$

The 2-D kernel, $\Phi_T^{(G)}(b, \beta)$, in (115) uniquely characterizes the generalized QTFR, $T^{(G)}$, and it is related to the kernel $\Gamma_T^{(G)}(b_1, b_2)$ in (107)–(111) as

$$\Gamma_T^{(G)}(b_1,b_2) = \left|\xi'\Big(\sqrt{|b_1 b_2|}\Big)\right| \Phi_T^{(G)}\Big(-\frac{\xi(b_1)+\xi(b_2)}{2}, \xi(b_1) - \xi(b_2)\Big). \tag{117}$$

We emphasize that the kernel $\Phi_T^{(G)}(b, \beta)$ is not arbitrary, in general, but is constrained by (109) using (117).

If $\xi(b) = b$ and $\tau(f) = \frac{1}{f_r}$, then the generalized normal form in (115) simplifies to the second affine class normal form in (19). If $\xi(b) = \ln b$ and $\tau(f) = \frac{1}{f}$, then it simplifies to the second hyperbolic class normal form in (48). Finally, if $\xi(b) = \xi_\kappa(b)$ in (68) and $\tau(f) = \tau_\kappa(f)$ in (69), then (115) simplifies to the power class normal form in (75). Note also that when these simplifications are made, the 2-D kernel $\Phi_T^{(G)}(b, \beta)$ simplifies to either the affine/power QTFR kernel, i.e., $\Phi_T^{(G)}(b, \beta) = \Phi_T^{(A)}(b, \beta)$, or to the hyperbolic QTFR kernel, i.e., $\Phi_T^{(G)}(b, \beta) = \Phi_T^{(H)}(b, \beta)$. Also, with the appropriate choice of the phase function, the signal product $V_X(b, \beta)$ simplifies to $U_X(b, \beta)$ in the affine class, $V_X^{(H)}(b, \beta)$ in the hyperbolic class, and $V_X^{(\kappa)}(b, \beta)$ in the power classes.

6.2 Generalized Signal Expansion

The TF geometry underlying the generalized time-shift covariance is related to the *generalized impulse* defined in the frequency domain as (cf. Table 9)

$$I_c(f) \triangleq \sqrt{|\tau(f)|}\, e^{-j2\pi c\, \xi(\frac{f}{f_r})}, \qquad c \in \mathbb{R}. \tag{118}$$

The generalized impulse has spectral energy density $|I_c(f)|^2 = |\tau(f)|$ and group delay $c\tau(f)$. The group delay of $I_c(f)$ reflects the dispersion characteristics of the corresponding generalized class. When the appropriate phase and group delay functions are used for each class as listed in Table 9, the generalized impulse in (118) simplifies to $I_c^{(A)}(f)$ in (24) for the affine class, it simplifies to $H_c(f)$ in (56) for the hyperbolic class, and it simplifies to $I_c^{(\kappa)}(f)$ in (82) for the κth power class. The generalized impulse is related to the dispersive time-shift operator $\mathcal{D}_c$ since generalized time-shifting the generalized impulses simply changes their parameter value:

$$(\mathcal{D}_{c_0} I_c)(f) = I_{c+c_0}(f)\,.$$

However, $I_c(f)$ will not, in general, satisfy a covariance property with respect to the scaling operator $\mathcal{C}_a$.

The family of generalized impulses covers the entire TF plane as the value of the parameter c varies from $-\infty$ to ∞. As a result, if the corresponding phase function, $\xi(b)$, is one-to-one with domain $\mathbb{R}$ and range $\mathbb{R}$, then any finite-energy signal $X(f)$ can be expanded in terms of generalized impulses

$$X(f) = \int_{-\infty}^{\infty} \rho_X(c)\, I_c(f)\, dc = \sqrt{|\tau(f)|} \int_{-\infty}^{\infty} \rho_X(c)\, e^{-j2\pi c\, \xi\left(\frac{f}{f_r}\right)}\, dc\,, \tag{119}$$

where the generalized coefficient function, $\rho_X(c)$, is the inner product of $X(f)$ with $I_c(f)$,

$$\rho_X(c) = \int_{-\infty}^{\infty} X(f)\, I_c^*(f)\, df = \int_{-\infty}^{\infty} X(f)\, \sqrt{|\tau(f)|}\, e^{j2\pi c\, \xi\left(\frac{f}{f_r}\right)}\, df\,. \tag{120}$$

The validity of the generalized signal expansion in (119)–(120) follows from the completeness property of the generalized impulses,

$$\int_{-\infty}^{\infty} I_c(f_1)\, I_c{}^*(f_2)\, dc = \delta(f_1 - f_2)\,.$$

We also note the orthogonality property

$$\int_{-\infty}^{\infty} I_{c_1}(f)\, I_{c_2}^*(f)\, df = \delta(c_1 - c_2)\,.$$

These two properties will be satisfied provided that the phase function, $\xi(b)$, is differentiable and one-to-one.

The generalized signal expansion in (119) constitutes a unitary, linear signal transform $X(f) \leftrightarrow \rho_X(c)$ which generalizes the corresponding transforms associated with the affine class in (25) and (26), the hyperbolic class in (58) and (59), and the power classes in (83) and (84). In particular, the generalized signal expansion generalizes both the Fourier transform (the "affine signal expansion" in Section 2.2) and the Mellin transform (the "hyperbolic signal expansion" in Section 3.3). Some important properties of the generalized signal expansion follow:

- Unitarity (cf. Parseval's theorem) states that the inner product of the generalized coefficients of two signals is the inner product of the signals,

$$\int_{-\infty}^{\infty} \rho_{X_1}(c)\, \rho_{X_2}^*(c)\, dc = \int_{-\infty}^{\infty} X_1(f)\, X_2^*(f)\, df\,.$$

- A generalized time-shift, $\mathcal{D}_{c_0}$, of the signal produces a simple translation by c_0 of the generalized coefficient, $\rho_{\mathcal{D}_{c_0}X}(c) = \rho_X(c - c_0)$.
- The generalized coefficient of a generalized impulse, $I_{c_0}(f)$, is a Dirac function centered at $c = c_0$, $\rho_{I_{c_0}}(c) = \delta(c - c_0)$.

An additional scaling property exists when the phase functions are logarithmic or power (see Sections 2.2, 3.3, and 4.3). The relevance of the generalized signal expansion to the generalized QTFR class is reflected by the QTFR properties P_6 (generalized marginal property) and P_8 (generalized localization property) in Table 10.

6.3 An Important Member, Properties, and Kernel Constraints

An important "central member" of the generalized class, denoted $W_X^{(G)}(t,f)$, corresponds to the kernel $\Phi_{W^{(G)}}^{(G)}(b,\beta) = \delta(b + \xi(1))$ or, equivalently, $\Gamma_{W^{(G)}}^{(G)}(b_1,b_2) = \left|\xi'\left(\sqrt{|b_1 b_2|}\right)\right| \; \delta\left(\frac{\xi(b_1)+\xi(b_2)}{2} - \xi(1)\right)$. Assuming (as is done throughout this section) that $\xi(b)$ is such that condition (109) is satisfied, the central member is generalized time-shift covariant and scale covariant, and it can be written as

$$W_X^{(G)}(t,f) = |f| \int_{-\infty}^{\infty} X\left(f\,\hat{l}(\beta)\right) X^*\left(f\,\hat{l}(-\beta)\right) e^{j2\pi \frac{tf}{f_r \tau(f_r)}\beta} \left| \frac{\xi'\left(\sqrt{|\hat{l}(\beta)\,\hat{l}(-\beta)|}\right)}{\xi'(\hat{l}(\beta))\,\xi'(\hat{l}(-\beta))} \right| d\beta \tag{121}$$

where $\hat{l}(\beta) = \xi^{-1}\left(\xi(1) + \frac{\beta}{2}\right)$ and $\xi^{-1}(b)$ is defined by $\xi^{-1}(\xi(b)) = b$. The central member, $W_X^{(G)}(t,f)$, reduces to the Wigner distribution in (10) when $\xi(b) = b$ and $\tau(f) = 1/f_r$ (affine class), it reduces to the Q-distribution in (54) when $\xi(b) = \ln b$ and $\tau(f) = 1/f$ (hyperbolic class), and it reduces to the κth power Wigner distribution in (80) when $\xi(b) = \xi_\kappa(b)$ and $\tau(f) = \tau_\kappa(f)$ as defined in (68) and (69), respectively (κth power class).

A list of desirable properties for the generalized class together with kernel constraints is given in Table 10. Since $W_X^{(G)}(t,f)$ in (121) has the simple kernel $\Phi_{W^{(G)}}^{(G)}(b,\beta) = \delta(b+\xi(1))$, it satisfies most of these properties. In particular, $W_X^{(G)}(t,f)$ is the only generalized QTFR that satisfies the frequency marginal property, P_5, and the frequency localization property, P_7. Furthermore, it can be shown that $W_X^{(G)}(t,f)$ also satisfies the generalized marginal property, P_6, and the generalized localization property, P_8, when the phase function $\xi(b)$ is a logarithmic or power function. Note that with the specific phase and group delay functions as in Table 9, the list of properties in Table 10 simplifies to the corresponding ones in Tables 2, 4, and 6 for the affine, hyperbolic, and κth power class, respectively.

6.4 Localized-Kernel Generalized Subclass

Generalizing the localized-kernel affine, hyperbolic, and power subclasses in Sections 2.4, 3.5, and 4.5, we can define a localized-kernel subclass of the generalized QTFR class by prescribing the following special form of the kernel $\Phi_T^{(G)}(b,\beta)$,

$$\Phi_T^{(G)}(b,\beta) = G_T^{(G)}(\beta)\; \delta\left(b - F_T^{(G)}(\beta)\right), \tag{122}$$

where the 1-D kernels $F_T^{(G)}(\beta) \in \mathbb{R}$ and $G_T^{(G)}(\beta)$ are arbitrary functions that characterize the localized-kernel generalized QTFR, $T^{(G)}$. Note that the kernel (122) is assumed to be compatible with (109) using (117).

This localized-kernel subclass is important since the 2-D kernel, $\Phi_T^{(G)}(b,\beta)$, is now parameterized in terms of two 1-D kernel functions which results in simpler formulations for the localized-kernel generalized QTFRs and simplifies the kernel constraints for the generalized class. Any localized-kernel generalized QTFR can be written as

$$T_X^{(G)}(t,f) = \left|\frac{f}{f_r}\right| \int_{-\infty}^{\infty} V_X\left(\frac{\xi\left(\frac{f}{f_r}d_1(\beta)\right)+\xi\left(\frac{f}{f_r}d_2(\beta)\right)}{2}, \xi\left(\frac{f}{f_r}d_1(\beta)\right)-\xi\left(\frac{f}{f_r}d_2(\beta)\right)\right) G_T^{(G)}(\beta)$$

$$\cdot e^{j2\pi\frac{t}{\tau(f)}\left(\xi\left(\frac{f}{f_r}d_1(\beta)\right)-\xi\left(\frac{f}{f_r}d_2(\beta)\right)\right)} \frac{|\xi'(\sqrt{|d_1(\beta)d_2(\beta)|})|\sqrt{|\xi'(\frac{f}{f_r}d_1(\beta))\xi'(\frac{f}{f_r}d_2(\beta))|}}{|\xi'(d_1(\beta))\xi'(d_2(\beta))|}\, d\beta \tag{123}$$

where $d_1(\beta) = \xi^{-1}(-F_T^{(G)}(\beta)+\frac{\beta}{2})$ and $d_2(\beta) = \xi^{-1}(-F_T^{(G)}(\beta)-\frac{\beta}{2})$. The QTFR expression (123) simplifies to that of the localized-kernel affine class in (39) when $\xi(b) = b$ and $\tau(f) = 1/f_r$, it simplifies to that of the localized-kernel hyperbolic class in (63) when $\xi(b) = \ln(b)$ and $\tau(f) = 1/f$, and finally it simplifies to that of the κth localized-kernel power class in (90) when $\xi(b) = \xi_\kappa(b)$ and $\tau(f) = \tau_\kappa(f)$ (see (68), (69)). An important member of the localized-kernel generalized class is the "central member" defined in (121) with kernels $F_{W^{(G)}}^{(G)}(\beta) = -\xi(1)$ and $G_{W^{(G)}}^{(G)}(\beta) = 1$. The kernel constraints for desirable QTFR properties can be reformulated in terms of the 1-D kernels $F_T^{(G)}(\beta)$ and $G_T^{(G)}(\beta)$ by substituting (122) in the second column of Table 10.

7 GENERALIZED WARPING

The generalized class considered in the last section is defined based on the generalized time-shift covariance property in (101) and the scale covariance property in (102) that the generalized QTFRs always satisfy. Thus, generalized QTFRs are useful when both of these properties are necessary in a particular application. If, however, scale covariance is not of interest and only the generalized time-shift covariance property is important, then other QTFR generalizations are possible. We can obtain generalized time-shift covariant QTFRs from Cohen's class or the affine class[9] using a unitary warping transformation. This warping is defined as

$$T_X^{(G\,W class)}(t,f) = T_{\mathcal{W}X}^{(class)}\left(\frac{t}{f_r\tau(f)}, f_r\,\xi\left(\frac{f}{f_r}\right)\right), \tag{124}$$

where the generalized warping operator $\mathcal{W}$ is given by

$$(\mathcal{W}X)(f) = \frac{1}{\sqrt{f_r\left|\tau\left(f_r\,\xi^{-1}\left(\frac{f}{f_r}\right)\right)\right|}}\, X\left(f_r\,\xi^{-1}\left(\frac{f}{f_r}\right)\right). \tag{125}$$

Here, the superscript $(class)$ indicates the QTFR class which undergoes the warping; this QTFR class is either Cohen's class, (C), or the affine class, (A). Depending upon which class is subject to the warping, we obtain the QTFRs $T_X^{(G\,WC)}(t,f)$ (generalized warped[10] Cohen's class QTFRs) or $T_X^{(G\,WA)}(t,f)$ (generalized warped affine

class QTFRs), both of which satisfy the generalized time-shift covariance property in (101). We emphasize that the phase function $\xi(b)$ need not satisfy (109) or (112); the only condition on the phase function is that it is a one-to-one, differentiable function. Next, we consider the generalized warped Cohen's class and the generalized warped affine class in detail.

