

ON MUTUAL INFORMATION AND OUTAGE FOR EXTENDED ALAMOUTI SPACE-TIME BLOCK CODES

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ABSTRACT

Extended Alamouti codes have proven to exhibit very desirable properties which enable the implementation of simple receiver types. We show analytically that the mean mutual information for extended Alamouti codes in i.i.d. Rayleigh fading is invariant to the number of transmit antennas. Numerical investigation indicates that the variance of the mutual information decreases. Therefore, outage probability decreases with the number of transmit antennas. Finally, we observe that the mutual information due to fading can be closely approximated by a Gaussian random variable.

1. INTRODUCTION

This contribution builds on the well-known scheme of Alamouti [1, 2] for two transmit antennas, its extension to four transmit antennas [3, 4, 5], as well as the generalization to more antennas [10, 11]. While mobiles are expected to support up to four antenna elements in the near future, base stations can be equipped with higher numbers of antenna elements. The paper starts with some basic results suitable for extended Alamouti schemes of $N_T = 2^m$ transmit antennas in Section 2. Section 3 reports on mean mutual information and Section 4 on outage probabilities.

2. EXTENDED ALAMOUTI CODES

Let $\mathbf{s} = (s_1, s_2, \dots, s_{N_T})^T$ be the $N_T \times 1$ vector of transmit symbols from N_T transmit antennas to a single receive antenna. Starting with the 2×2 -Alamouti scheme the following recursive construction rule (similar to the construction of a complex Walsh-Hadamard code) is applied

$$\begin{bmatrix} s_1 & s_2 \\ s_2^* & -s_1^* \end{bmatrix} \rightarrow \begin{bmatrix} S_1 & S_2 \\ S_2^* & -S_1^* \end{bmatrix}. \quad (1)$$

where the complex scalars s_1 and s_2 are replaced by the 2×2 matrices

$$S_1 = \begin{bmatrix} s_1 & s_2 \\ s_2^* & -s_1^* \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} s_3 & s_4 \\ s_4^* & -s_3^* \end{bmatrix}$$

and $*$ denotes complex conjugation without transposition. This results in the following symbol block \mathbf{S} for transmitting the four symbols $\mathbf{s} = [s_1, \dots, s_4]^T$:

$$\mathbf{S} = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2^* & -s_1^* & s_4^* & -s_3^* \\ s_3^* & s_4^* & -s_1^* & -s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{bmatrix}$$

The received vector can be expressed as $\mathbf{r} = \mathbf{Sh} + \mathbf{v}$. Converting the received vector by complex conjugation

$$\begin{aligned} y_1 &= r_1, & v_1 &= \bar{v}_1, \\ y_2 &= r_2^*, & v_2 &= \bar{v}_2^*, \\ y_3 &= r_3^*, & v_3 &= \bar{v}_3^*, \\ y_4 &= r_4, & v_4 &= \bar{v}_4. \end{aligned} \quad (2)$$

results in the following equivalent transmission scheme

$$\mathbf{y} = \mathbf{Hs} + \mathbf{v}, \quad (3)$$

in which

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ -\mathbf{H}_2^* & \mathbf{H}_1^* \end{bmatrix} \quad (4)$$

appears as MIMO channel transmission matrix with

$$\mathbf{H}_1 = \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix} \quad \text{and} \quad \mathbf{H}_2 = \begin{bmatrix} h_3 & h_4 \\ -h_4^* & h_3^* \end{bmatrix}.$$

If $\bar{\mathbf{v}}$ is a complex-valued Gaussian vector with i.i.d. elements then so is \mathbf{v} .

3. MEAN MUTUAL INFORMATION

The ergodic capacity of the MIMO channel is

$$\max_{\mathbf{Q} \geq \mathbf{O}} \frac{1}{N_T} \mathbf{E} \left[\log_2 \det \left(\mathbf{I} + \varrho \mathbf{HQH}^H \right) \right] \text{ b/s/Hz}, \quad (5)$$

with the normalization $\mathbf{E}[\text{tr}(\mathbf{HH}^H)] = \mathbf{E}[N_T h^2] = N_T$ and $\varrho = N_T \sigma_s^2 / \sigma_v^2$ denoting the SNR at the receive antenna (σ_s^2 and σ_v^2 denoting the variance of each entry of \mathbf{s} and \mathbf{v})

