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On Modal Subspaces of Extended Alamouti Space-Time Block Codes

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ABSTRACT

In this paper, two fundamental subspace invariances of Quasi-Orthogonal Space-Time Block Codes recursively derived from the Alamouti Code (Jafarkhani Codes, Extended Alamouti Codes) are proven: (i) an eigenbasis of the code word distance matrices can be found which is invariant with respect to the chosen modulation alphabet, and (ii) the eigenspaces of the virtual channel matrices are invariant with respect to the channel state realization. These subspace invariances lead to important conclusions: The subspace invariance (i) enables efficient transmission schemes with very small amount of channel state feedback. As a corrolary of (ii), an extremely tight analytical bound for the bit error rate can be obtained.

1 INTRODUCTION

Since the introduction of MIMO transmission concepts in wireless [1, 2], the research booms in this area. Currently Space-Time Codes (STC) are under heavy investigation [3] in order to find flexible schemes allowing various data rates while benefitting from the high potential diversity of MIMO systems. Although Orthogonal STC (OSTC) like the Alamouti Code [4] are optimal in the sense that they offer full diversity at full rate, only very few such OSTCs exist hindering scalability of data rates. Quasi-orthogonal schemes (QOSTC) on the other hand [5] often based on the Alamouti idea [6] and thus called Extended Alamouti Codes [7] (EAC), allow much more flexibility and higher data rates. However, they do not benefit in full of the diversity offered by the channel [8]. Many improvements have been proposed [9, 10] to overcome such drawbacks leaving the QOSTC becoming prominent members for MIMO transmission schemes. Although it was known that for the four [7] and even eight transmit antenna [11] scheme the virtual channel matrices obtained by EAC can easily be diagonalized with sparse matrices independent of the channel realization, such knowledge was not known for general antenna configurations of the form 2^k. This paper will close this gap by showing that indeed for all antenna configurations the modal matrices that diagonalize the virtual channel matrix are simple, sparse matrices independent of the channel realization. Consequences of this are that the modal matrices of EAC are independent of the instantaneous channel parameters. Efficient, low-complexity beamforming is feasible by feeding back the eigenvalues only *without* the eigenvectors. Furthermore, we show that the code distance matrices exhibit a corresponding invariance property. This greatly simplifies the computation of a lower bound [12].

1.1 Recursive Definition of Transmission Code Words

Starting with the standard Alamouti scheme:

$$\mathbf{S}_1 = \begin{pmatrix} s_1 & s_2 \\ s_2^* & -s_1^* \end{pmatrix}, \tag{1}$$

the first extension leads to the EAC matrix for $n_T = 2^k = 4$ transmit antennas:

$$\mathbf{S}_2 = \begin{pmatrix} \mathbf{S}_1^{(1)} & \mathbf{S}_1^{(2)} \\ \mathbf{S}_1^{(2)\,*} & -\mathbf{S}_1^{(1)\,*} \end{pmatrix},$$

where $\mathbf{S}_1^{(i)}, i = 1, 2$ are standard Alamouti matrices. In general, EAC matrices are defined as:

Definition 1: A general $(2^k \times 2^k)$ EAC matrix for $n_T = 2^k$ is recursively defined by:

$$\mathbf{S}_k = \begin{pmatrix} \mathbf{S}_{k-1}^{(1)} & \mathbf{S}_{k-1}^{(2)} \\ \mathbf{S}_{k-1}^{(2)\,*} & -\mathbf{S}_{k-1}^{(1)\,*} \end{pmatrix},$$

the initial value given by (1) above.