7.1 Generalized Time-Shift Covariant QTFRs from Cohen's Class

When Cohen's class QTFRs are warped using (124) and (125), the resulting generalized time-shift covariant QTFRs, $T_X^{(G\,WC)}(t,f)$, are obtained as

$$\begin{aligned} T_X^{(G\,WC)}(t,f) &= T_{\mathcal{W}X}^{(C)}\Big(\frac{t}{f_r\tau(f)},\, f_r\,\xi\Big(\frac{f}{f_r}\Big)\Big) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T^{(G\,WC)}\Big(\frac{t}{\tau(f)}-\frac{\hat{t}}{\tau(\hat{f})},\, \xi\Big(\frac{f}{f_r}\Big)-\xi\Big(\frac{\hat{f}}{f_r}\Big)\Big)\, W_X^{(G\,W)}(\hat{t},\hat{f})\, d\hat{t}\, d\hat{f}\,. \end{aligned} \tag{126}$$

Here, $\psi_T^{(G\,WC)}(c,b)$ is a 2-D kernel function that uniquely characterizes the QTFR $T^{(G\,WC)}$; it is related to the corresponding kernel in Cohen's class, $\psi_T^{(C)}(t,f)$ in (7), by $\psi_T^{(G\,WC)}(c,b) = \psi_T^{(C)}(\frac{c}{f_r}, f_r b)$. Furthermore, $W_X^{(G\,W)}(t,f)$ is the warped version of the Wigner distribution, $W_X(t,f)$ in (10),

$$\begin{aligned} W_X^{(G\,W)}(t,f) &= W_{\mathcal{W}X}\Big(\frac{t}{f_r\tau(f)},\, f_r\,\xi\Big(\frac{f}{f_r}\Big)\Big) \\ &= \int_{-\infty}^{\infty} V_X\Big(\xi\Big(\frac{f}{f_r}\Big),\beta\Big)\, e^{j2\pi\frac{t}{\tau(f)}\beta}\, d\beta \\ &= \int_{-\infty}^{\infty} X\Big(f_r\, d\Big(\xi\Big(\frac{f}{f_r}\Big),\beta\Big)\Big)\, X^*\Big(f_r\, d\Big(\xi\Big(\frac{f}{f_r}\Big),-\beta\Big)\Big)\, e^{j2\pi\frac{t}{\tau(f)}\beta}\, \frac{f_r}{\sqrt{\Big|\xi'\Big(d(\xi(\frac{f}{f_r}),\beta)\Big)\,\xi'\Big(d(\xi(\frac{f}{f_r}),-\beta)\Big)\Big|}}\, d\beta \end{aligned} \tag{127}$$

where

$$V_X(b,\beta) \;=\; f_r\, U_{\mathcal{W}X}(f_r b, f_r\beta) \;=\; X(f_r\, d(b,\beta))\, X^*(f_r\, d(b,-\beta))\, \frac{f_r}{\sqrt{|\xi'(d(b,\beta))\,\xi'(d(b,-\beta))|}}$$

(cf. (9) and (116)) and $d(b,\beta) = \xi^{-1}\Big(b+\frac{\beta}{2}\Big)$ with $\xi^{-1}(b)$ defined by $\xi^{-1}(\xi(b)) = b$. The generalized time-shift covariant version of the Wigner distribution, $W_X^{(G\,W)}(t,f)$, is a member of this generalized time-shift covariant QTFR class in (126) with kernel $\psi_{W^{(G\,W)}}^{(G\,WC)}(c,b) = \delta(c)\,\delta(b)$; it satisfies the generalized time-shift covariance property in (101). We emphasize that, in general, $W_X^{(G\,W)} \neq W_X^{(G)}$, where $W_X^{(G)}$ is defined in (121).

Any QTFR $T_X^{(G\,WC)}(t,f)$ satisfies two covariance properties. As mentioned above, one property is a covariance to generalized time shifts,

$$T_{\mathcal{D}_c X}^{(G\,WC)}(t,f) \;=\; T_X^{(G\,WC)}(t-c\,\tau(f),f), \tag{128}$$

where $\mathcal{D}_c$ is defined in (103). The second property is a covariance to generalized warped frequency shifts [35, 49]. These two covariance properties follow from the fact that the unitary warping in (124) maps the time-shift operator $\mathcal{S}_\tau$ and the frequency-shift operator $\mathcal{M}_\nu$ in (1) to two new operators,

$$\mathcal{W}^{-1}\mathcal{S}_{c/f_r}\mathcal{W} \;=\; \mathcal{D}_c\,, \tag{129}$$

$$\mathcal{W}^{-1}\mathcal{M}_\nu\mathcal{W} \;=\; \tilde{\mathcal{M}}_\nu\,. \tag{130}$$

Here, the inverse operator $\mathcal{W}^{-1}$, defined by $(\mathcal{W}^{-1}\mathcal{W}X)(f) = X(f)$, is given by $(\mathcal{W}^{-1}X)(f) = \sqrt{|\xi'(\frac{f}{f_r})|}\, X\Big(f_r\, \xi(\frac{f}{f_r})\Big)$. The first operator in (129) results in the generalized time-shift operator $\mathcal{D}_c$ in (103). The second operator in (130) produces a "warped frequency shift" according to [49]

$$(\tilde{\mathcal{M}}_\nu X)(f) \;=\; \sqrt{\frac{|\xi'(\frac{f}{f_r})|}{\Big|\xi'\Big(\xi^{-1}\Big(\xi(\frac{f}{f_r}) - \frac{\nu}{f_r}\Big)\Big)\Big|}}\; X\Big(f_r\, \xi^{-1}\Big(\xi\Big(\frac{f}{f_r}\Big) - \frac{\nu}{f_r}\Big)\Big)\,. \tag{131}$$

Note that when the phase function is logarithmic, i.e. $\xi(b) = \ln b$, this warped frequency shift reduces to a scaling transformation:

$$\tilde{\mathcal{M}}_\nu = \mathcal{C}_{e^{\nu/f_r}}\,, \quad \text{for } \xi(b) = \ln b\,,$$

where $\mathcal{C}_a$ is defined in (16). Two specific cases of generalized time-shift covariant QTFRs from Cohen's class are discussed in the following.

Hyperbolic class [29]: When $\xi(b) = \ln b$ and $\tau(f) = 1/f$, the generalized warping operator $\mathcal{W}$ in (125) simplifies to the constant-Q warping operator $\mathcal{W}_H$ in (46), and the generalized time-shift covariant class is the hyperbolic class,

$$T_X^{(GWC)}(t,f) \;\equiv\; T_X^{(H)}(t,f)\,, \quad \text{for } \xi(b) = \ln b\,.$$

Also, the generalized time-shift covariant version of the Wigner distribution, $W_X^{(GW)}(t,f)$, in (127) is the Altes-Marinovich Q-distribution $Q_X^{(H)}(t,f)$ in (54). The generalized time-shift covariance in (128) becomes the hyperbolic time-shift covariance in (41) and the warped frequency-shift covariance (i.e., the covariance to the operator $\tilde{\mathcal{M}}_\nu$ in (131)) becomes the scale covariance in (42). The warped operators in (129)-(130) characteristic of this class become, respectively, the hyperbolic time-shift operator and the scaling operator,

$$\mathcal{W}_H^{-1}\mathcal{S}_{c/f_r}\mathcal{W}_H = \mathcal{H}_c\,, \qquad \mathcal{W}_H^{-1}\mathcal{M}_{f_r \ln a}\mathcal{W}_H = \mathcal{C}_a.$$

Thus, as is evident from Section 3.1, time-shift covariance and frequency-shift covariance in Cohen's class map to hyperbolic time-shift covariance and scale covariance, respectively, in the hyperbolic class [29].

Power time-shift covariant class [33, 49]: When $\xi(b) = \xi_\kappa(b)$ and $\tau(f) = \tau_\kappa(f)$ (see (68), (69)), the corresponding generalized time-shift covariant class is not the κth power class discussed in Section 4. Instead, we obtain a class that is power time-shift covariant but not scale covariant. This follows from the fact that, although the generalized time-shift covariance in (128) simplifies to the power time-shift covariance in (65), the warped frequency-shift covariance does *not* simplify to the scale covariance in (66), since the signal transformation in (131) does not simplify to a scaling transformation.

7.2 Generalized Time-Shift Covariant QTFRs from the Affine Class

When the affine class QTFRs are warped using (124) and (125), the resulting generalized time-shift covariant QTFRs, $T_X^{(GWA)}(t,f)$, are obtained as

$$T_X^{(GWA)}(t,f) \;=\; T_{\mathcal{W}X}^{(A)}\Big(\frac{t}{f_r\tau(f)},\, f_r\,\xi\Big(\frac{f}{f_r}\Big)\Big)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T^{(A)}\left(\xi\Big(\frac{f}{f_r}\Big)\left(\frac{t}{\tau(f)} - \frac{\hat{t}}{\tau(\hat{f})}\right), -\frac{\xi(\frac{\hat{f}}{f_r})}{\xi(\frac{f}{f_r})}\right) W_X^{(G\,W)}(\hat{t},\hat{f})\, d\hat{t}\, d\hat{f} \tag{132}$$

where $\psi_T^{(A)}(c,b)$ is a 2-D kernel function that uniquely characterizes the QTFR, $T^{(G\,W\,A)}$. This kernel is identical to the corresponding one of the affine class in (20). Thus, the kernel of a particular affine QTFR is equal to that of its counterpart generalized warped affine QTFR. The generalized time-shift covariant version of the Wigner distribution, $W_X^{(G\,W)}(t,f)$, is defined in (127). $W_X^{(G\,W)}(t,f)$ is a member of the class in (132) with kernel function $\psi_{W(G\,W)}^{(A)}(c,b) = \delta(c)\,\delta(b+1)$. Note that $W_X^{(G\,W)}(t,f)$ is a member of both the generalized time-shift covariant class obtained by warping Cohen's class and the generalized time-shift covariant class obtained by warping the affine class. Thus, it is a member of the intersection of these two classes.

Any QTFR $T_X^{(G\,W\,A)}(t,f)$ satisfies two covariance properties. One property is a covariance to generalized time-shifts,

$$T_{\mathcal{D}_c X}^{(G\,W\,A)}(t,f) \;=\; T_X^{(G\,W\,A)}(t - c\,\tau(f), f)\,.$$

The second property is a covariance to generalized warped scalings [35, 49]. These two covariance properties follow from the fact that the unitary warping in (124) maps the time-shift operator, $\mathcal{S}_\tau$ in (1), and the scale operator, $\mathcal{C}_a$ in (16), to two new operators [35, 49]

$$\mathcal{W}^{-1}\mathcal{S}_{c/f_r}\mathcal{W} \;=\; \mathcal{D}_c\,, \tag{133}$$

$$\mathcal{W}^{-1}\mathcal{C}_a\mathcal{W} \;=\; \tilde{\mathcal{C}}_a\,. \tag{134}$$

The first operator in (133) results in the generalized time-shift operator $\mathcal{D}_c$ in (103). The second operator in (134) results in a "warped scaling" signal transformation given by [49]

$$(\tilde{\mathcal{C}}_a X)(f) \;=\; \sqrt{\frac{|\xi'(\frac{f}{f_r})|}{\Big|a\,\xi'\Big(\xi^{-1}\Big(\frac{\xi(\frac{f}{f_r})}{a}\Big)\Big)\Big|}}\; X\left(f_r\,\xi^{-1}\Big(\frac{\xi(\frac{f}{f_r})}{a}\Big)\right). \tag{135}$$

Note that when the phase function is a power law, i.e. $\xi(b) = \xi_\kappa(b)$ in (68), the warped scaling in (135) simplifies to a conventional scaling transformation

$$\tilde{\mathcal{C}}_a = \mathcal{C}_{\xi_\kappa^{-1}(a)}\,, \quad \text{for } \xi(b) = \xi_\kappa(b)\,.$$

Two specific cases of generalized time-shift covariant QTFRs from the affine class are discussed in the following.

κth Power class [38]: When $\xi(b) = \xi_\kappa(b)$ and $\tau(f) = \tau_\kappa(f)$ (see (68), (69)), then the generalized warping operator $\mathcal{W}$ in (125) simplifies to the power warping operator $\mathcal{W}_\kappa$ in (72). Furthermore, the generalized time-shift covariant class becomes the κth power class,

$$T_X^{(G\,W\,A)}(t,f) \;\equiv\; T_X^{(\kappa)}(t,f)\,, \quad \text{for } \xi(b) = \xi_\kappa(b)\,.$$

Also, the generalized time-shift covariant version of the Wigner distribution, $W_X^{(GW)}(t,f)$ in (127), is the κth power Wigner distribution $W_X^{(\kappa)}(t,f)$ in (80). The generalized time-shift covariance in (128) simplifies to the power time-shift covariance in (65) and the warped scale covariance (i.e., the covariance to the operator $\tilde{\mathcal{C}}_a$ in (135)) is simply the scale covariance in (66). The warped operators in (133)-(134) characteristic of this class simplify, respectively, to the power time-shift operator and the scaling operator,

$$\mathcal{W}_\kappa^{-1}\mathcal{S}_{c/f_r}\mathcal{W}_\kappa = \mathcal{D}_c^{(\kappa)}, \qquad \mathcal{W}_\kappa^{-1}\mathcal{C}_{\xi_\kappa(a)}\mathcal{W}_\kappa = \mathcal{C}_a.$$

Thus, as is evident from Section 4.1, time-shift covariance and scale covariance in the affine class map to power time-shift covariance and scale covariance, respectively, in the power classes [38].

Hyperbolic time-shift covariant class [33, 49]: When $\xi(b) = \ln b$ and $\tau(f) = 1/f$, the corresponding generalized time-shift covariant class is not the hyperbolic class discussed in Section 3; instead, it is a class that is hyperbolic time-shift covariant but not scale covariant. This follows from the fact that, although the generalized time-shift covariance in (128) simplifies to the hyperbolic time-shift covariance in (41), the warped scale covariance does not simplify to the scale covariance in (42), since the signal transformation in (135) does not simplify to a scaling transformation.

7.3 Intersection Subclass of the Two Generalized Time-Shift Covariant Classes

The generalized time-shift covariant class obtained by warping Cohen's class and the generalized time-shift covariant class obtained by warping the affine class intersect to form a new subclass of QTFRs [49, 50]. This subclass is obtained from the intersection between Cohen's class and the affine class (the shift-scale covariant class [10]) using the warping in (124). The QTFRs of this subclass satisfy three covariance properties: the generalized time-shift covariance, the warped frequency-shift covariance, and the warped scale covariance. In particular, the subclass contains the generalized time-shift covariant version of the affine generalized Wigner distribution in (27) defined as

$$W_X^{(GW)(\alpha)}(t,f) = \int_{-\infty}^{\infty} V_X\Big(\xi\Big(\frac{f}{f_r}\Big) - \alpha\beta,\, \beta\Big)\, e^{j2\pi\frac{t}{\tau(f)}\beta}\, d\beta. \tag{136}$$

Note that $W_X^{(GW)(\alpha)}(t,f)$ for $\alpha = 0$ is the generalized time-shift covariant version of the Wigner distribution in (127). Any QTFR member of this intersection, $T_X^{(GWC\cap GWA)}(t,f)$, can be written in terms of $W_X^{(GW)(\alpha)}(t,f)$ as (cf. [10])

$$T_X^{(GWC\cap GWA)}(t,f) \;=\; \int_{-\infty}^{\infty} r_T(\eta)\, W_X^{(GW)(\eta)}(t,f)\, d\eta\,,$$

where $r_T(\eta)$ is a 1-D kernel function that uniquely characterizes the QTFR $T_X^{(GWC\cap GWA)}$. For $W_X^{(GW)(\alpha)}(t,f)$ in (136), this kernel is $r_{W^{(GW)(\alpha)}}(\eta) = \delta(\eta - \alpha)$.

8 SIMULATION RESULTS

We now demonstrate the importance of some of the QTFR classes studied in the paper by considering their application in analyzing various signals. In particular, we will use affine or Cohen's class QTFRs (see Sections 1 and 2) to analyze Dirac impulses (24), hyperbolic class QTFRs (see Section 3) to analyze hyperbolic impulses (56), and κth power class QTFRs (see Section 4) to analyze κth power impulses (82).

Figure 4 shows the Wigner distribution and the pseudo Wigner distribution [4] of the sum of two Dirac impulses (24) with $c = 8$ and $c = 18$, i.e. $X(f) = \frac{1}{\sqrt{f_r}}(e^{-j2\pi 8 \frac{f}{f_r}} + e^{-j2\pi 18 \frac{f}{f_r}})$. Both of these QTFRs are members of Cohen's class, and they are well-suited for analyzing Dirac impulses as they preserve constant group delay changes. In particular, the Wigner distribution is well localized along the constant group delay of the signal, but has cross terms between the two signal components. The pseudo Wigner distribution removes the cross terms with a small loss of TF concentration.

Figure 5 analyzes the sum of two hyperbolic impulses defined in (56) with parameters $c = 3$ and $c = 7$, i.e. $X(f) = \frac{1}{\sqrt{f}}(e^{-j2\pi 3 \ln \frac{f}{f_r}} + e^{-j2\pi 7 \ln \frac{f}{f_r}})$, $f > 0$. Hyperbolic QTFRs are well matched to signals like hyperbolic impulses with hyperbolic time-frequency structure. Thus, the Altes-Marinovich Q-distribution (54) in Figure 5c has good TF concentration along the two hyperbolae $t = 3/f$ and $t = 7/f$. However, it also results in cross terms along the mean hyperbola $t = 5/f$ [30]. The smoothed pseudo Q-distribution (61) in Figure 5d removes the cross terms with only some loss of time-frequency resolution. Cohen's class QTFRs, such as the Wigner distribution in 5a and the smoothed pseudo Wigner distribution in 5b, are not well matched to hyperbolic impulses. The Wigner distribution results in complicated cross terms between the two signal components as well as inner interference terms. In comparison to the smoothed pseudo Q-distribution in Figure 5d, the smoothed pseudo Wigner distribution in Figure 5b has a larger loss of TF resolution and it is as successful at removing all the cross terms.