in (3), respectively). The factor $\frac{1}{N_T}$ in (5) stems from the fact that N_T symbol intervals are used for transmitting an $N_T \times N_T$ space-time code block \mathbf{S} . The maximization over \mathbf{Q} in (5) is constrained to positive semi-definite hermitian matrices. Instead of (5), we study the mean mutual information of the channel,

$$\bar{C}_{N_T} = \frac{1}{N_T} \mathbb{E} \left[\log_2 \det \left(\mathbf{I} + \varrho \mathbf{H} \mathbf{H}^H \right) \right] \text{ b/s/Hz.} \quad (6)$$

Diagonalization of $\mathbf{G} = \mathbf{H} \mathbf{H}^H$ is applied, leading to

$$\bar{C}_{N_T} = \frac{1}{N_T} \mathbb{E} \left[\sum_{i=1}^{N_T} \log_2 (1 + \varrho \xi_i) \right] \text{ b/s/Hz.} \quad (7)$$

We need the marginal distribution of the eigenvalue $\xi_i = h^2 \lambda_i$ ($i = 1, \dots, N_T$) of \mathbf{G} for the evaluation of the mean mutual information.

For the Alamouti scheme ($N_T = 2$), we have $\lambda_i = 1$ and $\xi_i = h^2$. For flat Rayleigh fading, h^2 is a χ^2_4 variate with four degrees of freedom rescaled to $\mathbb{E}[h^2] = 1$, i.e. the density of ξ_i is given by $f_{\xi,2}(\xi) = 4\xi e^{-2\xi}$ for $\xi \geq 0$. The mean mutual information can be evaluated analytically,

$$\begin{aligned} \bar{C}_2 &= \int_0^\infty \log_2 (1 + \varrho \xi) f_{\xi,2}(\xi) d\xi, \\ &= \frac{1}{\ln 2} \left[1 + \left(1 - \frac{2}{\varrho} \right) e^{2/\varrho} E_1(2/\varrho) \right], \end{aligned} \quad (8)$$

where $E_1(x)$ denotes the exponential integral

$$E_1(x) \triangleq \int_1^\infty \frac{e^{-xt}}{t} dt, \quad \text{for } \operatorname{Re}(x) > 0.$$

For the four antenna case ($N_T = 4$), the distributions of λ_i and ξ_i were reported in [5]. We summarize the required result as a lemma here:

Lemma 1: Given an extended Alamouti code for $N_T = 4$. Let the channel coefficients h_i ($i = 1, \dots, 4$) be i.i.d. complex-valued Gaussian variates with variance $\frac{1}{4}$ such that $\mathbb{E}[\sum_{i=1}^4 |h_i|^2] = 1$ and let λ be an eigenvalue of \mathbf{G}/h^2 and let ξ be an eigenvalue of \mathbf{G} . The scaled eigenvalue $\lambda/2$ is Beta(2,2)-distributed, i.e. the probability density of λ is given by $f_{\lambda,4}(\lambda) = \frac{3}{4}\lambda(2-\lambda)$ for $0 < \lambda < 2$ and zero elsewhere. The probability density of $\xi = h^2 \lambda$ is given by $f_{\xi,4}(\xi) = 4\xi e^{-2\xi}$.

We can now evaluate the mean mutual information of the four antenna scheme and observe that it is the same as for the Alamouti scheme with $N_T = 2$

$$\bar{C}_4 = \int_0^\infty \log_2 (1 + \varrho \xi) f_{\xi,4}(\xi) d\xi = \bar{C}_2. \quad (9)$$

The extension to the eight antenna case ($N_T = 8$) is enabled by the following two lemmas.

Lemma 2: The eigenvalues of $\mathbf{H}^H \mathbf{H}/h^2$ for $N_T = 8$ occur in pairs and can be expressed as

$$\begin{aligned} \lambda_1 = \lambda_2 &= (1 - X) + (Y - Z), \\ \lambda_3 = \lambda_4 &= (1 + X) - (Y + Z), \\ \lambda_5 = \lambda_6 &= (1 + X) + (Y + Z), \\ \lambda_7 = \lambda_8 &= (1 - X) - (Y - Z), \end{aligned}$$

where $-1 < X, Y, Z < 1$.