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As first stated in [3], the performance of STCs strongly depends on the eigenvalues of the so-called distance matrix $\mathbf{A}_k = \mathbf{B}_k^H \mathbf{B}_k$. Where the difference matrix $\mathbf{B}_k$ is defined as the difference between two allowed STC words $\mathbf{S}_k - \tilde{\mathbf{S}}_k$. It is noteworthy that if $\mathbf{S}_k$ and $\tilde{\mathbf{S}}_k$ are EAC matrices according to Definition 1, then also the code word difference

$$\mathbf{B}_k = \mathbf{S}_k - \tilde{\mathbf{S}}_k = \begin{pmatrix} \mathbf{B}_{k-1}^{(1)} & \mathbf{B}_{k-1}^{(2)} \\ \mathbf{B}_{k-1}^{(2)\,*} & -\mathbf{B}_{k-1}^{(1)\,*} \end{pmatrix} \quad (2)$$

exhibits the same structure as the EAC matrix.

Before stating some important properties on the distance matrix $\mathbf{A}_k$ two additional matrices and their recursive structure have to be defined:

Definition 2: The matrices $\mathbf{A}_k^{(i)}$, $\mathbf{X}_k^{(i,j)}$ and $\mathbf{Y}_k^{(i,j)}$ are defined as:

$$\begin{aligned} \mathbf{A}_k^{(i)} &= \mathbf{B}_k^{(i)^H} \mathbf{B}_k^{(i)} \\ \mathbf{X}_k^{(i,j)} &= \mathbf{B}_k^{(i)^H} \mathbf{B}_k^{(j)} - \mathbf{B}_k^{(j)^T} \mathbf{B}_k^{(i)^*} \\ \mathbf{Y}_k^{(i,j)} &= \mathbf{B}_k^{(i)^T} \mathbf{B}_k^{(j)^*} + \mathbf{B}_k^{(j)^T} \mathbf{B}_k^{(i)^*} \end{aligned}$$

Note that in the following the superscripts (i, j) are sometimes omitted. These matrices exhibit the following simple but important properties:

Lemma 1: The matrices $\mathbf{A}_k$, $\mathbf{X}_k$ and $\mathbf{Y}_k$ are real valued and additionally

$$\begin{aligned} \mathbf{A}_k^T &= \mathbf{A}_k \\ \mathbf{X}_k^T &= -\mathbf{X}_k \\ \mathbf{Y}_k^T &= \mathbf{Y}_k \end{aligned}$$

holds.

Proof: The first properties of Lemma 1 are shown by induction. Starting with a joint proof of $\mathbf{X}_k$ and $\mathbf{Y}_k$ to be real by showing:

$$\begin{aligned} \mathbf{X}_k &= \mathbf{X}_k^* \\ \mathbf{Y}_k &= \mathbf{Y}_k^* \end{aligned}$$

This can be done by inserting the recursive definition of $\mathbf{B}_k$. With this, it can be shown that $\mathbf{A}_k$ is real in the same way.

The remaining properties can simply be shown by inserting the definition of the matrices (Definition 2) and applying the first part of the property (the matrices are real valued).□

With Definition 2, the matrices $\mathbf{A}_k$, $\mathbf{X}_k$ and $\mathbf{Y}_k$ are recursively defined by:

$$\mathbf{A}_k^{(i)} = \begin{pmatrix} \mathbf{A}_{k-1}^{(i1)} + \mathbf{A}_{k-1}^{(i2)} & \mathbf{X}_{k-1}^{(i1,i2)} \\ -\mathbf{X}_{k-1}^{(i1,i2)} & \mathbf{A}_{k-1}^{(i1)} + \mathbf{A}_{k-1}^{(i2)} \end{pmatrix} \quad (3)$$

$$\mathbf{X}_k^{(i,j)} = \begin{pmatrix} \mathbf{X}_{k-1}^{(i1,j1)} + \mathbf{X}_{k-1}^{(i2,j2)} & \mathbf{Y}_{k-1}^{(i1,j2)} - \mathbf{Y}_{k-1}^{(i2,j1)} \\ -\left(\mathbf{Y}_{k-1}^{(i1,j2)} - \mathbf{Y}_{k-1}^{(i2,j1)}\right) & \mathbf{X}_{k-1}^{(i1,j1)} + \mathbf{X}_{k-1}^{(i2,j2)} \end{pmatrix}$$