Figure 6 analyzes the sum of two power impulses, defined in (82) with $\kappa = 3$ and parameters $c = 8$ and $c = 23$, i.e. $X(f) = \sqrt{|\tau_3(f)|}\,(e^{-j2\pi 8\, \xi_3(\frac{f}{f_r})} + e^{-j2\pi 23\, \xi_3(\frac{f}{f_r})})$. The power Wigner distribution (defined in (80) with $\kappa = 3$) in Figure 6c is well matched to the two power impulses and has very good TF concentration but large cross terms. These cross terms are effectively suppressed in the smoothed pseudo power Wigner distribution (defined in (86) with $\kappa = 3$) with only a moderate loss of TF concentration. Note that the power parameter of the two power QTFRs is matched to that of the power impulses. The Wigner distribution (defined in (10)) and the affine smoothed pseudo Wigner distribution (defined in (29)), shown in Figure 6a and Figure 6b, respectively, are both members of the affine class. The Wigner distribution in not matched to the power impulses, displaying complicated cross terms between the impulses and inner interference terms in the concave region under the bottom power impulse. The affine smoothed pseudo Wigner distribution does not remove all the cross terms and has a larger loss of TF concentration than the smoothed pseudo power Wigner distribution. Note that all QTFRs shown are scale covariant. The reason why the results of the two power class QTFRs are better than the results of the two affine class QTFRs is that the former are optimally matched to the group delays of the

two signal components [56, 57].

In the last three examples, we showed that Cohen's class QTFRs are matched to signals with constant group delay, hyperbolic class QTFRs are matched to signals with $1/f$ group delay, and power class QTFRs are matched to signals with powers of frequency group delay. We have also shown that Cohen's class or affine class QTFRs are not well suited for analyzing signals with dispersive group delay. In the next example, we further show that hyperbolic and power QTFRs are not well suited for analyzing signals with constant group delay. Figure 7 analyzes the sum of the two Dirac impulses in Figure 4 using hyperbolic class QTFRs and $\kappa = 3$ power class QTFRs. The Q-distribution in Figure 7a and the κth power Wigner distribution with $\kappa = 3$ in Figure 7c both suffer from oscillatory cross terms just like the Wigner distribution in Figure 4a, but they are not matched to the signal's TF structure resulting in a loss of TF resolution. The smoothed pseudo Q-distribution in Figure 7b is not successful at removing oscillatory cross terms, and forces a hyperbolic structure to the second impulse. The κth smoothed pseudo power Wigner distribution with $\kappa = 3$ in Figure 7d largely distorts the signal's TF structure, and does not appear to remove any cross terms. Furthermore, the smoothing imposes a power TF structure to the resulting QTFR.

9 CONCLUSION

There exist a large number of QTFRs that satisfy different desirable properties. However, no single QTFR exists that satisfies all possible desirable properties and that can be used successfully in all applications involving nonstationary signals. In this paper, we have established a generalized framework of QTFRs that is based on the scale covariance property and the generalized time-shift covariance property. These two properties are potentially useful in a large number of applications. The scale covariance property is important in multiscale or multiresolution analysis, for self-similar signals [42], scale-covariant systems [43], and in the context of the wideband Doppler effect [44]. The generalized time-shift covariance property is important when analyzing signals propagating through systems with specific dispersive characteristics. In particular, the nondispersive time-shift covariance with constant group delay (conventional time-shift) is useful in a broad range of applications (e.g., speech analysis), the dispersive time-shift covariance with hyperbolic group delay (hyperbolic time-shift) is useful for the analysis of Doppler-invariant signals similar to the signals used by bats and dolphins for echolocation [48], and the dispersive time-shift covariance with power group delay (power time-shift) is useful for the analysis of signals whose TF localization is related to a power-law TF geometry.

The generalized class of QTFRs proposed in the paper provides a unifying framework for previously considered classes of scale covariant QTFRs. These classes are obtained with specific choices of the phase and group delay functions occurring in the generalized time-shift covariance property. In particular, the linear phase function $\xi(b) = b$ characterizes the affine class, the hyperbolic phase function $\xi(b) = \ln b$ characterizes the hyperbolic class, and the power phase function $\xi(b) = \xi_\kappa(b) = \mathrm{sgn}(b)\,|b|^\kappa$, $\kappa \neq 0$, characterizes the κth power class. Table 11 demonstrates how the generalized class, and the functions and transformations corresponding to it, simplify to

the different classes considered in the paper. These particular classes exhaust all scale-covariant and generalized time-shift covariant QTFR classes that are parameterized in terms of an *arbitrary* 2-D kernel. For other phase functions, the kernel is constrained by the restrictive condition (109), which, for most phase functions, will not have any solution.

Some of the classes considered are related by unitary frequency-domain warpings of the signal combined with area-preserving TF warpings [29, 33, 35–38]. In particular, such a one-to-one correspondence exists between the hyperbolic class and Cohen's class [29], between the power classes and the affine class, and between different power classes [38]. The importance of these unitary mappings lies in the fact that the various QTFR properties relevant to the class that is being warped, map onto a new set of QTFR properties relevant to the new class obtained through the warping. For example, the time-shift covariance in (2) and the frequency-shift covariance in (3) defining Cohen's class map (through the constant-Q warping in (45)) to the hyperbolic time-shift covariance in (41) and the scale covariance in (42), respectively, defining the hyperbolic class.

In (124), we provided the unitary warping transformation that results in generalized time-shift covariant QTFRs [49]. Through this warping, the time-shift covariance in (2) or (14) maps to the generalized time-shift covariance in (101). However, neither the frequency-shift covariance in (3) nor the scale covariance in (15) map to the scale covariance in (102) except in the specific cases of power group delay functions. Thus, if scale covariance is not important for a particular application, but generalized time-shift covariance is necessary, then either Cohen's class or affine class QTFRs can be mapped using (124) to obtain QTFRs that are suitable for this application. On the other hand, if scale covariance is of equal importance as generalized time-shift covariance then the only dispersive characteristics possible are the non-dispersive characteristic offered by the affine class (see Section 2), the hyperbolic dispersive characteristic offered by the hyperbolic class (see Section 3), the power-law dispersive characteristic offered by the power classes (see Section 4), and any other dispersive characteristics (i.e. phase functions) for which there exists a kernel satisfying condition (109).

A Appendix: Group Delay Functions Satisfying Condition (112)

In the following, it is shown that the condition (112),

$$\frac{\xi(\alpha b) - \xi(\alpha b_0)}{\alpha\, \xi'(\alpha)} \equiv \frac{\xi(b) - \xi(b_0)}{\xi'(1)}\,, \qquad \forall\, b, b_0, \alpha\,, \tag{A.1}$$

is only satisfied if the derivative of the phase function $\xi(b)$ is proportional to a power function. The proof given here is a simplified version of a proof due to P. Flandrin [58].

Taking the derivative with respect to b of both sides of (A.1), we obtain $\frac{\alpha\, \xi'(\alpha b)}{\alpha\, \xi'(\alpha)} = \frac{\xi'(b)}{\xi'(1)}$ and further, for $\alpha \neq 0$, $\xi'(\alpha b) = \xi'(\alpha)\, \frac{\xi'(b)}{\xi'(1)}$. Dividing both sides through $\xi'(1)$ yields

$$\overline{\xi'}(\alpha b) \;=\; \overline{\xi'}(\alpha)\, \overline{\xi'}(b) \quad \text{with} \quad \overline{\xi'}(b) \;\triangleq\; \frac{\xi'(b)}{\xi'(1)}\,.$$

This is satisfied if and only if $\overline{\xi'}(b)$ is a power function, i.e., $\overline{\xi'}(b) = b^p$ or $\overline{\xi'}(b) = |b|^p$. We shall adopt the second solution since for $b < 0$ the first solution is not real for all $p \in \mathbb{R}$. It follows that $\xi'(b) = \xi'(1)\, \overline{\xi'}(b) = K\, |b|^{\kappa - 1}$

with $K \neq 0$ arbitrary (note that we have substituted $p = \kappa - 1$). For $\kappa \neq 0$, setting $K = \kappa$ and recalling that $\tau(f) = \frac{1}{f_r}\,\xi'\!\left(\frac{f}{f_r}\right)$, we obtain the group delay function of the power classes,

$$\tau(f) \;=\; \frac{\kappa}{f_r}\left|\frac{f}{f_r}\right|^{\kappa-1} \;=\; \tau_\kappa(f)\,, \qquad \kappa \neq 0\,.$$

For $\kappa = 0$, setting $K = 1$ we obtain the group delay function $\tau(f) = 1/|f|$; for $f > 0$, this is the group delay function of the hyperbolic class.

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List of Footnotes

[1]The advantages and disadvantages of the covariance and warping approaches in obtaining QTFRs are discussed in [41].

[2]Note that the QTFR kernel $F_T^{(A)}(\beta)$ is the negative of the corresponding kernel in [5]. For example, the $F_T^{(A)}(\beta)$ kernel for the Wigner distribution, $W_X(t,f)$, in this paper is $F_W^{(A)}(\beta) = -1$; the corresponding kernel for the Wigner distribution in [5] is $+1$.

[3]We note that the mapping $(\mathcal{W}_\kappa X)(f)$ has been introduced in [54] in relation to wavelet transforms. Also, a generalization of the power warping concept is discussed and applied to Cohen's class in [33–35, 37, 49, 50].

[4]*Area-preserving* means that the transform's Jacobian is 1.

[5]The kernels $\phi_T^{(A)}(c,\zeta)$, $\Phi_T^{(A)}(b,\beta)$, and $\psi_T^{(A)}(c,b)$ in (74)-(76) and in (18)-(20) differ from the corresponding kernels in [38] in that the c and b arguments are the negative of the corresponding ones in [38].

[6]Note that if the powergram is computed from its affine counterpart, the scalogram, using the warping in (71), then the scalogram in (28) should use a window $\Theta(f) = (\mathcal{W}_\kappa \tilde{\Theta})(f)$.

[7]The mth power warped version of the Bertrand P_κ-distributions is obtained by warping the affine Bertrand P_κ-distributions with κ replaced with m in (71), but not in the Bertrand P_κ-distributions formulation in (33). We use the power m instead of κ in the warping in order to avoid any confusion with the parameter κ of the Bertrand P_κ-distributions. In particular, when $m = 1$, $P_{\kappa_X}^{(m)}(t,f)$ in (88) equals the Bertrand P_κ-distributions, $P_{\kappa_X}(t,f)$ in (33).

[8]In the generalized class, the range of the frequency f depends on the domain of the phase function $\xi(\frac{f}{f_r})$, where $f_r > 0$. Thus, although all integrations in the generalized class with respect to frequency are shown to be from $-\infty$ to ∞, they are actually over the domain of $\xi(\frac{f}{f_r})$, i.e., the set of all f for which $\xi(\frac{f}{f_r})$ is defined.

[9]The idea of general unitary warpings from Cohen's class or the affine class has been proposed in [35–37]. Here, we are considering the special warping that leads to the generalized time-shift covariance property in (101) [33, 34, 49, 50].

[10]Here, by "generalized warped" we do not mean that the warping is generalized but that it leads to QTFRs satisfying the generalized time-shift covariance property in (101).

List of Figure Captions

Figure 1: A pictorial summary of the different classes of QTFRs considered in the paper: Cohen's class, affine class, hyperbolic class, and κth power class ($\kappa \neq 0, 1$) together with their intersection subclasses and some important QTFR members. The spectrogram, SPEC, the Wigner distribution, WD, the generalized WD, GWD, the smoothed pseudo-WD, SPWD, the Cohen-Bertrand P_0, CP_0, the Choi-Williams exponential distribution, ED, the generalized exponential distribution, GED, and the reduced interference distributions (RID) are members of Cohen's class. The WD, the GWD, the ED, the RID, the scalogram, SCAL, the Flandrin D distribution, FD, the passive Unterberger distribution, PUD, the active Unterberger distribution, AUD, the Bertrand P_0-distribution, the Bertrand P_1-distribution, and the Bertrand P_κ-distributions, $\kappa \neq 0, 1$, are members of the affine class. The affine-Cohen's intersection (**AC**) contains the WD, the GWD, the ED, and the RID. The hyperbologram, HYP, the Altes-Marinovich Q-distribution, AD, the generalized AD, GAD, the pseudo AD, PAD, the smoothed pseudo AD, SPAD, the P_0-distribution, and the power Bertrand P_0-distribution, $P_0^{(\kappa)}$, are members of the hyperbolic class. The affine-hyperbolic intersection (**AH**) contains the P_0-distribution. The powergram, POW, the power Wigner distribution, PWD, the power GWD, PGWD, the P_κ-distributions, the $P_0^{(\kappa)}$-distributions, the power FD, PFD, the power PUD, PPUD, and the power AUD, PAUD, are all members of the power classes. The affine-power intersection (**AP**) contains the P_κ-distributions. The hyperbolic-power intersection (**HP**) contains the $P_0^{(\kappa)}$-distributions. Note that the hyperbolic and Cohen's classes, and the affine and power classes are related through unitary mappings.

Figure 2: A pictorial summary of the affine class, the hyperbolic class, and the κth power class ($\kappa \neq 0, 1$) that adds to Figure 1 the localized-kernel affine subclass (**LAC**), the localized-kernel hyperbolic subclass (**LHC**), and the localized-kernel κth power class (**LPC**). Recall from Figure 1 that **AH** stands for the affine-hyperbolic intersection, **AP** stands for the affine-power intersection, and **HP** stands for the hyperbolic-power intersection. The acronyms for the different QTFRs shown are defined in the caption of Figure 1.

Figure 3: The generalized time-shift operator $\mathcal{D}_c$ corresponds to a group delay curve $t = c\tau(f)$ in the TF plane. (In this figure, c is assumed to be positive.)

Figure 4: Time-Frequency analysis of the sum of two Dirac impulses. (a) Wigner distribution, (b) pseudo Wigner distribution.

Figure 5: Time-Frequency analysis of the sum of two hyperbolic impulses. (a) Wigner distribution, (b) smoothed pseudo Wigner distribution, (c) Altes-Marinovich Q-distribution, (d) smoothed pseudo Q-distribution.

Figure 6: Time-Frequency analysis of the sum of two power impulses with $\kappa = 3$. (a) Wigner distribution, (b) affine smoothed pseudo Wigner distribution, (c) κth power Wigner distribution with $\kappa = 3$, (d) κth smoothed pseudo power Wigner distribution with $\kappa = 3$.

Figure 7: Time-Frequency analysis of the sum of two Dirac impulses. (a) Q-distribution, (b) smoothed pseudo Q-distribution, (c) κth power Wigner distribution with $\kappa = 3$, (d) κth smoothed pseudo power Wigner distribution with $\kappa = 3$.

Table 1: Classification of QTFRs based on covariance properties with respect to elementary signal transformations. The covariance property in the fourth column is guaranteed to hold for all QTFRs in the classes that are checked off in the last column. Cohen's class QTFRs satisfy the time-shift covariance (i) and the frequency-shift covariance (ii); the affine class QTFRs satisfy the time-shift covariance (i) and the scale covariance (iii); the hyperbolic class QTFRs satisfy the hyperbolic time-shift covariance (iv) and the scale covariance (iii); the power classes QTFRs satisfy the power time-shift covariance (v) and the scale covariance (iii); the generalized class QTFRs satisfy the generalized time-shift covariance (vi) and the scale covariance (iii). Note that f_r is a positive reference frequency.