Proof: Applying rule (1) two times in succession results in the 8×8 scheme

$$\mathbf{H} = \left[\begin{array}{cccc|cccc} h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ -h_2^* & h_1^* & -h_4^* & h_3^* & -h_6^* & h_5^* & -h_8^* & h_7^* \\ -h_3^* & -h_4^* & h_1^* & h_2^* & -h_7^* & -h_8^* & h_5^* & h_6^* \\ h_4 & -h_3 & -h_2 & h_1 & h_8 & -h_7 & -h_6 & h_5 \\ \hline -h_5^* & -h_6^* & -h_7^* & -h_8^* & h_1^* & h_2^* & h_3^* & h_4^* \\ h_6 & -h_5 & h_8 & -h_7 & -h_2 & h_1 & -h_4 & h_3 \\ h_7 & h_8 & -h_5 & -h_6 & -h_3 & -h_4 & h_1 & h_2 \\ -h_8^* & h_7^* & h_6^* & -h_5^* & h_4^* & -h_3^* & -h_2^* & h_1^* \end{array} \right].$$

It is straightforward (but somewhat tedious) to verify that the Grammian $\mathbf{G} = \mathbf{H}^H \mathbf{H}$ is given by

$$\mathbf{G} = h^2 \left[\begin{array}{cccc} \mathbf{I}_2 & X \mathbf{J}_2 & -Z \mathbf{J}_2 & Y \mathbf{I}_2 \\ -X \mathbf{J}_2 & \mathbf{I}_2 & -Y \mathbf{I}_2 & -Z \mathbf{J}_2 \\ Z \mathbf{J}_2 & -Y \mathbf{I}_2 & \mathbf{I}_2 & X \mathbf{J}_2 \\ Y \mathbf{I}_2 & Z \mathbf{J}_2 & -X \mathbf{J}_2 & \mathbf{I}_2 \end{array} \right]. \quad (10)$$

with $h^2 = \|\mathbf{h}\|^2 = \sum_{k=1}^8 |h_k|^2$,

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{J}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad (11)$$

and

$$\begin{aligned} X &= 2 \operatorname{Re}(h_1 h_4^* - h_2 h_3^* + h_5 h_8^* - h_6 h_7^*)/h^2, \\ Y &= 2 \operatorname{Re}(h_1 h_7^* - h_3 h_5^* + h_2 h_8^* - h_4 h_6^*)/h^2, \\ Z &= 2 \operatorname{Re}(h_2 h_5^* - h_1 h_6^* + h_4 h_7^* - h_3 h_8^*)/h^2, \end{aligned}$$

The Grammian $\mathbf{G} = \mathbf{H}^H \mathbf{H}$ is diagonalized by $\mathbf{V}^T \mathbf{G} \mathbf{V}$ with the orthogonal modal matrix \mathbf{V}

$$\mathbf{V} = \frac{1}{2} \begin{bmatrix} \mathbf{I}_2 & \mathbf{J}_2 & \mathbf{J}_2 & \mathbf{I}_2 \\ \mathbf{J}_2 & \mathbf{I}_2 & -\mathbf{I}_2 & -\mathbf{J}_2 \\ \mathbf{J}_2 & \mathbf{I}_2 & \mathbf{I}_2 & \mathbf{J}_2 \\ -\mathbf{I}_2 & -\mathbf{J}_2 & \mathbf{J}_2 & \mathbf{I}_2 \end{bmatrix}, \quad (12)$$

We note that \mathbf{V} does not depend on \mathbf{h} . We can show by induction that the Grammians \mathbf{G} corresponding to recursively extended Alamouti schemes of arbitrary order can be diagonalized by *constant* orthogonal matrices \mathbf{V} with elements taken solely from $\{-1, 0, +1\}$ [?].