$$\mathbf{Y}_k^{(i,j)} = \begin{pmatrix} \mathbf{Y}_{k-1}^{(i1,j1)} + \mathbf{Y}_{k-1}^{(i2,j2)} & \mathbf{X}_{k-1}^{(i1,j2)} - \mathbf{X}_{k-1}^{(i2,j1)} \\ -\left(\mathbf{X}_{k-1}^{(i1,j2)} - \mathbf{X}_{k-1}^{(i2,j1)}\right) & \mathbf{Y}_{k-1}^{(i1,j1)} + \mathbf{Y}_{k-1}^{(i2,j2)} \end{pmatrix}$$

Note that due to the structure of the distance matrix $\mathbf{A}_k$ the eigenvalues exhibits specific properties:

Lemma 2: The distance matrix $\mathbf{A}_k$ defined in Definition 2 has pairs of identical eigenvalues.

Proof: Using the general definition of eigenvalues λ and eigenvectors $\mathbf{e}$: $\mathbf{A}_k \mathbf{e} = \lambda \mathbf{e}$ and the structure of $\mathbf{A}_k$ shown in (3), it can be shown that if a eigenvector $(\mathbf{a}^T\ \mathbf{b}^T)$ exists for a given eigenvalue λ, then the eigenvector $(-\mathbf{b}^T\ \mathbf{a}^T)$ is also an eigenvector for the same eigenvalue λ.□

1.2 Recursive Definition of Virtual Channel Matrix

A common model for MIMO transmission is:

$$\mathbf{y} = \mathbf{h}\,\mathbf{S}^T + \mathbf{n}\,,$$

where $\mathbf{y} = (y_1, y_2, ..., y_{n_T})$, $\mathbf{h} = (h_1, h_2, ..., h_{n_T})$, $\mathbf{n} = (n_1, n_2, ..., n_{n_T})$. If $\mathbf{S}$ is the well-known Alamouti code matrix, an equivalent description, and for some cases a more suitable model, is:

$$\tilde{\mathbf{y}} = \mathbf{H}\,\mathbf{s} + \tilde{\mathbf{n}}\,, \quad (4)$$

where $\tilde{\mathbf{y}} = (y_1, y_2^*, ..., y_{n_T})^T$, $\mathbf{s} = (s_1, s_2, ..., s_{n_T})^T$, $\tilde{\mathbf{n}} = (n_1, n_2^*, ..., n_{n_T})^T$. The so called virtual channel matrix $\mathbf{H}$ can recursively be generated, starting with:

$$\mathbf{H} = \begin{pmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{pmatrix}. \quad (5)$$

Definition 3: A general $(2^k \times 2^k)$ EA virtual channel matrix for $n_T = 2^k$ and $n_R = 1$ is recursively defined by:

$$\mathbf{H}_k = \begin{pmatrix} \mathbf{H}_{k-1}^{(1)} & \mathbf{H}_{k-1}^{(2)} \\ -\mathbf{H}_{k-1}^{(2)\,*} & \mathbf{H}_{k-1}^{(1)\,*} \end{pmatrix},$$

starting with (5) above.

2 MAIN RESULTS

Theorem 1: If $\mathbf{S}_{k-1}^{(1)}$ and $\mathbf{S}_{k-1}^{(2)}$ are two different EAC matrices defining a new code word $\mathbf{S}_k$ according to Definition 1, then the following property for $\mathbf{A}_k$ holds. The distance matrix $\mathbf{A}_k$ can be diagonalized by the matrices $\mathbf{V}_k$ and $\mathbf{W}_k$ starting with $k = 2, 3, ..$:

$$\begin{aligned} (1) \quad \mathbf{\Lambda_{A_k}} &= 2^{-(k-1)} \mathbf{V_k^T A_k V_k} \\ (2) \quad \mathbf{\Lambda_{A_k}} &= 2^{-(k-1)} \mathbf{W_k^T A_k W_k} \end{aligned}$$