Name of covariance property	Operator symbol	Effect of operator on signal $X(f)$	Covariance property	QTFR classes				
				Cohen's	Affine	Hyperbolic	Power	Generalized
(i) Time-shift	$\mathcal{S}_\tau$	$(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f}\, X(f)$	$T_{\mathcal{S}_\tau X}(t,f) = T_X(t-\tau, f)$	$\surd$	$\surd$			
(ii) Frequency shift	$\mathcal{M}_\nu$	$(\mathcal{M}_\nu X)(f) = X(f-\nu)$	$T_{\mathcal{M}_\nu X}(t,f) = T_X(t, f-\nu)$	$\surd$				
(iii) Scale	$\mathcal{C}_a$	$(\mathcal{C}_a X)(f) = \frac{1}{\sqrt{\lvert a\rvert}}\, X(\frac{f}{a})$	$T_{\mathcal{C}_a X}(t,f) = T_X(at, \frac{f}{a})$		$\surd$	$\surd$	$\surd$	$\surd$
(iv) Hyperbolic time-shift	$\mathcal{H}_c$	$(\mathcal{H}_c X)(f) = e^{-j2\pi c \ln\frac{f}{f_r}}\, X(f)$	$T_{\mathcal{H}_c X}(t,f) = T_X\left(t - \frac{c}{f}, f\right)$			$\surd$		
(v) Power time-shift	$\mathcal{D}_c^{(\kappa)}$	$(\mathcal{D}_c^{(\kappa)} X)(f) = e^{-j2\pi c\, \xi_\kappa(\frac{f}{f_r})}\, X(f)$ $\xi_\kappa(b) = \mathrm{sgn}(b)\, \lvert b\rvert^\kappa$	$T_{\mathcal{D}_c^{(\kappa)} X}(t,f) = T_X\left(t - c\,\tau_\kappa(f), f\right)$ $\tau_\kappa(f) = \frac{d}{df}\, \xi_\kappa(\frac{f}{f_r})$				$\surd$	
(vi) Generalized time-shift	$\mathcal{D}_c$	$(\mathcal{D}_c X)(f) = e^{-j2\pi c\, \xi(\frac{f}{f_r})}\, X(f)$	$T_{\mathcal{D}_c X}(t,f) = T_X\left(t - c\,\tau(f), f\right)$ $\tau(f) = \frac{d}{df}\, \xi(\frac{f}{f_r})$					$\surd$

Table 2: Desirable properties and corresponding kernel constraints in the affine class and the localized-kernel affine subclass. Note that $\rho_X^{(A)}(c)$ is defined in (26). For the localized-kernel affine class, the corresponding kernel is given by $\Phi_T^{(A)}(b,\beta) = G_T^{(A)}(\beta)\,\delta\Big(b - F_T^{(A)}(\beta)\Big)$. Also, $F_T^{(A)\prime}(\beta) = \frac{d}{d\beta}F_T^{(A)}(\beta)$ and $G_T^{(A)\prime}(\beta) = \frac{d}{d\beta}G_T^{(A)}(\beta)$.

AFFINE CLASS QTFR PROPERTY	AFFINE CLASS KERNEL CONSTRAINT	LOCALIZED-KERNEL AFFINE CLASS KERNEL CONSTRAINT
P_1 Scale covariance: $(\mathcal{C}_a X)(f) = \frac{1}{\sqrt{\lvert a\rvert}} X\big(\frac{f}{a}\big) \;\Rightarrow$ $T_{\mathcal{C}_a X}^{(A)}(t,f) = T_X^{(A)}\Big(at, \frac{f}{a}\Big)$	always satisfied	always satisfied
P_2 Time-shift covariance: $(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f} X(f) \;\Rightarrow$ $T_{\mathcal{S}_\tau X}^{(A)}(t,f) = T_X^{(A)}(t-\tau, f)$	always satisfied	always satisfied
P_3 Real-valuedness: $T_X^{(A)}(t,f) = T_X^{(A)*}(t,f)$	$\Phi_T^{(A)}(b,\beta) = \Phi_T^{(A)*}(b,-\beta)$	$F_T^{(A)}(\beta) = F_T^{(A)}(-\beta)$ $G_T^{(A)}(\beta) = G_T^{(A)*}(-\beta)$
P_4 Energy distribution: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_X^{(A)}(t,f)\,dt\,df =$ $\int_{-\infty}^{\infty} \lvert X(f)\rvert^2\,df$	$\int_{-\infty}^{\infty} \Phi_T^{(A)}(b,0)\frac{db}{\lvert b\rvert} = 1$	$G_T^{(A)}(0) = -F_T^{(A)}(0)$
P_5 Frequency marginal: $\int_{-\infty}^{\infty} T_X^{(A)}(t,f)\,dt = \lvert X(f)\rvert^2$	$\Phi_T^{(A)}(b,0) = \delta(b+1)$	$G_T^{(A)}(0) = -F_T^{(A)}(0) = 1$
P_6 Time marginal: $\frac{1}{f_r}\int_{-\infty}^{\infty} T_X^{(A)}\big(\frac{c}{f_r}, f\big)\,df = \lvert\rho_X^{(A)}(c)\rvert^2$	$\int_{-\infty}^{\infty} \Phi_T^{(A)}(b,\alpha b)\frac{db}{\lvert b\rvert} = 1, \;\; \forall\alpha$	$\int_{-\infty}^{\infty} G_T^{(A)}(\alpha b)\,\delta\big(b - F_T^{(A)}(\alpha b)\big)$ $\cdot\frac{db}{\lvert b\rvert} = 1, \;\; \forall\alpha$
P_7 Frequency localization: $X(f) = \delta(f-\hat{f}) \Rightarrow$ $T_X^{(A)}(t,f) = \delta(f-\hat{f})$	$\Phi_T^{(A)}(b,0) = \delta(b+1)$	$G_T^{(A)}(0) = -F_T^{(A)}(0) = 1$
P_8 Time localization: $I_c^{(A)}(f) = \frac{1}{\sqrt{f_r}}\,e^{-j2\pi c\frac{f}{f_r}} \;\Rightarrow$ $T_{I_c^{(A)}}^{(A)}(t,f) = \frac{1}{f_r}\,\delta\big(t - \frac{c}{f_r}\big)$	$\int_{-\infty}^{\infty} \Phi_T^{(A)}(b,\beta)\,db = 1, \;\; \forall\beta$	$G_T^{(A)}(\beta) = 1, \;\; \forall\beta$
P_9 Moyal's formula/unitarity [25, 27]: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{X_1}^{(A)}(t,f)\,T_{X_2}^{(A)*}(t,f)\,dt\,df =$ $\Big\lvert\int_{-\infty}^{\infty} X_1(f)\,X_2^*(f)\,df\Big\rvert^2$	$\int_{-\infty}^{\infty} \Phi_T^{(A)*}(b\beta,\alpha\beta)\,\Phi_T^{(A)}(\beta,\alpha\beta)\,d\beta$ $= \delta(b-1), \quad \forall\alpha$	$\int_{-\infty}^{\infty} \delta\big(b\beta - F_T^{(A)}(\alpha\beta)\big)$ $\cdot\,\delta\big(\beta - F_T^{(A)}(\alpha\beta)\big)\,\lvert G_T^{(A)}(\alpha\beta)\rvert^2$ $\cdot\,d\beta = \delta(b-1), \quad \forall\alpha$
P_{10} Group delay: $\frac{\int_{-\infty}^{\infty} t\,T_X^{(A)}(t,f)\,dt}{\int_{-\infty}^{\infty} T_X^{(A)}(t,f)\,dt} = -\frac{1}{2\pi}\,\frac{d}{df}\arg X(f)$	$\Phi_T^{(A)}(b,0) = \delta(b+1)$ and $\frac{\partial}{\partial\beta}\Phi_T^{(A)}(b,\beta)\Big\rvert_{\beta=0} = 0$	$G_T^{(A)}(0) = -F_T^{(A)}(0) = 1$ and $G_T^{(A)\prime}(0) = F_T^{(A)\prime}(0) = 0$
P_{11} Finite frequency support: $X(f) = 0$ for $f \notin [f_1, f_2] \;\Rightarrow$ $T_X^{(A)}(t,f) = 0$ for $f \notin [f_1, f_2]$	$\Phi_T^{(A)}(b,\beta) = 0$ for $\big\lvert\frac{b+1}{\beta}\big\rvert > \frac{1}{2}$	$G_T^{(A)}(\beta) = 0$ for $\big\lvert F_T^{(A)}(\beta) + 1\big\rvert > \big\lvert\frac{\beta}{2}\big\rvert$
P_{12} Finite time support: $\rho_X^{(A)}(c) = 0$ for $c \notin [c_1, c_2] \;\Rightarrow$ $T_X^{(A)}(t,f) = 0$ for $t \notin [\frac{c_1}{f_r}, \frac{c_2}{f_r}]$	$\phi_T^{(A)}(c,\zeta) = 0$ for $\big\lvert\frac{c}{\zeta}\big\rvert > \frac{1}{2}$	$\int_{-\infty}^{\infty} G_T^{(A)}(\beta)\,e^{j2\pi\zeta F_T^{(A)}(\beta)}$ $\cdot\,e^{j2\pi c\beta}d\beta = 0$ for $\big\lvert\frac{c}{\zeta}\big\rvert > \frac{1}{2}$
P_{13} Frequency-shift covariance: $(\mathcal{M}_\nu X)(f) = X(f-\nu) \;\Rightarrow$ $T_{\mathcal{M}_\nu X}^{(A)}(t,f) = T_X^{(A)}(t, f-\nu)$	$\Psi_T^{(A)}(\zeta,\beta) = S_T(\zeta\beta)\,e^{-j2\pi\zeta}$	$F_T^{(A)}(\beta) = -1$ $G_T^{(A)}(\beta) = S_T(\alpha\beta), \;\; \forall\alpha$

Table 3: Kernels and properties (defined in Table 2) of some QTFRs of the affine class. Note that $u_X(t,\tau)$ and $U_X(f,\nu)$, $W_X(t,f)$, and $A_X(\tau,\nu)$ are defined in (9), (10), and (11), respectively. Also, $s(c)$ and $S(\beta)$, and $h(\zeta)$ and $H(b)$ are Fourier transform pairs.

AFFINE QTFR	KERNELS	PROPERTIES SATISFIED
Wigner distribution, W	$\phi_W^{(A)}(c,\zeta) = e^{-j2\pi\zeta}\,\delta(c)$ $\Phi_W^{(A)}(b,\beta) = \delta(b+1)$ $\psi_W^{(A)}(c,b) = \delta(c)\,\delta(b+1)$ $\Psi_W^{(A)}(\zeta,\beta) = e^{-j2\pi\zeta}$	P_1–P_{13}
Generalized Wigner distribution, $W^{(\alpha)}$	$\phi_{W^{(\alpha)}}^{(A)}(c,\zeta) = e^{-j2\pi\zeta}\,\delta(c+\alpha\zeta)$ $\Phi_{W^{(\alpha)}}^{(A)}(b,\beta) = \delta(b+1-\alpha\beta)$ $\psi_{W^{(\alpha)}}^{(A)}(c,b) = \frac{1}{\lvert\alpha\rvert}\,e^{j2\pi\frac{c(b+1)}{\alpha}}$ $\Psi_{W^{(\alpha)}}^{(A)}(\zeta,\beta) = e^{j2\pi\alpha\zeta\beta}\,e^{-j2\pi\zeta}$	P_1, P_2, P_4–P_9, P_{13} P_{11} and P_{12} if $\lvert\alpha\rvert < \frac{1}{2}$
Scalogram, SCAL with analysis wavelet, $\Theta(f)$	$\phi_{\mathrm{SCAL}}^{(A)}(c,\zeta) = \frac{1}{f_r}\,u_\Theta(-\frac{c}{f_r},-\frac{\zeta}{f_r})$ $\Phi_{\mathrm{SCAL}}^{(A)}(b,\beta) = f_r\,U_\Theta(-f_r b,-f_r\beta)$ $\psi_{\mathrm{SCAL}}^{(A)}(c,b) = W_\Theta(-\frac{c}{f_r},-f_r b)$ $\Psi_{\mathrm{SCAL}}^{(A)}(\zeta,\beta) = A_\Theta(-\frac{\zeta}{f_r},-f_r\beta)$	P_1–P_3, P_4 if $\int_{-\infty}^{\infty}\lvert\Theta(f)\rvert^2\,\frac{df}{\lvert f\rvert} = \frac{1}{f_r}$
Affine smoothed pseudo Wigner distribution, ASPW	$\phi_{\mathrm{ASPW}}^{(A)}(c,\zeta) = s(c)h(\zeta)$ $\Phi_{\mathrm{ASPW}}^{(A)}(b,\beta) = H(b)S(\beta)$ $\psi_{\mathrm{ASPW}}^{(A)}(c,b) = s(c)H(b)$ $\Psi_{\mathrm{ASPW}}^{(A)}(\zeta,\beta) = h(\zeta)S(\beta)$	P_1, P_2 P_3 if $H(b), s(c) \in R$ P_4 if $S(0)\int_{-\infty}^{\infty} H(b)\frac{db}{\lvert b\rvert} = 1$
Bertrand P_0-distribution, P_0	$\phi_{P_0}^{(A)}(c,\zeta) = \int_{-\infty}^{\infty} G_{P_0}^{(A)}(\beta)\,e^{j2\pi(c\beta+\zeta F_{P_0}^{(A)}(\beta))}\,d\beta$ $\Phi_{P_0}^{(A)}(b,\beta) = G_{P_0}^{(A)}(\beta)\,\delta(b-F_{P_0}^{(A)}(\beta))$ $\psi_{P_0}^{(A)}(c,b) = \int_{-\infty}^{\infty} G_{P_0}^{(A)}(\beta)\,\delta(b-F_{P_0}^{(A)}(\beta))\,e^{j2\pi c\beta}\,d\beta$ $\Psi_{P_0}^{(A)}(\zeta,\beta) = G_{P_0}^{(A)}(\beta)\,e^{j2\pi\zeta F_{P_0}^{(A)}(\beta)}$ with $F_{P_0}^{(A)}(\beta) = -\frac{\beta}{2}\coth(\frac{\beta}{2})$, $G_{P_0}^{(A)}(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$	P_1–P_5, P_7, P_9–P_{11}
Flandrin D-distribution, D	same kernels as P_0 with $F_D^{(A)}(\beta) = -1-(\frac{\beta}{4})^2$, $G_D^{(A)}(\beta) = 1-(\frac{\beta}{4})^2$	P_1–P_7, P_{10}
Passive Unterberger distribution, PUD	same kernels as P_0 with $F_{\mathrm{PUD}}^{(A)}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}$, $G_{\mathrm{PUD}}^{(A)}(\beta) = \frac{1}{\sqrt{1+(\frac{\beta}{2})^2}}$	P_1–P_7, P_{10}, P_{11}
Active Unterberger distribution, AUD	same kernels as P_0 with $F_{\mathrm{AUD}}^{(A)}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}$, $G_{\mathrm{AUD}}^{(A)}(\beta) = 1$	P_1–P_5, P_7, P_8, P_{10}, P_{11}
Bertrand P_κ-distributions, P_κ	$\Phi_{P_\kappa}^{(A)}(b,\beta) = \int_{-\infty}^{\infty}\delta(-b+\frac{\beta}{2}-\lambda_\kappa(u))$ $\delta(-b-\frac{\beta}{2}-\lambda_\kappa(-u))\,\mu(u)du$	P_1, P_2, other properties depend on $\mu(u)$ [20]

Table 4: Desirable QTFR properties and corresponding kernel constraints in the hyperbolic class and the localized-kernel hyperbolic subclass. For the localized-kernel hyperbolic class, the corresponding kernel is given by $\Phi_T^{(H)}(b,\beta) = G_T^{(H)}(\beta)\,\delta\Big(b - F_T^{(H)}(\beta)\Big)$. Also, $F_T^{(H)\prime}(\beta) = \frac{d}{d\beta}F_T^{(H)}(\beta)$ and $G_T^{(H)\prime}(\beta) = \frac{d}{d\beta}G_T^{(H)}(\beta)$.