Lemma 3: Given an extended Alamouti Code for $N_T = 8$. Let the channel coefficients h_i ($i = 1, \dots, 8$) be i.i.d. complex-valued Gaussian variates with variance $\frac{1}{8}$ such that $\mathbb{E}[\mathbf{h}^H \mathbf{h}] = 1$ and let λ be an eigenvalue of \mathbf{G}/h^2 and let ξ be an eigenvalue of \mathbf{G} . The scaled eigenvalue $\lambda/4$ is Beta(2,6)-distributed, i.e. the probability density of λ is given by $f_{\lambda,8}(\lambda) = \frac{21}{8192} \lambda(4-\lambda)^5$ for $0 < \lambda < 4$ and zero elsewhere. The probability density of $\xi = h^2 \lambda$ is given by $f_{\xi,8}(\xi) = 4\xi e^{-2\xi}$,

Proof: It is sufficient to give the proof for one eigenvalue, say $h^2 \lambda_5$ for the eight antenna case. The proof for the remaining eigenvalues proceeds similarly. By completing the squares $h^2 \lambda_5/4$ can be regarded as the sum of two χ_n^2 -distributed variables with $n = 2$ degrees of freedom each, i.e.

$$\left| \frac{h_1 + h_4 - h_6 + h_7}{2} \right|^2 + \left| \frac{h_2 - h_3 + h_5 + h_8}{2} \right|^2.$$

Using Lemma 3 to evaluate (7) for $N_T = 8$, we obtain the same mean mutual information as for the Alamouti scheme:

$$\overline{C}_8 = \int_0^\infty \log_2 (1 + \varrho \xi) f_{\xi,8}(\xi) d\xi = \overline{C}_2. \quad (13)$$

4. DISCUSSION OF OUTAGE CAPACITY

Although the mean mutual information does not change with the number of transmit antennas, the corresponding outage probabilities differ for varying numbers of transmit antennas at any chosen level of mutual information. This is due to the variance-reduction in mutual information with increasing number of transmit antennas. Figure 1 displays the 10% outage quantile of mutual information for $N_T = 2, 4$, and 8 antennas obtained from Monte Carlo simulations in flat i.i.d. Rayleigh fading (marked by symbols \diamond , $*$, and \times). The result for $N_T = 4$ was already obtained by simulations in [4]. The mean mutual information \overline{C}_2 from (8) is shown as a dashed line for comparison. The Monte Carlo simulation results for 10% outage levels in mutual information are shown in this figure. They can be compared with the 10%-percentiles of hypothetical Gaussian distributions, all with the same mean $\mu_C = \overline{C}_2$ and numerically calculated standard deviations (denoted as “semi-analytical” in the figure). We see that the Gaussian approximation to the distribution of mutual information fits the 10% outage quantile extremely well.

5. DECORRELATING PROPERTIES

An important property of Extended Alamouti Codes has not been paid much attention to in the past: their decorrelating behavior. In a realistic wireless transmission scenario, the antennas are coupled. For antenna spacing larger than half

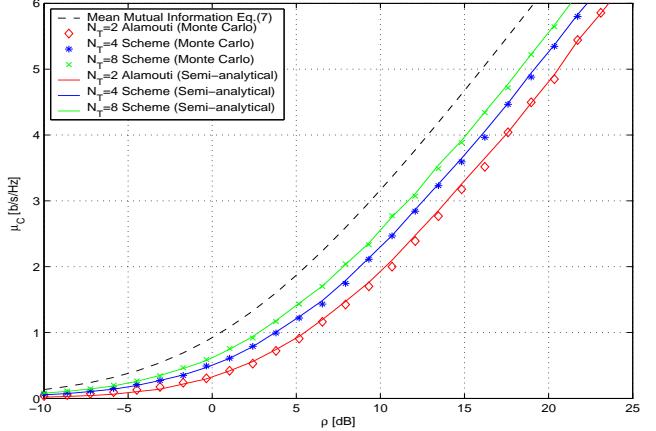


Fig. 1. Mean mutual information and 10% outage levels for extended Alamouti schemes with $N_T = 2, 4, 8$.

a wavelength, the coupling is small, so that the impact on coupling is often omitted. However, with larger and larger arrays it may not be possible to place the antennas sufficiently close. Also base stations are often located under roof tops causing additional antenna coupling so that even large antenna spacing can cause severe correlation between the antenna signals. Due to their structure, Extended Alamouti Codes are capable of decorrelating the channels [12, ?].