where $\mathbf{V}_k$ and $\mathbf{W}_k$ is recursively defined by:

$$\begin{aligned} \mathbf{V}_k &= \begin{pmatrix} \mathbf{V}_{k-1} & \mathbf{W}_{k-1} \\ -\mathbf{W}_{k-1} & \mathbf{V}_{k-1} \end{pmatrix}, \\ \mathbf{W}_k &= \begin{pmatrix} \mathbf{W}_{k-1} & \mathbf{V}_{k-1} \\ \mathbf{V}_{k-1} & -\mathbf{W}_{k-1} \end{pmatrix}, \end{aligned}$$

with the initial values

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{W}_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Additional properties are:

$$\begin{array}{ll} (3) & \mathbf{W}_k^T \mathbf{X}_k^{(i,j)} \mathbf{W}_k + \mathbf{V}_k^T \mathbf{X}_k^{(i,j)} \mathbf{V}_k = \mathbf{0} \\ (4) & \mathbf{W}_k^T \mathbf{Y}_k^{(i,j)} \mathbf{V}_k - \mathbf{V}_k^T \mathbf{Y}_k^{(i,j)} \mathbf{W}_k = \mathbf{0} \\ (5) & \mathbf{W}_k^T \mathbf{A}_k^{(i)} \mathbf{V}_k - \mathbf{V}_k^T \mathbf{A}_k^{(i)} \mathbf{W}_k = \mathbf{0} \\ (6) & \mathbf{W}_k^T \mathbf{X}_k^{(i,j)} \mathbf{V}_k - \mathbf{V}_k^T \mathbf{X}_k^{(i,j)} \mathbf{W}_k = 2^k \mathbf{\Lambda}_{\mathbf{X}_k} \\ (7) & \mathbf{W}_k^T \mathbf{Y}_k^{(i,j)} \mathbf{W}_k + \mathbf{V}_k^T \mathbf{Y}_k^{(i,j)} \mathbf{V}_k = 2^k \mathbf{\Lambda}_{\mathbf{Y}_k} \end{array}$$

The proof of this theorem is shown in Appendix A. Note that in this form the theorem requires a scaling term $2^{-(k-1)}$. The authors are well aware that this term could be consumed in the matrices, however, in the form presented, the entries of the matrices $\mathbf{V}_k, \mathbf{W}_k$ are always in $\{-1, 0, 1\}$ and thus very suitable for implementation. The formulation of the theorem with an explicit scaling term reflects such properties.

A very similar statement can be proven for the virtual channel matrix. In order to do so we have to redefine the following terms:

$$\begin{aligned} \mathbf{A}_k^{(i)} &= \mathbf{H}_k^{(i)H} \mathbf{H}_k^{(i)} \\ \mathbf{X}_k^{(i,j)} &= \mathbf{H}_k^{(i)H} \mathbf{H}_k^{(j)} - \mathbf{H}_k^{(j)T} \mathbf{H}_k^{(i)*} \\ \mathbf{Y}_k^{(i,j)} &= \mathbf{H}_k^{(i)T} \mathbf{H}_k^{(j)*} + \mathbf{H}_k^{(j)T} \mathbf{H}_k^{(i)*}, \end{aligned}$$

Theorem 2: If $\mathbf{H}_{k-1}^{(1)}$ and $\mathbf{H}_{k-1}^{(2)}$ are two different virtual channel matrices defining a new virtual channel matrix $\mathbf{H}_k$ according to Definition 3, then the following property for $\mathbf{H}_k$ holds. The virtual channel matrix $\mathbf{H}_k$ can be diagonalized by the matrices $\mathbf{V}_k$ and $\mathbf{W}_k$:

$$\begin{array}{ll} (1) & \mathbf{\Lambda_{H_k}} = 2^{-(k-1)} \mathbf{V_k^T H_k^T H_k V_k} \\ (2) & \mathbf{\Lambda_{H_k}} = 2^{-(k-1)} \mathbf{W_k^T H_k^T H_k W_k} \end{array}$$

where $\mathbf{V}_k$ and $\mathbf{W}_k$ follow the same recursions as defined in Theorem 1. Furthermore, the properties (3)-(7) of Theorem 1 hold correspondingly. The proof follows identical arguments as Theorem 1 and is thus not repeated.