HYPERBOLIC CLASS QTFR PROPERTY	HYPERBOLIC CLASS KERNEL CONSTRAINT	LOCALIZED-KERNEL HYPERBOLIC CLASS KERNEL CONSTRAINT
P_1 Scale Covariance: $(\mathcal{C}_a X)(f) = \frac{1}{\sqrt{\lvert a\rvert}} X\big(\frac{f}{a}\big) \;\Rightarrow$ $T_{\mathcal{C}_a X}^{(H)}(t,f) = T_X^{(H)}\Big(at, \frac{f}{a}\Big)$	always satisfied	always satisfied
P_2 Hyperbolic time-shift covariance: $(\mathcal{H}_c X)(f) = e^{-j2\pi c \ln\frac{f}{f_r}} X(f) \;\Rightarrow$ $T_{\mathcal{H}_c X}^{(H)}(t,f) = T_X^{(H)}\big(t - \frac{c}{f}, f\big)$	always satisfied	always satisfied
P_3 Real-valuedness: $T_X^{(H)}(t,f) = T_X^{(H)*}(t,f)$	$\Phi_T^{(H)}(b,\beta) = \Phi_T^{(H)*}(b,-\beta)$	$F_T^{(H)}(\beta) = F_T^{(H)}(-\beta)$ $G_T^{(H)}(\beta) = G_T^{(H)*}(-\beta)$
P_4 Energy distribution: $\int_{-\infty}^{\infty}\int_0^{\infty} T_X^{(H)}(t,f)\,dt\,df = \int_0^{\infty} \lvert X(f)\rvert^2\,df$	$\int_{-\infty}^{\infty} \Phi_T^{(H)}(b,0)\,db = 1$	$G_T^{(H)}(0) = 1$
P_5 Frequency marginal: $\int_{-\infty}^{\infty} T_X^{(H)}(t,f)\,dt = \lvert X(f)\rvert^2$	$\Phi_T^{(H)}(b,0) = \delta(b)$	$F_T^{(H)}(0) = 0$ $G_T^{(H)}(0) = 1$
P_6 Hyperbolic marginal: $\int_0^{\infty} T_X^{(H)}\big(\frac{c}{f}, f\big)\,\frac{df}{f} = \lvert\rho_X^{(H)}(c)\rvert^2$	$\int_{-\infty}^{\infty} \Phi_T^{(H)}(b,\beta)\,db = 1, \;\forall\beta$	$G_T^{(H)}(\beta) = 1, \;\forall\beta$
P_7 Frequency localization: $X(f) = \delta(f-\hat{f}) \Rightarrow T_X^{(H)}(t,f) = \delta(f-\hat{f})$	$\Phi_T^{(H)}(b,0) = \delta(b)$	$F_T^{(H)}(0) = 0$ $G_T^{(H)}(0) = 1$
P_8 Hyperbolic localization: $H_c(f) = \frac{1}{\sqrt{f}}\,e^{-j2\pi c\ln\frac{f}{f_r}} \;\Rightarrow$ $T_{H_c}^{(H)}(t,f) = \frac{1}{f}\,\delta\big(t - \frac{c}{f}\big), \;\; f > 0$	$\int_{-\infty}^{\infty} \Phi_T^{(H)}(b,\beta)\,db = 1, \;\forall\beta$	$G_T^{(H)}(\beta) = 1, \;\forall\beta$
P_9 Moyal's formula/unitarity [25,27]: $\int_0^{\infty}\int_{-\infty}^{\infty} T_{X_1}^{(H)}(t,f)\,T_{X_2}^{(H)*}(t,f)\,dt\,df =$ $\Big\lvert\int_0^{\infty} X_1(f)\,X_2^*(f)\,df\Big\rvert^2$	$\int_{-\infty}^{\infty} \Phi_T^{(H)*}(b+\alpha,\beta)$ $\cdot\,\Phi_T^{(H)}(b,\beta)\,db = \delta(\alpha), \;\forall\beta$	$\lvert G_T^{(H)}(\beta)\rvert = 1, \;\forall\beta$
P_{10} Group delay: $\frac{\int_{-\infty}^{\infty} t\,T_X^{(H)}(t,f)\,dt}{\int_{-\infty}^{\infty} T_X^{(H)}(t,f)\,dt} = -\frac{1}{2\pi}\,\frac{d}{df}\arg X(f)$	$\Phi_T^{(H)}(b,0) = \delta(b)$ and $\frac{\partial}{\partial\beta}\Phi_T^{(H)}(b,\beta)\Big\rvert_{\beta=0} = 0$	$G_T^{(H)}(0) = 1$, $F_T^{(H)}(0) = 0$, $G_T^{(H)\prime}(0) = F_T^{(H)\prime}(0) = 0$
P_{11} Finite frequency support: $X(f) = 0$ for $f \notin [f_1, f_2] \;\Rightarrow$ $T_X^{(H)}(t,f) = 0$ for $f \notin [f_1, f_2]$	$\Phi_T^{(H)}(b,\beta) = 0$ for $\big\lvert\frac{b}{\beta}\big\rvert > \frac{1}{2}$	$G_T^{(H)}(\beta) = 0$ for $\lvert F_T^{(H)}(\beta)\rvert > \big\lvert\frac{\beta}{2}\big\rvert$
P_{12} Finite hyperbolic support: $\rho_X^{(H)}(c) = 0 \;\;$ for $\;\; c \notin [c_1, c_2] \;\Rightarrow$ $T_X^{(H)}(t,f) = 0 \;\;$ for $\;\; t \notin [\frac{c_1}{f}, \frac{c_2}{f}]$	$\phi_T^{(H)}(c,\zeta) = 0$ for $\big\lvert\frac{c}{\zeta}\big\rvert > \frac{1}{2}$	$\int_{-\infty}^{\infty} G_T^{(H)}(\beta)\,e^{j2\pi\zeta F_T^{(H)}(\beta)}$ $\cdot\,e^{j2\pi c\beta}d\beta = 0$ for $\big\lvert\frac{c}{\zeta}\big\rvert > \frac{1}{2}$
P_{13} Time-shift covariance: $(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f} X(f) \;\Rightarrow$ $T_{\mathcal{S}_\tau X}^{(H)}(t,f) = T_X^{(H)}(t-\tau, f)$	$\Phi_T^{(H)}(b,\beta) =$ $G_T^{(H)}(\beta)\,\delta\Big(b - \ln(\frac{\sinh(\beta/2)}{\beta/2})\Big)$, $G_T^{(H)}(\beta)$ arbitrary	$F_T^{(H)}(\beta) = \ln(\frac{\sinh(\beta/2)}{\beta/2})$, $G_T^{(H)}(\beta)$ arbitrary

Table 5: Kernels and properties (defined in Table 4) of some QTFRs of the hyperbolic class. Note that $v_X^{(H)}(c,\zeta)$, $V_X^{(H)}(b,\beta)$, $Q_X(t,f)$, $A_X^{(H)}(\zeta,\beta)$, and $\rho_X^{(H)}(c)$ are defined in (51), (52), (54), (55), and (53), respectively. Also, $s(c)$ and $S(\beta)$ are a Fourier transform pair.

HYPERBOLIC QTFR	KERNELS	PROPERTIES SATISFIED
Altes-Marinovich Q-distribution, Q	$\phi_Q^{(H)}(c,\zeta)=\delta(c)$ $\Phi_Q^{(H)}(b,\beta)=\delta(b)$ $\psi_Q^{(H)}(c,b)=\delta(c)\,\delta(b)$ $\Psi_Q^{(H)}(\zeta,\beta)=1$	P_1–P_{12}
Generalized Altes-Marinovich Q-distribution, $Q^{(\alpha)}$	$\phi_{Q^{(\alpha)}}^{(H)}(c,\zeta)=\delta(c+\alpha\zeta)$ $\Phi_{Q^{(\alpha)}}^{(H)}(b,\beta)=\delta(b-\alpha\beta)$ $\psi_{Q^{(\alpha)}}^{(H)}(c,b)=\frac{1}{\lvert\alpha\rvert}\,e^{j2\pi\frac{cb}{\alpha}}$ $\Psi_{Q^{(\alpha)}}^{(H)}(\zeta,\beta)=e^{j2\pi\alpha\zeta\beta}$	P_1, P_2, P_4–P_9, P_{11} and P_{12} if $\lvert\alpha\rvert<\frac{1}{2}$
Power Bertrand P_0-distribution, $P_0^{(\kappa)}$ (reduces to the Bertrand P_0-distribution for $\kappa=1$)	$\phi_{P_0^{(\kappa)}}^{(H)}(c,\zeta)=\int_{-\infty}^{\infty}e^{j2\pi(c\beta-\zeta F_{P_0^{(\kappa)}}^{(H)}(\beta))}\,d\beta$ $\Phi_{P_0^{(\kappa)}}^{(H)}(b,\beta)=\delta(b-F_{P_0^{(\kappa)}}^{(H)}(\beta))$ $\psi_{P_0^{(\kappa)}}^{(H)}(c,b)=\int_{-\infty}^{\infty}\delta(b-F_{P_0^{(\kappa)}}^{(H)}(\beta))\,e^{j2\pi c\beta}\,d\beta$ $\Psi_{P_0^{(\kappa)}}^{(H)}(\zeta,\beta)=e^{-j2\pi\zeta F_{P_0^{(\kappa)}}^{(H)}(\beta)}$ with $F_{P_0^{(\kappa)}}^{(H)}(\beta)=\frac{1}{\kappa}\ln\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}$	P_1–P_{11}, P_{12}??[a] P_{13} for $\kappa=1$ only
Hyperbologram, Y with analysis wavelet, $\Theta(f)$	$\phi_Y^{(H)}(c,\zeta)=v_\Theta^{(H)}(-c,-\zeta)$ $\Phi_Y^{(H)}(b,\beta)=V_\Theta^{(H)}(-b,-\beta)$ $\psi_Y^{(H)}(c,b)=Q_\Theta\left(\frac{-c}{f_r e^{-b}},f_r e^{-b}\right)$ $\Psi_Y^{(H)}(\zeta,\beta)=A_\Theta^{(H)}(-\zeta,-\beta)$	P_1–P_3, P_4 if $\int_0^\infty\lvert\Theta(f)\rvert^2\,df=1$
Pseudo Altes-Marinovich Q-distribution, PAD with analysis wavelet, $\Theta(f)$	$\phi_{\mathrm{PAD}}^{(H)}(c,\zeta)=f_r\,\delta(c)\,v_\Theta^{(H)}(0,\zeta)$ $\Phi_{\mathrm{PAD}}^{(H)}(b,\beta)=f_r\,Q_\Theta(0,f_r e^b)$ $\psi_{\mathrm{PAD}}^{(H)}(c,b)=f_r\,\delta(c)\,Q_\Theta(0,f_r e^b)$ $\Psi_{\mathrm{PAD}}^{(H)}(\zeta,\beta)=f_r\,v_\Theta^{(H)}(0,\zeta)$	P_1–P_3, P_{12}; P_4, P_6, P_8 if $\lvert\rho_\Theta^{(H)}(0)\rvert^2=\frac{1}{f_r}$
Smoothed pseudo Altes-Marinovich Q-distribution, SPAD with analysis wavelet, $\Theta(f)$, and smoothing function, $s(c)$	$\phi_{\mathrm{SPAD}}^{(H)}(c,\zeta)=f_r\,s(c)\,v_\Theta^{(H)}(0,\zeta)$ $\Phi_{\mathrm{SPAD}}^{(H)}(b,\beta)=f_r\,S(\beta)\,Q_\Theta(0,f_r e^b)$ $\psi_{\mathrm{SPAD}}^{(H)}(c,b)=f_r\,s(c)\,Q_\Theta(0,f_r e^b)$ $\Psi_{\mathrm{SPAD}}^{(H)}(\zeta,\beta)=f_r\,S(\beta)\,v_\Theta^{(H)}(0,\zeta)$	P_1, P_2; P_3 if $s(c)\in\mathbb{R}$, P_4 if $S(0)\,\lvert\rho_\Theta^{(H)}(0)\rvert^2=\frac{1}{f_r}$

[a]It is still unknown whether property P_{12} is satisfied by the power Bertrand P_0-distribution.

Table 6: Desirable QTFR properties and corresponding kernel constraints in the κth power class and the κth localized-kernel power subclass. Note that $\xi_\kappa(f)$, $\tau_\kappa(f)$, and $\rho_X^{(\kappa)}(c)$ are defined in (68), (69), and (84), respectively. For the localized-kernel power classes, $\Phi_T^{(A)}(b,\beta) = G_T^{(A)}(\beta)\,\delta\Big(b - F_T^{(A)}(\beta)\Big)$. Also, $F_T^{(A)\prime}(\beta) = \frac{d}{d\beta}F_T^{(A)}(\beta)$ and $G_T^{(A)\prime}(\beta) = \frac{d}{d\beta}G_T^{(A)}(\beta)$.