For simulations with correlated channels, a uniform linear array with eight antenna elements (ULA-8) is assumed at the transmitter. The channel coefficients h_i are complex Gaussian zero-mean variates with a covariance matrix $\mathbf{C}_h = \mathbb{E}[\mathbf{h}\mathbf{h}^H]$. Following [6], we assume isotropic coherency loss. We consider the case when the loss across the array is the same for all wave fronts, irrespective of their directions of departure θ . Random fluctuations in time-variant propagation environments may result in situations where this assumption is a reasonable approximation. In short, this model builds on

$$\mathbf{C}_h = \mathbf{B} \odot \mathbf{d} \mathbf{d}^H, \quad (14)$$

where “ \odot ” denotes the Schur-Hadamard (element-by-element) product and the plane wave steering vector of the ULA-8 is defined as

$$\mathbf{d} = \frac{1}{\sqrt{8}} (1, z, z^2, \dots, z^7)^T, \quad \text{with } z = e^{-j\frac{\omega}{d} \cos \theta}. \quad (15)$$

The wave front coherency across the array is described by a real symmetric positive-definite Toeplitz matrix \mathbf{B} with elements

$$\mathbf{B}_{ij} = \rho^{|i-j|}, \quad (i, j = 1, \dots, 8), \quad (16)$$

where $0 \leq \rho \leq 1$ describes the channel correlation between adjacent elements. For the extreme case of $\rho = 0$, this models independent flat Rayleigh fading with $\mathbf{C}_h = \mathbf{I}_8$ for all

directions $0^\circ < \theta < 180^\circ$, whereas $\rho = 1$ models the fully coherent case, where $\mathbf{C}_h = \mathbf{d}\mathbf{d}^H$ has rank one. Note that this case is equivalent to the recently much favoured Kronecker channel model [7] which generates random realizations for the channel coefficients \mathbf{h} through

$$\mathbf{h} = \mathbf{R}_T^{\frac{1}{2}} \mathbf{g} \mathbf{R}_R^{\frac{1}{2}} \quad (17)$$

where $\mathbf{R}_T = \mathbf{B}$ and $\mathbf{R}_R = 1$ are constant, and \mathbf{g} is a complex zero-mean random vector with eight i.i.d. Gaussian elements.

We investigated the uncoded bit error rate (BER) of zero-forcing (ZF) and maximum-likelihood (ML) receivers by simulation runs over 2000 independent realizations of the channel coefficients \mathbf{h} . The symbol alphabet in the simulations is based on the QPSK constellation. For the standard QPSK constellation, it turns out that this space-time block code does not fulfill the rank criterion [?] which would effect a loss of diversity order. Let \mathbf{S}_i and \mathbf{S}_j denote two space-time block code symbols corresponding to the symbol vectors s_i and s_j , respectively. For the standard QPSK constellation, it turns out that pairs of symbol vectors exist with $i \neq j$ such that the difference matrix $\mathbf{S}_i - \mathbf{S}_j$ is rank-deficient. This singularity problem is easily circumvented by rotating the standard QPSK constellations for the symbols s_1, \dots, s_8 in the complex plane cf. [5, 8]. To be more specific, we selected

$$s_k \in \{e^{jk\phi}, je^{jk\phi}, -e^{jk\phi}, -je^{jk\phi}\}$$

for $\phi = \frac{32}{180}\pi$ rad. This particular rotation angle was found to minimize the worst-case l_2 -condition number of difference matrices for distinct pairs of code vectors. Results from the random search are shown in Fig. 2. The search revealed that two distinct rotation angles $\frac{32}{180}\pi$ rad and $\frac{57}{180}\pi$ rad solve the minimax problem

$$\min_{0 < \phi < \frac{\pi}{2}} \left(\max_{i>j} (\text{cond}(\mathbf{S}_i - \mathbf{S}_j)) \right),$$

where $\text{cond}(\cdot)$ denotes Matlab's condition number estimate.

We assumed broadside configuration ($\theta = \pi/2$ rad) from the Tx array and various values for the correlation $\rho \in \{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}\}$. From Figure 3, we see that the bit error rate is not sensitive to variations in ρ as long as $\rho \leq 0.75$. The dashed line (—) in Figure 3 shows the BER curves of a maximum-likelihood receiver for diversity order 8, see [9], Eq.(14.4-15). The circles (\circ) in Figure 3 mark the BER performance of a maximum-likelihood receiver for 8th order diversity with an additional E_b/N_o offset of $10 \log_{10}(\frac{7}{4}) = 2.43$ dB according to the noise enhancement (n.e.)