3 CONCLUSIONS

The implication of both theorems are manyfold as outlined in the following:

1. The BER behavior is governed by the eigenvalues of the distance matrix $\mathbf{A}$ as shown in [3]. More importantly, the *constant* matrix $\mathbf{V}$ (or $\mathbf{W}$) diagonalizes the distance matrix $\mathbf{A}$, i.e., irrespective of the involved code word pairs and alphabet. Due to this property an extraordinary tight BER approximation can be analytically calculated as shown in [12]. These important eigenvalues can be calculated with the properties of Theorem 1. As an illustrative example, the eigenvalues of $\mathbf{A}_2$ for $n_T = 4$ (k=2) read:

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

with

$$\begin{aligned} \lambda_1 = \lambda_4 &= |(s_1 - \tilde{s}_1) + (s_4 - \tilde{s}_4)|^2 + |(s_2 - \tilde{s}_2) - (s_3 - \tilde{s}_3)|^2 \\ \lambda_2 = \lambda_3 &= |(s_1 - \tilde{s}_1) - (s_4 - \tilde{s}_4)|^2 + |(s_2 - \tilde{s}_2) + (s_3 - \tilde{s}_3)|^2 \end{aligned}$$

A direct consequence of such specific eigenvalues is that the maximum rank is four and its minimum two. It can be seen in the above equations that zero rank is not possible since in the case all eigenvalues are zero, no symbol error can occur. As already shown in [3], the asymptotic slope of the pairwise error probability (PEP) curves for high signal to noise ratios is determined by the minimum rank. Thus, some PEP curves exhibit a slope of two and therefore the total BER performance curve has slope two, which can be seen in [12].

2. Similarly the eigenvalues of the virtual channel matrix appear in pairs. In the case of four transmit antennas they are determined by two values h^2 and X (see [7])

$$\lambda_1 = \lambda_4 = h^2(1 - X), \quad \lambda_2 = \lambda_3 = h^2(1 + X).$$

If for example beamforming is to implement, the power of the eigenvalue is of less importance and the knowledge of $X \in (-1, 1)$ alone is sufficient channel state information to be fed back for optimal beamforming. For larger antenna schemes, it is common to implement beamforming only on the largest eigenvalue. Since the largest eigenvalue comes in pairs with two eigenvectors, beamforming methods utilizing such QSTBCs are expected to be very efficient. Next to the largest eigenvalue it is required to select the corresponding eigenvector. If all eigenvectures are labeled by numbers, the selection information can be performed by re-transmitting only $\log_2(n_T) - 1$ additional feedback bits where n_T is the number of transmit antennas.

3. From an implementational viewpoint, we wish to highlight a beneficial property of the recursively defined modal matrices $\mathbf{V}_k$ and $\mathbf{W}_k$: The elements take values from $\{-1, 0, 1\}$. Thus, the diagonalization of the virtual channel matrix can be realized without multipliers.