POWER CLASSES QTFR PROPERTY	POWER CLASSES KERNEL CONSTRAINT	LOCALIZED-KERNEL POWER CLASSES KERNEL CONSTRAINT
P_1 Scale covariance: $(\mathcal{C}_a X)(f) = \frac{1}{\sqrt{\lvert a\rvert}} X\big(\frac{f}{a}\big) \;\Rightarrow$ $T_{\mathcal{C}_a X}^{(\kappa)}(t,f) = T_X^{(\kappa)}\Big(at, \frac{f}{a}\Big)$	always satisfied	always satisfied
P_2 Power time-shift covariance: $(\mathcal{D}_c^{(\kappa)} X)(f) = e^{-j2\pi c\,\xi_\kappa(\frac{f}{f_r})} X(f) \;\Rightarrow$ $T_{\mathcal{D}_c^{(\kappa)} X}^{(\kappa)}(t,f) = T_X^{(\kappa)}\big(t - c\tau_\kappa(f), f\big)$	always satisfied	always satisfied
P_3 Real-valuedness: $T_X^{(\kappa)}(t,f) = T_X^{(\kappa)*}(t,f)$	$\Phi_T^{(A)}(b,\beta) = \Phi_T^{(A)*}(b,-\beta)$	$F_T^{(A)}(\beta) = F_T^{(A)}(-\beta)$ $G_T^{(A)}(\beta) = G_T^{(A)*}(-\beta)$
P_4 Energy distribution: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_X^{(\kappa)}(t,f)\,dt\,df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$	$\int_{-\infty}^{\infty} \Phi_T^{(A)}(b,0)\frac{db}{\lvert b\rvert} = 1$	$G_T^{(A)}(0) = -F_T^{(A)}(0)$
P_5 Frequency marginal: $\int_{-\infty}^{\infty} T_X^{(\kappa)}(t,f)\,dt = \lvert X(f)\rvert^2$	$\Phi_T^{(A)}(b,0) = \delta(b+1)$	$G_T^{(A)}(0) = -F_T^{(A)}(0) = 1$
P_6 Power marginal: $\int_{-\infty}^{\infty} T_X^{(\kappa)}\big(c\tau_\kappa(f), f\big)\,\lvert\tau_\kappa(f)\rvert\,df = \lvert\rho_X^{(\kappa)}(c)\rvert^2$	$\int_{-\infty}^{\infty} \Phi_T^{(A)}(b,\alpha b)\frac{db}{\lvert b\rvert} = 1, \;\; \forall\alpha$	$\int_{-\infty}^{\infty} G_T^{(A)}(\alpha b)\,\delta\big(b - F_T^{(A)}(\alpha b)\big)$ $\cdot\,\frac{db}{\lvert b\rvert} = 1, \;\; \forall\alpha$
P_7 Frequency localization: $X(f) = \delta(f-\hat{f}) \Rightarrow T_X^{(\kappa)}(t,f) = \delta(f-\hat{f})$	$\Phi_T^{(A)}(b,0) = \delta(b+1)$	$G_T^{(A)}(0) = -F_T^{(A)}(0) = 1$
P_8 Power localization: $I_c^{(\kappa)}(f) = \sqrt{\lvert\tau_\kappa(f)\rvert}\,e^{-j2\pi c\,\xi_\kappa(\frac{f}{f_r})} \;\Rightarrow$ $T_{I_c^{(\kappa)}}^{(\kappa)}(t,f) = \lvert\tau_\kappa(f)\rvert\,\delta\big(t - c\tau_\kappa(f)\big)$	$\int_{-\infty}^{\infty} \Phi_T^{(A)}(b,\beta)\,db = 1, \;\; \forall\beta$	$G_T^{(A)}(\beta) = 1, \;\; \forall\beta$
P_9 Moyal's formula/unitarity: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{X_1}^{(\kappa)}(t,f)\,T_{X_2}^{(\kappa)*}(t,f)\,dt\,df =$ $\Big\lvert\int_{-\infty}^{\infty} X_1(f)\,X_2^*(f)\,df\Big\rvert^2$	$\int_{-\infty}^{\infty} \Phi_T^{(A)*}(b\beta,\alpha\beta)\,\Phi_T^{(A)}(\beta,\alpha\beta)$ $d\beta = \delta(b-1), \quad \forall\alpha$	$\int_{-\infty}^{\infty} \delta\big(b\beta - F_T^{(A)}(\alpha\beta)\big)$ $\delta\big(\beta - F_T^{(A)}(\alpha\beta)\big)\,\lvert G_T^{(A)}(\alpha\beta)\rvert^2$ $\cdot\,d\beta = \delta(b-1), \quad \forall\alpha$
P_{10} Group delay: $\frac{\int_{-\infty}^{\infty} t\,T_X^{(\kappa)}(t,f)\,dt}{\int_{-\infty}^{\infty} T_X^{(\kappa)}(t,f)\,dt} = -\frac{1}{2\pi}\,\frac{d}{df}\arg X(f)$	$\Phi_T^{(A)}(b,0) = \delta(b+1)$ and $\frac{\partial}{\partial\beta}\Phi_T^{(A)}(b,\beta)\Big\rvert_{\beta=0} = 0$	$G_T^{(A)}(0) = -F_T^{(A)}(0) = 1,$ $G_T^{(A)\prime}(0) = F_T^{(A)\prime}(0) = 0$
P_{11} Finite frequency support: $X(f) = 0$ for $f \notin [f_1, f_2] \;\Rightarrow$ $T_X^{(\kappa)}(t,f) = 0$ for $f \notin [f_1, f_2]$	$\Phi_T^{(A)}(b,\beta) = 0$ for $\big\lvert\frac{b+1}{\beta}\big\rvert > \frac{1}{2}$	$G_T^{(A)}(\beta) = 0$ for $\big\lvert F_T^{(A)}(\beta) + 1\big\rvert > \big\lvert\frac{\beta}{2}\big\rvert$
P_{12} Finite power support: $\rho_X^{(\kappa)}(c) = 0$ for $c \notin [c_1, c_2] \;\Rightarrow$ $T_X^{(\kappa)}\big(t,f\big) = 0$ for $t \notin [c_1\tau_\kappa(f), c_2\tau_\kappa(f)]$	$\phi_T^{(A)}(c,\zeta) = 0$ for $\big\lvert\frac{c}{\zeta}\big\rvert > \frac{1}{2}$	$\int_{-\infty}^{\infty} G_T^{(A)}(\beta)\,e^{j2\pi\zeta F_T^{(A)}(\beta)}$ $\cdot\,e^{j2\pi c\beta}d\beta = 0$ for $\big\lvert\frac{c}{\zeta}\big\rvert > \frac{1}{2}$
P_{13} Time-shift covariance: $(\mathcal{S}_\tau X)(f) = e^{-j2\pi\tau f} X(f) \;\Rightarrow$ $T_{\mathcal{S}_\tau}^{(\kappa)}(t,f) = T_X^{(\kappa)}(t-\tau, f)$	see Section 5.2	see Section 5.2

Table 7: Kernels and properties (defined in Table 6) of some QTFRs of the power classes. Note that $\xi_\kappa(f)$, $\tau_\kappa(f)$, $v_X^{(\kappa)}(c,\zeta)$, $V_X^{(\kappa)}(b,\beta)$, $Q_X^{(\kappa)}(t,f)$, and $A_X^{(\kappa)}(\zeta,\beta)$ are defined in (68), (69), (78), (79), (80), and (81), respectively. Also, $s(c)$ and $S(\beta)$, and $h(\zeta)$ and $H(b)$ are Fourier transform pairs.

POWER QTFR	KERNELS	PROPERTIES SATISFIED
Power Wigner distribution, $W^{(\kappa)}$	$\phi^{(A)}_{W^{(\kappa)}}(c,\zeta) = e^{-j2\pi\zeta}\,\delta(c)$ $\Phi^{(A)}_{W^{(\kappa)}}(b,\beta) = \delta(b+1)$ $\psi^{(A)}_{W^{(\kappa)}}(c,b) = \delta(c)\,\delta(b+1)$ $\Psi^{(A)}_{W^{(\kappa)}}(\zeta,\beta) = e^{-j2\pi\zeta}$	P_1–P_{12}
Generalized power Wigner distribution, $W^{(\kappa)(\alpha)}$	$\phi^{(A)}_{W^{(\kappa)(\alpha)}}(c,\zeta) = e^{-j2\pi\zeta}\,\delta(c+\alpha\zeta)$ $\Phi^{(A)}_{W^{(\kappa)(\alpha)}}(b,\beta) = \delta(b+1-\alpha\beta)$ $\psi^{(A)}_{W^{(\kappa)(\alpha)}}(c,b) = \frac{1}{\lvert\alpha\rvert}\,e^{j2\pi\frac{c(b+1)}{\alpha}}$ $\Psi^{(A)}_{W^{(\kappa)(\alpha)}}(\zeta,\beta) = e^{j2\pi\alpha\zeta\beta}\,e^{-j2\pi\zeta}$	P_1, P_2, P_4–P_9, P_{11} and P_{12} if $\lvert\alpha\rvert < \frac{1}{2}$
Powergram, $Y^{(\kappa)}$ with analysis wavelet, $\tilde{\Theta}(f)$	$\phi^{(A)}_{Y^{(\kappa)}}(c,\zeta) = v^{(\kappa)}_{\tilde{\Theta}}(-c,-\zeta)$ $\Phi^{(A)}_{Y^{(\kappa)}}(b,\beta) = V^{(\kappa)}_{\tilde{\Theta}}(-b,-\beta)$ $\psi^{(A)}_{Y^{(\kappa)}}(c,b) = W^{(\kappa)}_{\tilde{\Theta}}\Big(-\tau_\kappa(f_r\xi_\kappa^{-1}(b))\,c, -f_r\xi_\kappa^{-1}(b)\Big)$ $\Psi^{(A)}_{Y^{(\kappa)}}(\zeta,\beta) = A^{(\kappa)}_{\tilde{\Theta}}(-\zeta,-\beta)$	P_1–P_3, P_4 if $\int_{-\infty}^{\infty}\lvert\frac{f_r}{f}\rvert^\kappa\lvert\tilde{\Theta}(f)\rvert^2 df = 1$
Smoothed pseudo power Wigner distribution, $\mathrm{ASPW}^{(\kappa)}$	$\phi^{(A)}_{\mathrm{ASPW}^{(\kappa)}}(c,\zeta) = s(-c)h(-\zeta)$ $\Phi^{(A)}_{\mathrm{ASPW}^{(\kappa)}}(b,\beta) = H(-b)S(-\beta)$ $\psi^{(A)}_{\mathrm{ASPW}^{(\kappa)}}(c,b) = s(-c)H(-b)$ $\Psi^{(A)}_{\mathrm{ASPW}^{(\kappa)}}(\zeta,\beta) = h(-\zeta)S(-\beta)$	P_1, P_2 P_3 if $H(b), s(c) \in R$ P_4 if $S(0)\int_{-\infty}^{\infty} H(b)\frac{db}{\lvert b\rvert} = 1$
Power Bertrand P_0-distribution, $P_0^{(\kappa)}$	$\phi^{(A)}_{P_0^{(\kappa)}}(c,\zeta) = \int_{-\infty}^{\infty} G^{(A)}_{P_0^{(\kappa)}}(\beta)\,e^{j2\pi(c\beta+\zeta F^{(A)}_{P_0^{(\kappa)}}(\beta))}\,d\beta$ $\Phi^{(A)}_{P_0^{(\kappa)}}(b,\beta) = G^{(A)}_{P_0^{(\kappa)}}(\beta)\,\delta(b - F^{(A)}_{P_0^{(\kappa)}}(\beta))$ $\psi^{(A)}_{P_0^{(\kappa)}}(c,b) = \int_{-\infty}^{\infty} G^{(A)}_{P_0^{(\kappa)}}(\beta)\,\delta(b - F^{(A)}_{P_0^{(\kappa)}}(\beta))\,e^{j2\pi c\beta}\,d\beta$ $\Psi^{(A)}_{P_0^{(\kappa)}}(\zeta,\beta) = G^{(A)}_{P_0^{(\kappa)}}(\beta)\,e^{j2\pi\zeta F^{(A)}_{P_0^{(\kappa)}}(\beta)}$ with $F^{(A)}_{P_0^{(\kappa)}}(\beta) = -\frac{\beta}{2}\coth(\frac{\beta}{2})$, $G^{(A)}_{P_0^{(\kappa)}}(\beta) = \frac{\beta/2}{\sinh(\beta/2)}$	P_1–P_5, P_7, P_9–P_{11}, P_{13} for $\kappa = 1$ only
Power Flandrin D-distribution, $D^{(\kappa)}$	same kernels as $P_0^{(\kappa)}$ with $F^{(A)}_{D^{(\kappa)}}(\beta) = -1 - (\frac{\beta}{4})^2$, $G^{(A)}_{D^{(\kappa)}}(\beta) = 1 - (\frac{\beta}{4})^2$	P_1–P_7, P_{10}
Power passive Unterberger distribution, $\mathrm{PUD}^{(\kappa)}$	same kernels as $P_0^{(\kappa)}$ with $F^{(A)}_{\mathrm{PUD}^{(\kappa)}}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}$, $G^{(A)}_{\mathrm{PUD}^{(\kappa)}}(\beta) = \frac{1}{\sqrt{1+(\frac{\beta}{2})^2}}$	P_1–P_7, P_{10}, P_{11}
Power active Unterberger distribution, $\mathrm{AUD}^{(\kappa)}$	same kernels as $P_0^{(\kappa)}$ with $F^{(A)}_{\mathrm{AUD}^{(\kappa)}}(\beta) = -\sqrt{1+(\frac{\beta}{2})^2}$, $G^{(A)}_{\mathrm{AUD}^{(\kappa)}}(\beta) = 1$	P_1–P_5, P_7, P_8, P_{10}, P_{11}
Bertrand P_κ-distributions, P_κ	$\Gamma^{(\kappa)}_{P_\kappa}(b_1,b_2) = \int_{-\infty}^{\infty}\delta(b_1 - \lambda_\kappa(u))$ $\delta(b_2 - \lambda_\kappa(-u))\,\mu(u)du$	P_1, P_2, P_{13}, other properties depending on $\mu(u)$ [20]

Table 8: Desirable hyperbolic class QTFR properties as defined in Table 4 and corresponding kernel constraints in the affine-hyperbolic subclass and the κth hyperbolic-power subclass. We note that $F^{(H)}_{P_0}(\beta) = \ln\left(\frac{\sinh(\beta/2)}{\beta/2}\right)$ and $F^{(H)}_{P_0^{(\kappa)}}(\beta) = \frac{1}{\kappa}\ln\left(\frac{\sinh(\kappa\beta/2)}{\kappa\beta/2}\right)$.

HYPERBOLIC CLASS QTFR PROPERTY	AFFINE-HYPERBOLIC KERNEL CONSTRAINT	HYPERBOLIC-POWER KERNEL CONSTRAINT
P_1 Scale covariance:	always satisfied	always satisfied
P_2 Hyperbolic time-shift covariance:	always satisfied	always satisfied
P_3 Real-valuedness:	$G_T^{(H)}(\beta) = G_T^{(H)*}(-\beta)$	$G_T^{(H)}(\beta) = G_T^{(H)*}(-\beta)$
P_4 Energy distribution:	$G_T^{(H)}(0) = 1$	$G_T^{(H)}(0) = 1$
P_5 Frequency marginal:	$G_T^{(H)}(0) = 1$	$G_T^{(H)}(0) = 1$
P_6 Hyperbolic marginal:	$G_T^{(H)}(\beta) = 1$	$G_T^{(H)}(\beta) = 1$
P_7 Frequency localization:	$G_T^{(H)}(0) = 1$	$G_T^{(H)}(0) = 1$
P_8 Hyperbolic localization:	$G_T^{(H)}(\beta) = 1$	$G_T^{(H)}(\beta) = 1$
P_9 Moyal's formula/unitarity:	$\lvert G_T^{(H)}(\beta)\rvert = 1$	$\lvert G_T^{(H)}(\beta)\rvert = 1$
P_{10} Group delay:	$G_T^{(H)}(0) = 1, \quad G_T^{(H)\prime}(0) = 0$	$G_T^{(H)}(0) = 1, \quad G_T^{(H)\prime}(0) = 0$
P_{11} Finite frequency support:	always satisfied	always satisfied
P_{12} Finite hyperbolic support:	$\int_{-\infty}^{\infty} G_T^{(H)}(\beta)\, e^{j2\pi\left(c\beta+\zeta F^{(H)}_{P_0}(\beta)\right)} d\beta = 0 \quad \text{for } \lvert\frac{c}{\zeta}\rvert > \frac{1}{2}$	$\int_{-\infty}^{\infty} G_T^{(H)}(\beta)\, e^{j2\pi\left(c\beta+\zeta F^{(H)}_{P_0^{(\kappa)}}(\beta)\right)} d\beta = 0 \quad \text{for } \lvert\frac{c}{\zeta}\rvert > \frac{1}{2}$
P_{13} Time-shift covariance:	always satisfied	satisfied only when $\kappa = 1$
P_{14} κth power time-shift covariance:	satisfied only when $\kappa = 1$	always satisfied

Table 9: The phase function, group delay function, and impulse associated with the different QTFR classes.

Class of QTFRs	Phase function	Group delay function	Impulse
Affine class	b	$\frac{1}{f_r}$	$\frac{1}{\sqrt{f_r}}\, e^{-j2\pi c \frac{f}{f_r}}$
Hyperbolic class	$\ln b, \quad b > 0$	$\frac{1}{f}, \quad f > 0$	$\frac{1}{\sqrt{f}}\, e^{-j2\pi c \ln \frac{f}{f_r}}, \quad f > 0$
κth Power class	$\xi_\kappa(b) = \mathrm{sgn}(b)\, \lvert b \rvert^\kappa, \;\; \kappa \neq 0$	$\tau_\kappa(f) = \frac{\kappa}{f_r} \left\lvert \frac{f}{f_r} \right\rvert^{\kappa-1}, \;\; \kappa \neq 0$	$\sqrt{\lvert \tau_\kappa(f) \rvert}\, e^{-j2\pi c \xi_\kappa\left(\frac{f}{f_r}\right)}$
Generalized class	$\xi(b)$	$\tau(f) = \frac{d}{df}\, \xi\left(\frac{f}{f_r}\right)$	$\sqrt{\lvert \tau(f) \rvert}\, e^{-j2\pi c \xi\left(\frac{f}{f_r}\right)}$

Table 10: Desirable QTFR properties in the generalized class and corresponding kernel constraints. Note that $d(b,\beta) = \xi^{-1}(b + \frac{\beta}{2})$, and that it is assumed that $\Phi_T^{(G)}(b,\beta)$ satisfies (109) using (117).