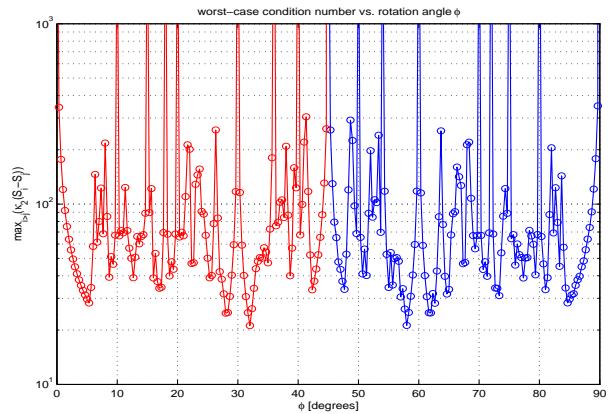


Fig. 2. Worst case condition number of space-time code block differences for (8×1) antenna scheme.

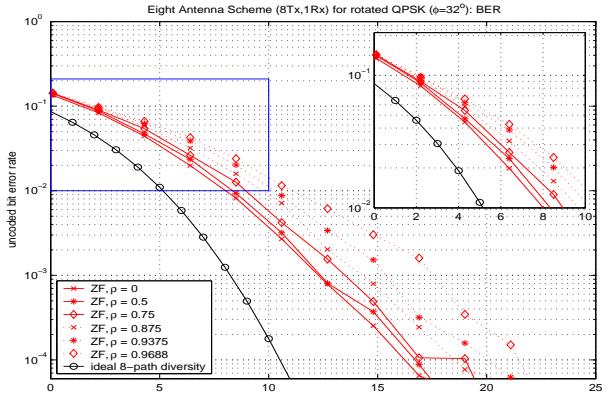


Fig. 3. BER for (8×1) antenna scheme for zero-forcing under correlated fading channels.

6. CONCLUSION

We evaluated the mean mutual information for extended Alamouti space-time block codes in i.i.d. Rayleigh fading analytically. It is shown through Monte Carlo simulations that a Gaussian approximation can be used to evaluate outage levels of mutual information for this recursively defined family of space-time block codes.

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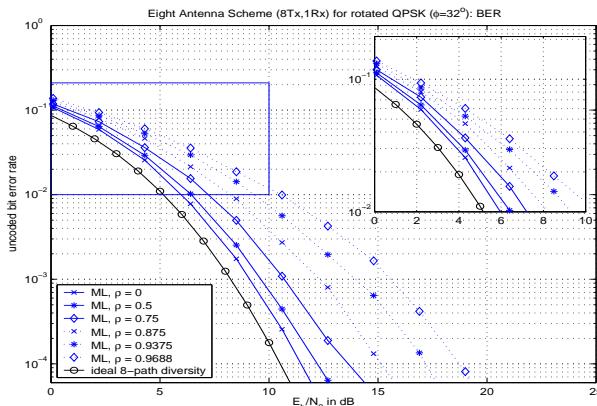


Fig. 4. BER for (8×1) antenna scheme for maximum-likelihood receiver under correlated fading channels.

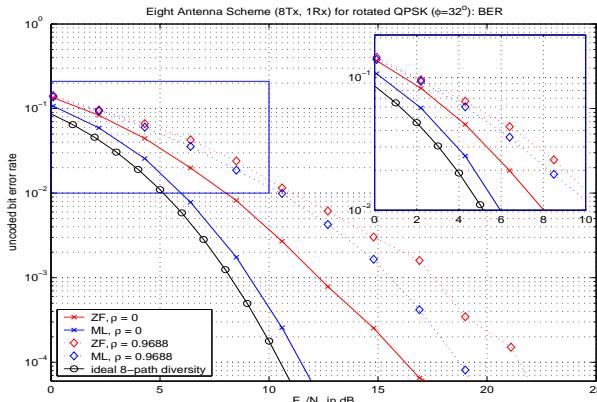


Fig. 5. BER for (8×1) antenna scheme for maximum-likelihood and zero-forcing receivers under correlated fading channels.

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