Appendix A

Proof of Theorem 1: All properties are shown by induction. The results for k=1 are easily obtained by inserting the initial values for all matrices involved. Starting with a joint proof of property (3) and (4). In order to get a simpler notation in the following we drop the indices (k-1) and (i,j). Inserting the recursions for $\mathbf{W}_k$,$\mathbf{V}_k$ and $\mathbf{X}_k$ into property (3) yields

$$\mathbf{W}_k^T\mathbf{X}_k^{(i,j)}\mathbf{W}_k + \mathbf{V}_k^T\mathbf{X}_k^{(i,j)}\mathbf{V}_k = \begin{pmatrix} \boldsymbol{\alpha}_{11} & \boldsymbol{\alpha}_{12} \\ \boldsymbol{\alpha}_{21} & \boldsymbol{\alpha}_{22} \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{\alpha}_{11}=\boldsymbol{\alpha}_{22}=&2\left[\mathbf{W}^T(\mathbf{X}-\tilde{\mathbf{X}})\mathbf{W}+\mathbf{V}^T(\mathbf{X}-\tilde{\mathbf{X}})\mathbf{V}\right] \\ &+2\left[\mathbf{W}^T(\mathbf{Y}-\tilde{\mathbf{Y}})\mathbf{V}-\mathbf{V}^T(\mathbf{Y}-\tilde{\mathbf{Y}})\mathbf{W}\right] \\ \boldsymbol{\alpha}_{12}=\boldsymbol{\alpha}_{21}=&\mathbf{0} \end{aligned}$$

Here, $\mathbf{X}$,$\tilde{\mathbf{X}}$,$\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ denote $\mathbf{X}^{(i1,j1)}$, $\mathbf{X}^{(i2,j2)}$, $\mathbf{Y}^{(i1,j2)}$ and $\mathbf{Y}^{(i2,j1)}$, respectively. Property (4) yields:

$$\mathbf{W}_k^T\mathbf{Y}_k^{(i,j)}\mathbf{V}_k - \mathbf{V}_k^T\mathbf{Y}_k^{(i,j)}\mathbf{W}_k = \begin{pmatrix} \boldsymbol{\beta}_{11} & \boldsymbol{\beta}_{12} \\ \boldsymbol{\beta}_{21} & \boldsymbol{\beta}_{22} \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{\beta}_{11}=-\boldsymbol{\beta}_{22}=&2\left[\mathbf{W}^T(\mathbf{Y}+\tilde{\mathbf{Y}})\mathbf{V}-\mathbf{V}^T(\mathbf{Y}+\tilde{\mathbf{Y}})\mathbf{W}\right] \\ &-2\left[\mathbf{W}^T(\mathbf{X}-\tilde{\mathbf{X}})\mathbf{W}+\mathbf{V}^T(\mathbf{X}-\tilde{\mathbf{X}})\mathbf{V}\right] \\ \boldsymbol{\beta}_{12}=\boldsymbol{\beta}_{21}=&\mathbf{0} \end{aligned}$$

Here, $\mathbf{X}$,$\tilde{\mathbf{X}}$,$\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ denote $\mathbf{X}^{(i1,j2)}$, $\mathbf{X}^{(i2,j1)}$, $\mathbf{Y}^{(i1,j1)}$ and $\mathbf{Y}^{(i2,j2)}$, respectively. We jointly show that property (3) and (4) hold for the (k)-step under the assumption of fulfilling the properties for the (k-1)-step. If both properties (3) and (4) are fulfill for the (k-1)-step, the variables $\boldsymbol{\alpha}_{11}$, $\boldsymbol{\alpha}_{22}$, $\boldsymbol{\beta}_{11}$ and $\boldsymbol{\beta}_{22}$ become the zero matrix $\mathbf{0}$ either.

Property (5) is shown in same way:

$$\mathbf{W}_k^T\mathbf{A}_k^{(i)}\mathbf{V}_k - \mathbf{V}_k^T\mathbf{A}_k^{(i)}\mathbf{W}_k = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

$$\begin{aligned} \gamma_{11}=-\gamma_{22}=&2\overbrace{\left[\mathbf{W}^T(\mathbf{A}^{(i1)}+\mathbf{A}^{(i2)})\mathbf{V}-\mathbf{V}^T(\mathbf{A}^{(i1)}+\mathbf{A}^{(i2)})\mathbf{W}\right]}^{=\mathbf{0}\text{ with Property (5)}} \\ &-2\underbrace{\left[\mathbf{W}^T\mathbf{X}^{(i1,i2)}\mathbf{W}+\mathbf{V}^T\mathbf{X}^{(i1,i2)}\mathbf{V}\right]}_{=\mathbf{0}\text{ with Property (3)}} \\ \gamma_{12}=\gamma_{21}=&\mathbf{0} \end{aligned}$$