GENERALIZED CLASS QTFR PROPERTY	GENERALIZED CLASS KERNEL CONSTRAINT
P_1 Scale covariance: $(\mathcal{C}_a X)(f) = \frac{1}{\sqrt{\lvert a\rvert}} X(\frac{f}{a}) \;\Rightarrow$ $T^{(G)}_{\mathcal{C}_a X}(t,f) = T^{(G)}_X\left(at, \frac{f}{a}\right)$	always satisfied
P_2 Generalized time-shift covariance: $(\mathcal{D}_c X)(f) = e^{-j2\pi c\,\xi(\frac{f}{f_r})} X(f) \;\Rightarrow$ $T^{(G)}_{\mathcal{D}_c X}(t,f) = T^{(G)}_X\big(t - c\tau(f), f\big)$	always satisfied
P_3 Real-valuedness: $T^{(G)}_X(t,f) = T^{(G)*}_X(t,f)$	$\Phi^{(G)}_T(b,\beta) = \Phi^{(G)*}_T(b,-\beta)$
P_4 Energy distribution: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T^{(G)}_X(t,f)\,dt\,df = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2\,df$	$\int_{-\infty}^{\infty} \Phi^{(G)}_T(-b,0) \left\lvert \frac{\tau(f/\xi^{-1}(b))}{\tau(f)\,\xi^{-1}(b)} \right\rvert db = 1$
P_5 Frequency marginal: $\int_{-\infty}^{\infty} T^{(G)}_X(t,f)\,dt = \lvert X(f)\rvert^2$	$\Phi^{(G)}_T(b,0) = \delta(b + \xi(1))$
P_6 Generalized marginal: $\int_{-\infty}^{\infty} T^{(G)}_X\big(c\tau(f), f\big)\,\lvert\tau(f)\rvert\,df = \lvert\rho_X(c)\rvert^2$	$\int_{-\infty}^{\infty} \Phi^{(G)}_T\Big(-\frac{\xi(f_1/f)+\xi(f_2/f)}{2}, \xi(f_1/f) - \xi(f_2/f)\Big) \cdot \left\lvert \frac{\tau(f)}{f}\,\xi'\Big(\sqrt{\lvert f_1 f_2/f^2\rvert}\Big)\right\rvert df = \sqrt{\lvert\tau(f_1)\,\tau(f_2)\rvert}$
P_7 Frequency localization: $X(f) = \delta(f - \hat{f}) \Rightarrow T^{(G)}_X(t,f) = \delta(f - \hat{f})$	$\Phi^{(G)}_T(b,0) = \delta(b + \xi(1))$
P_8 Generalized localization: $I_c(f) = \sqrt{\lvert\tau(f)\rvert}\,e^{-j2\pi c\xi(\frac{f}{f_r})} \;\Rightarrow$ $T^{(G)}_{I_c}(t,f) = \lvert\tau(f)\rvert\,\delta\big(t - c\tau(f)\big)$	$\int_{-\infty}^{\infty} \Phi^{(G)}_T\left(-\frac{\xi(\frac{f_r}{f}d(b,\beta))+\xi(\frac{f_r}{f}d(b,-\beta))}{2}, \xi\Big(\frac{f_r}{f}d(b,\beta)\Big) - \xi\Big(\frac{f_r}{f}d(b,-\beta)\Big)\right) \cdot \frac{\left\lvert\xi'(\frac{f_r}{f}\sqrt{\lvert d(b,\beta)\,d(b,-\beta)\rvert})\right\rvert}{\sqrt{\lvert\xi'(d(b,\beta))\,\xi'(d(b,-\beta))\rvert}} db = \left\lvert\frac{f}{f_r}\right\rvert, \;\; \forall\beta$
P_9 Moyal's formula/unitarity: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T^{(G)}_{X_1}(t,f)\,T^{(G)*}_{X_2}(t,f)\,dt\,df = \Big\lvert \int_{-\infty}^{\infty} X_1(f)\,X_2^*(f)\,df \Big\rvert^2$	$\int_{-\infty}^{\infty} \left\lvert\frac{f_r}{f}\right\rvert^2 \lvert\tau(f)\rvert \frac{\left\lvert\xi'(\frac{f_r}{f}\sqrt{\lvert d(b_1,\beta)d(b_1,-\beta)\rvert})\,\xi'(\frac{f_r}{f}\sqrt{\lvert d(b_2,\beta)d(b_2,-\beta)\rvert})\right\rvert}{\sqrt{\lvert\xi'(d(b_1,\beta))\,\xi'(d(b_1,-\beta))\,\xi'(d(b_2,\beta))\,\xi'(d(b_2,-\beta))\rvert}} \cdot \Phi^{(G)}_T\left(-\frac{\xi(\frac{f_r}{f}d(b_1,\beta))+\xi(\frac{f_r}{f}d(b_1,-\beta))}{2}, \xi\Big(\frac{f_r}{f}d(b_1,\beta)\Big) - \xi\Big(\frac{f_r}{f}d(b_1,-\beta)\Big)\right) \cdot \Phi^{(G)*}_T\left(-\frac{\xi(\frac{f_r}{f}d(b_2,\beta))+\xi(\frac{f_r}{f}d(b_2,-\beta))}{2}, \xi\Big(\frac{f_r}{f}d(b_2,\beta)\Big) - \xi\Big(\frac{f_r}{f}d(b_2,-\beta)\Big)\right) df = \delta(b_1 - b_2), \;\; \forall\beta$
P_{10} Group delay: $\frac{\int_{-\infty}^{\infty} t\,T^{(G)}_X(t,f)\,dt}{\int_{-\infty}^{\infty} T^{(G)}_X(t,f)\,dt} = -\frac{1}{2\pi}\,\frac{d}{df}\arg X(f)$	$\Phi^{(G)}_T(b,0) = \delta(b + \xi(1))$ and $\frac{\partial}{\partial\beta}\Phi^{(G)}_T(b,\beta)\Big\rvert_{\beta=0} = 0$ if $\frac{\partial}{\partial\beta}\lvert\xi'(d(b,\beta))\,\xi'(d(b,-\beta))\rvert^{-1/2}\Big\rvert_{\beta=0} = 0$ and $\frac{\partial}{\partial\beta}\lvert\xi'(\sqrt{\lvert d(b,\beta)\,d(b,-\beta)\rvert})\rvert\Big\rvert_{\beta=0} = 0$
P_{11} Finite frequency support: $X(f) = 0$ for $f \notin [f_1, f_2] \;\Rightarrow$ $T^{(G)}_X(t,f) = 0$ for $f \notin [f_1, f_2]$	$\Phi^{(G)}_T(b,\beta) = 0$ for b,β such that $\left\lvert\frac{\xi(\frac{f}{f_r}d(-b,\beta))+\xi(\frac{f}{f_r}d(-b,-\beta))}{2} - \xi(\frac{f}{f_r})\right\rvert > \left\lvert\frac{\xi(\frac{f}{f_r}d(-b,\beta))-\xi(\frac{f}{f_r}d(-b,-\beta))}{2}\right\rvert$

Table 11: Affine, hyperbolic, and power classes as special cases of the generalized class. By choosing the specific phase functions $\xi(b)$ and group delay functions $\tau(f)$ in the second through fourth columns, the generalized class simplifies to the affine class, the hyperbolic class, and the κth power class, respectively. Thus, all functions and operators in the first column of the generalized class simplify to the corresponding functions and operators listed in each column. Note that the definitions of these functions and operators are provided in the text in the indicated equations.

GENERALIZED CLASS Phase function, $\xi(b)$: Group delay function, $\tau(f)$:	AFFINE CLASS $\xi(b) = b$ $\tau(f) = 1/f_r$	HYPERBOLIC CLASS $\xi(b) = \ln b$ $\tau(f) = 1/f$	κTH POWER CLASS $\xi(b) = \xi_\kappa(b)$ (68) $\tau(f) = \tau_\kappa(f)$ (69)
Dispersion operator, $\mathcal{D}_c$ (103):	$\mathcal{S}_{c/f_r}$ (1)	$\mathcal{H}_c$ (43)	$\mathcal{D}_c^{(\kappa)}$ (67)
Generalized impulse, $I_c(f)$ (118):	$I_c^{(A)}(f)$ (24)	$H_c(f)$ (56)	$I_c^{(\kappa)}(f)$ (82)
Warped signal, $(\mathcal{W}X)(f)$ (125):	$(\mathcal{W}_1 X)(f) = X(f)$	$(\mathcal{W}_H X)(f)$ (46)	$(\mathcal{W}_\kappa X)(f)$ (72)
Signal product, $V_X(b,\beta)$ (116):	$V_X^{(1)}(b,\beta) =$ $f_r U_X(f_r b, f_r \beta)$ (9)	$V_X^{(H)}(b,\beta)$ (52)	$V_X^{(\kappa)}(b,\beta)$ (79)
Generalized version of the Wigner distribution, $W_X^{(G)}(t,f)$ (121):	$W_X^{(1)}(t,f) =$ $W_X(t,f)$ (10)	$Q_X(t,f)$ (54)	$W_X^{(\kappa)}(t,f)$ (80)
Generalized time-shift Wigner distribution, $W_X^{(GW)}(t,f)$ (127):	$W_X^{(1)}(t,f) =$ $W_X(t,f)$ (10)	$Q_X(t,f)$ (54)	$W_X^{(\kappa)}(t,f)$ (80)
Generalized coefficient $\rho_X(c)$ (120) :	$\rho_X^{(A)}(c)$ (26)	$\rho_X^{(H)}(c)$ (59)	$\rho_X^{(\kappa)}(c)$ (84)
Generalized QTFR, $T_X^{(G)}(t,f)$ (110),(111),(115):	$T_X^{(1)}(t,f) =$ $T_X^{(A)}(t,f)$ (17),(19)	$T_X^{(H)}(t,f)$ (44),(48)	$T_X^{(\kappa)}(t,f)$ (70),(75)
Kernel $\Phi_T^{(G)}(b,\beta)$ (115):	$\Phi_T^{(A)}(b,\beta)$ (19)	$\Phi_T^{(H)}(b,\beta)$ (48)	$\Phi_T^{(A)}(b,\beta)$ (75)
Kernel $\Gamma_T^{(G)}(b_1,b_2)$ (110),(111):	$\Gamma_T^{(A)}(b_1,b_2) =$ $\Gamma_T^{(1)}(b_1,b_2)$ (17)	$\Gamma_T^{(H)}(b_1,b_2)$ (44)	$\Gamma_T^{(\kappa)}(b_1,b_2)$ (70)

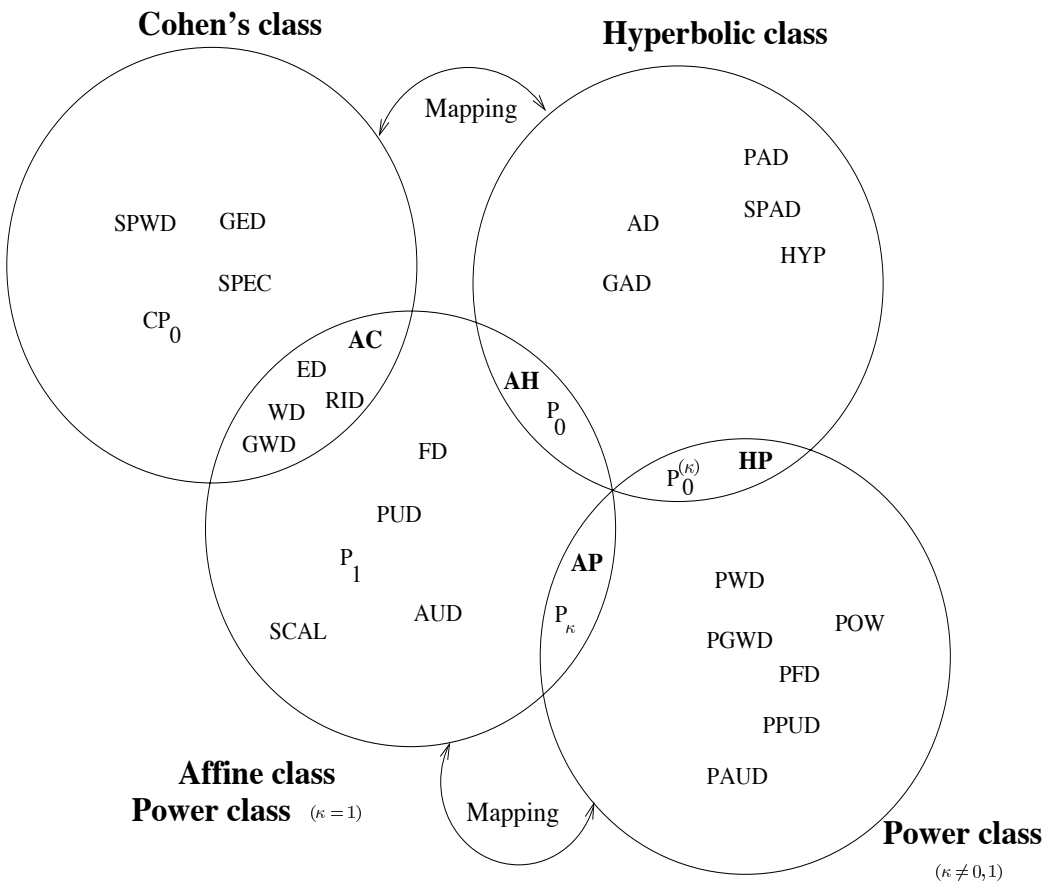

Figure 1: A pictorial summary of the different classes of QTFRs considered in the paper: Cohen's class, affine class, hyperbolic class, and κth power class ($\kappa \neq 0, 1$) together with their intersection subclasses and some important QTFR members. The spectrogram, SPEC, the Wigner distribution, WD, the generalized WD, GWD, the smoothed pseudo-WD, SPWD, the Cohen-Bertrand P_0, CP_0, the Choi-Williams exponential distribution, ED, the generalized exponential distribution, GED, and the reduced interference distributions (RID) are members of Cohen's class. The WD, the GWD, the ED, the RID, the scalogram, SCAL, the Flandrin D distribution, FD, the passive Unterberger distribution, PUD, the active Unterberger distribution, AUD, the Bertrand P_0-distribution, the Bertrand P_1-distribution, and the Bertrand P_κ-distributions, $\kappa \neq 0, 1$, are members of the affine class. The affine-Cohen's intersection (**AC**) contains the WD, the GWD, the ED, and the RID. The hyperbologram, HYP, the Altes-Marinovich Q-distribution, AD, the generalized AD, GAD, the pseudo AD, PAD, the smoothed pseudo AD, SPAD, the P_0-distribution, and the power Bertrand P_0-distribution, $P_0^{(\kappa)}$, are members of the hyperbolic class. The affine-hyperbolic intersection (**AH**) contains the P_0-distribution. The powergram, POW, the power Wigner distribution, PWD, the power GWD, PGWD, the P_κ-distributions, the $P_0^{(\kappa)}$-distributions, the power FD, PFD, the power PUD, PPUD, and the power AUD, PAUD, are all members of the power classes. The affine-power intersection (**AP**) contains the P_κ-distributions. The hyperbolic-power intersection (**HP**) contains the $P_0^{(\kappa)}$-distributions. Note that the hyperbolic and Cohen's classes, and the affine and power classes are related through unitary mappings.

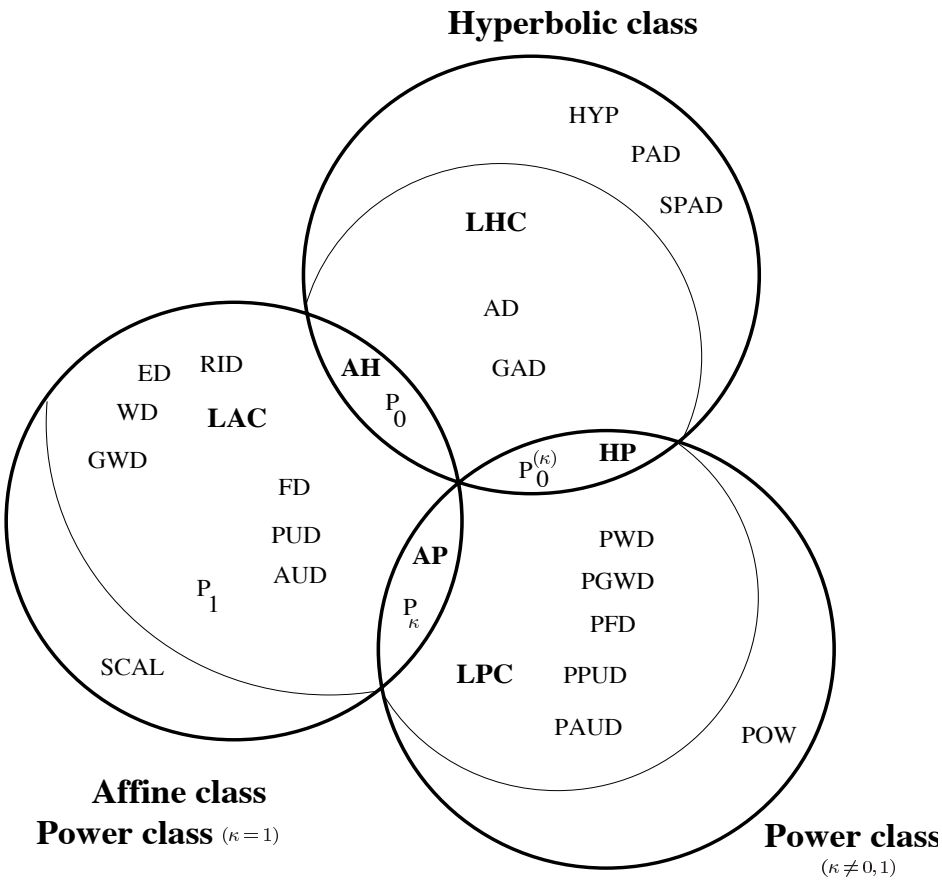

Figure 2: A pictorial summary of the affine class, the hyperbolic class, and the κth power class ($\kappa \neq 0, 1$) that adds to Figure 1 the localized-kernel affine subclass (**LAC**), the localized-kernel hyperbolic subclass (**LHC**), and the localized-kernel κth power class (**LPC**). Recall from Figure 1 that **AH** stands for the affine-hyperbolic intersection, **AP** stands for the affine-power intersection, and **HP** stands for the hyperbolic-power intersection. The acronyms for the different QTFRs shown are defined in the caption of Figure 1.

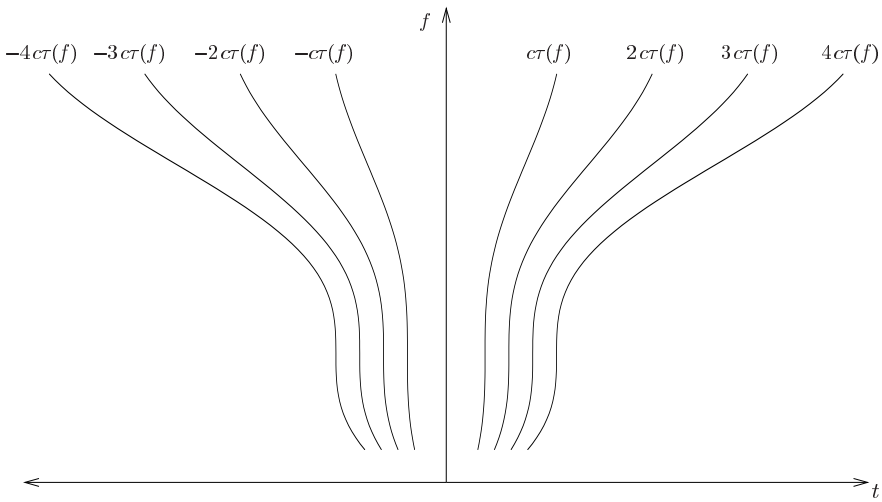

Figure 3: The generalized time-shift operator $\mathcal{D}_c$ corresponds to a group delay curve $t = c\tau(f)$ in the TF plane. (In this figure, c is assumed to be positive.)

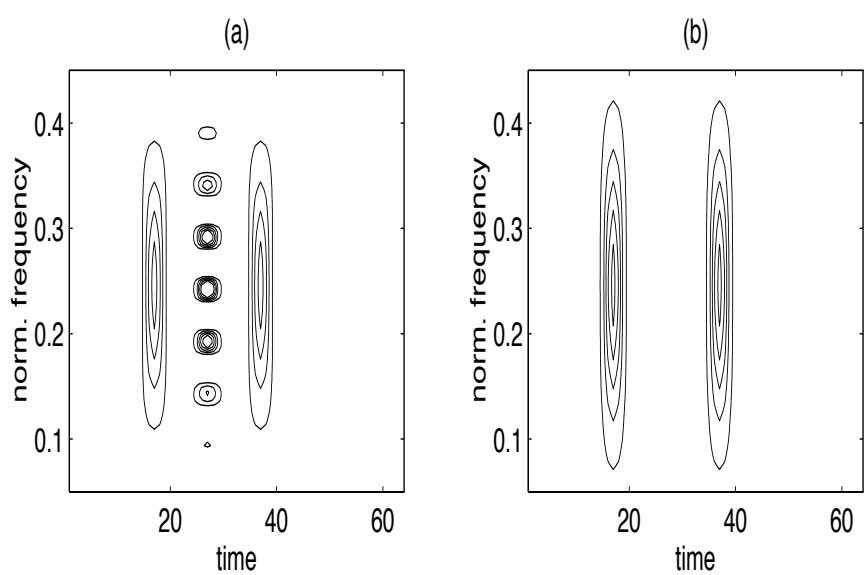

Figure 4: Time-Frequency analysis of the sum of two Dirac impulses. (a) Wigner distribution, (b) pseudo Wigner distribution.

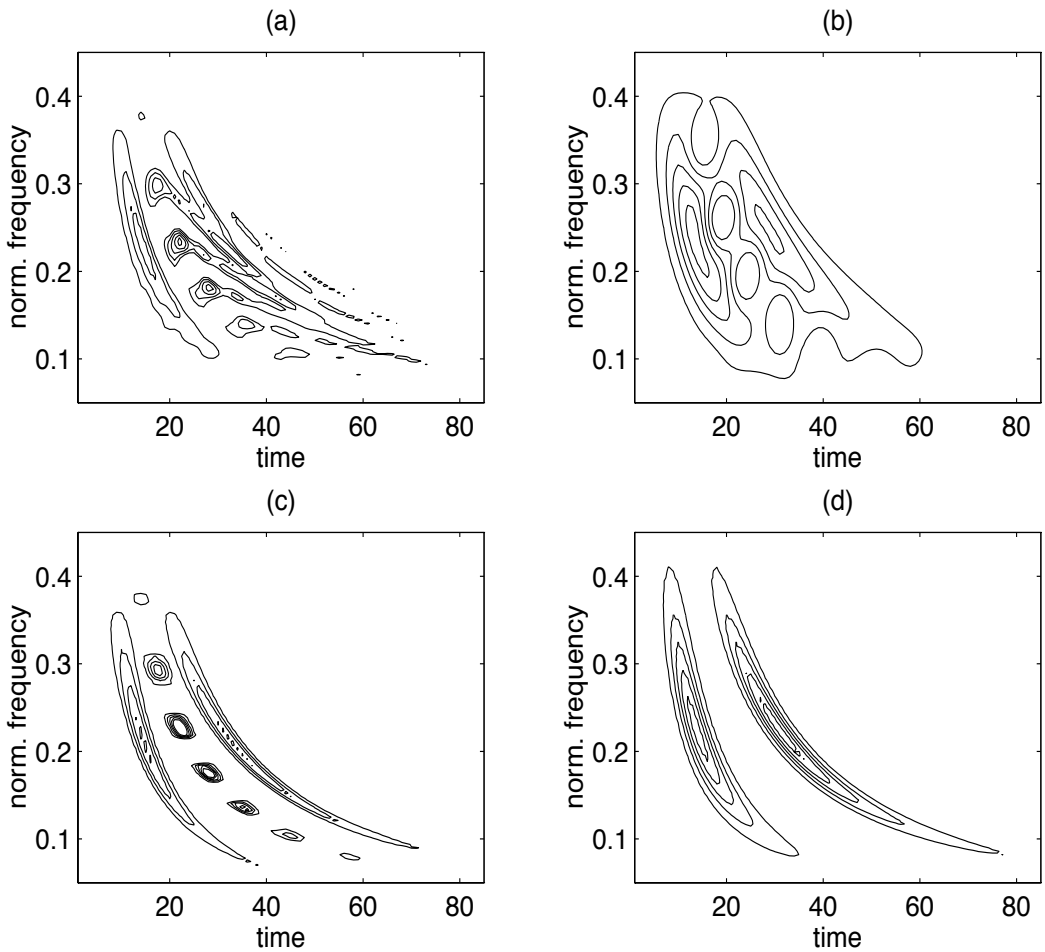

Figure 5: Time-Frequency analysis of the sum of two hyperbolic impulses. (a) Wigner distribution, (b) smoothed pseudo Wigner distribution, (c) Altes-Marinovich Q-distribution, (d) smoothed pseudo Q-distribution.

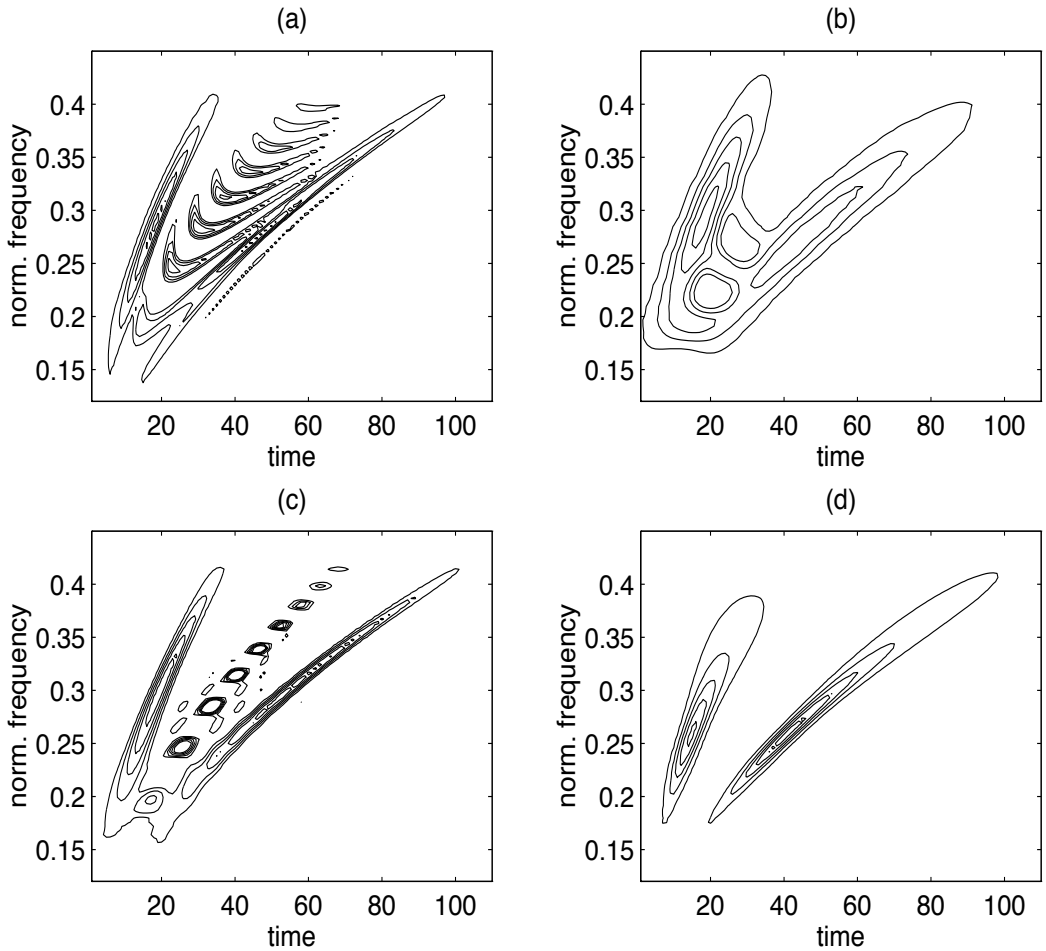

Figure 6: Time-Frequency analysis of the sum of two power impulses with $\kappa = 3$. (a) Wigner distribution, (b) affine smoothed pseudo Wigner distribution, (c) κth power Wigner distribution with $\kappa = 3$, (d) κth smoothed pseudo power Wigner distribution with $\kappa = 3$.

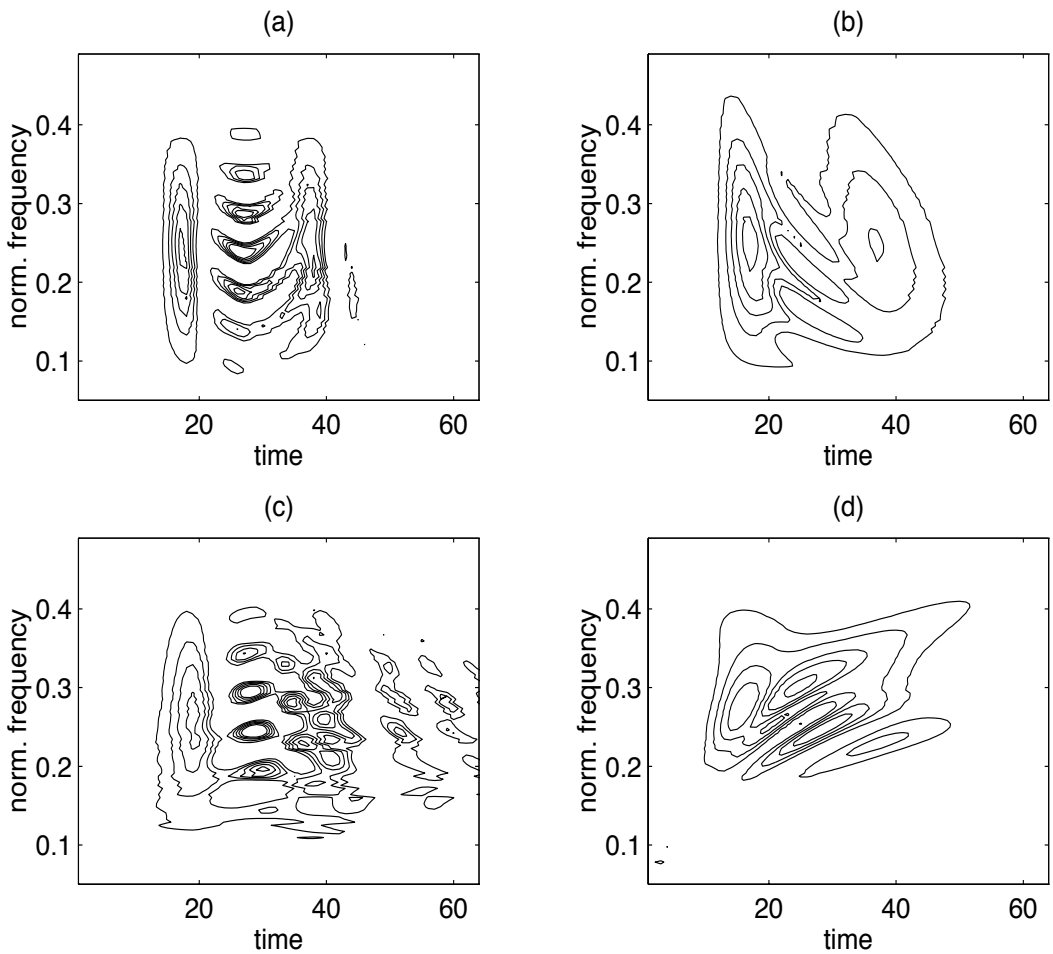

Figure 7: Time-Frequency analysis of the sum of two Dirac impulses. (a) Q-distribution, (b) smoothed pseudo Q-distribution, (c) κth power Wigner distribution with $\kappa = 3$, (d) κth smoothed pseudo power Wigner distribution with $\kappa = 3$.