Property (6) and Property (7) have to be shown jointly. Starting with (6):

$$\mathbf{W}_k^T\mathbf{X}_k^{(i,j)}\mathbf{V}_k - \mathbf{V}_k^T\mathbf{X}_k^{(i,j)}\mathbf{W}_k = \begin{pmatrix} \boldsymbol{\delta}_{11} & \boldsymbol{\delta}_{12} \\ \boldsymbol{\delta}_{21} & \boldsymbol{\delta}_{22} \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{\delta}_{11}=-\boldsymbol{\delta}_{22}=&2\overbrace{\left[\mathbf{W}^T(\mathbf{X}-\tilde{\mathbf{X}})\mathbf{V}-\mathbf{V}^T(\mathbf{X}-\tilde{\mathbf{X}})\mathbf{W}\right]}^{=2^{k-1}(\boldsymbol{\Lambda}_{\mathbf{X}}-\boldsymbol{\Lambda}_{\tilde{\mathbf{X}}})\text{ with property (6)}} \\ &-2\underbrace{\left[\mathbf{V}^T(\mathbf{Y}-\tilde{\mathbf{Y}})\mathbf{V}+\mathbf{W}^T(\mathbf{Y}-\tilde{\mathbf{Y}})\mathbf{W}\right]}_{=2^{k-1}(\boldsymbol{\Lambda}_{\mathbf{Y}}-\boldsymbol{\Lambda}_{\tilde{\mathbf{Y}}})\text{ with property (7)}} \\ \boldsymbol{\delta}_{12}=\boldsymbol{\delta}_{21}=&\mathbf{0} \end{aligned}$$

Here, $\mathbf{X}$,$\tilde{\mathbf{X}}$,$\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ denote $\mathbf{X}^{(i1,j1)}$, $\mathbf{X}^{(i2,j2)}$, $\mathbf{Y}^{(i1,j2)}$ and $\mathbf{Y}^{(i2,j1)}$, respectively. Property (7) yields:

$$\mathbf{W}_k^T\mathbf{Y}_k^{(i,j)}\mathbf{W}_k + \mathbf{V}_k^T\mathbf{Y}_k^{(i,j)}\mathbf{V}_k = \begin{pmatrix} \boldsymbol{\epsilon}_{11} & \boldsymbol{\epsilon}_{12} \\ \boldsymbol{\epsilon}_{21} & \boldsymbol{\epsilon}_{22} \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{\epsilon}_{11}=\boldsymbol{\epsilon}_{22}=&2\overbrace{\left[\mathbf{W}^T(\mathbf{Y}+\tilde{\mathbf{Y}})\mathbf{W}+\mathbf{V}^T(\mathbf{Y}+\tilde{\mathbf{Y}})\mathbf{V}\right]}^{=2^{k-1}(\boldsymbol{\Lambda}_{\mathbf{Y}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{Y}}})\text{ with property (7)}} \\ &+2\underbrace{\left[\mathbf{W}^T(\mathbf{X}+\tilde{\mathbf{X}})\mathbf{V}-\mathbf{V}^T(\mathbf{X}+\tilde{\mathbf{X}})\mathbf{W}\right]}_{=2^{k-1}(\boldsymbol{\Lambda}_{\mathbf{X}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{X}}})\text{ with property (6)}} \\ \boldsymbol{\epsilon}_{12}=\boldsymbol{\epsilon}_{21}=&\mathbf{0} \end{aligned}$$

Here, $\mathbf{X}$,$\tilde{\mathbf{X}}$,$\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ denote $\mathbf{X}^{(i1,j2)}$, $\mathbf{X}^{(i2,j1)}$, $\mathbf{Y}^{(i1,j1)}$ and $\mathbf{Y}^{(i2,j2)}$, respectively. We jointly show that property (6) and (7) hold for the (k)-step under the assumption of fulfilling the properties for the (k-1)-step. If both properties (6) and (7) are fulfill for the (k-1)-step, the variables $\boldsymbol{\delta}_{11}$, $\boldsymbol{\delta}_{22}$, $\boldsymbol{\epsilon}_{11}$ and $\boldsymbol{\epsilon}_{22}$ changes to the values indicated by the brackets. Thus, the following recursion hold for $\boldsymbol{\Lambda}_{\mathbf{X}_k^{(i,j)}}$:

$$\boldsymbol{\Lambda}_{\mathbf{X}_k^{(i,j)}}=\begin{pmatrix} \boldsymbol{\Lambda}_{\mathbf{X}}-\boldsymbol{\Lambda}_{\tilde{\mathbf{X}}}-\boldsymbol{\Lambda}_{\mathbf{Y}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{Y}}} & \mathbf{0} \\ \mathbf{0} & -(\boldsymbol{\Lambda}_{\mathbf{X}}-\boldsymbol{\Lambda}_{\tilde{\mathbf{X}}}-\boldsymbol{\Lambda}_{\mathbf{Y}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{Y}}}) \end{pmatrix}$$

($\mathbf{X}$,$\tilde{\mathbf{X}}$,$\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ denote $\mathbf{X}^{(i1,j1)}$, $\mathbf{X}^{(i2,j2)}$, $\mathbf{Y}^{(i1,j2)}$ and $\mathbf{Y}^{(i2,j1)}$, respectively.) and for $\boldsymbol{\Lambda}_{\mathbf{Y}_k^{(i,j)}}$

$$\boldsymbol{\Lambda}_{\mathbf{Y}_k^{(i,j)}}=\begin{pmatrix} \boldsymbol{\Lambda}_{\mathbf{Y}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{Y}}}+\boldsymbol{\Lambda}_{\mathbf{X}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{X}}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_{\mathbf{Y}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{Y}}}+\boldsymbol{\Lambda}_{\mathbf{X}}+\boldsymbol{\Lambda}_{\tilde{\mathbf{X}}}) \end{pmatrix}$$

($\mathbf{X}$,$\tilde{\mathbf{X}}$,$\mathbf{Y}$ and $\tilde{\mathbf{Y}}$ denote $\mathbf{X}^{(i1,j2)}$, $\mathbf{X}^{(i2,j1)}$, $\mathbf{Y}^{(i1,j1)}$ and $\mathbf{Y}^{(i2,j2)}$, respectively.) Finally, we can proof the properties (1) and (2) jointly by inserting the recursion formula for $\mathbf{A}_k^{(i)}$. With the aid of the above proven properties (3) - (7) it is easy to show that the properties (1) and (2) hold. The result of the proof is a practical recursion for $\boldsymbol{\Lambda}_{\mathbf{A}_k^{(i)}}$:

$$\boldsymbol{\Lambda}_{\mathbf{A}_k^{(i)}}=\begin{pmatrix} \boldsymbol{\Lambda}_{\mathbf{A}_{k-1}^{(i1)}}+\boldsymbol{\Lambda}_{\mathbf{A}_{k-1}^{(i2)}}+\boldsymbol{\Lambda}_{\mathbf{X}_{k-1}^{(i1,i2)}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_{\mathbf{A}_{k-1}^{(i1)}}+\boldsymbol{\Lambda}_{\mathbf{A}_{k-1}^{(i2)}}+\boldsymbol{\Lambda}_{\mathbf{X}_{k-1}^{(i1,i2)}} \end{pmatrix}$$

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