Performance Analysis of Unbiased Finite-Length DFE Receivers

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Abstract

We consider the problem of equalization of linear (finite-impulse-response - FIR) channels with the use of desicion feedback equalizers (DFE's). In recent contributions [2], [3], it was shown that an arbitrary FIR channel can be perfectly equalized in the absence of noise with the use of an FIR-DFE receiver (whether it is fractionally-spaced (FS) or not). In such cases, by properly choosing the filter lengths, as well as recovery delay parameters, it is possible to null perfectly the residual intersymbol interference (ISI) at the decision device (DD) output, without compromising the performance. The simplicity and practical importance of this case call for a performance analysis of these so-called unbiased *DFE receivers. In the absence of residual ISI, the key issue that determines bit error rate (BER) performance is* error propagation. *We derive both approximative and exact closed-form expressions for the BER of unbiased DFE receivers. These expressions allow for a simple calculation of the equalizer output BER at steady-state, as well as for a quantitative description of issues such as instability and erratic behavior. In support to our findings, we provide computer simulation results that verify the validity of our results.*

1 Introduction

Since their introduction by Austin [1] more than thirty years ago, decision feedback equalizers have been founding an increasing number of applications, that range from high-speed wireline to satellite and cellular channel modems. Combining a performance that is superior to linear equalizers and a complexity far smaller than that of the Viterbi algorithm (VA), as well as close-to-optimal performance when used with suitable coding schemes, it is foreseeable that DFE's will continue to play an important role in modem design for both fixed and wireless communication systems, as well as in non-communication (such as magnetic storage) applications for years to come.

As evidenced by the large number of recent theoretical contributions on the performance analysis of DFE's, there are still a number of issues that remain open in the understanding of DFE receiver performance. In particular, there is an increased interest in the theoretical BER evaluation of DFE's for specific channels, especially at low BER regions where simulation is either inefficient or excessively time-consuming. Such evaluations may allow for a rapid assessment of the performance which could lead to important choices of receiver parameters and design criteria that affect cost, complexity, and performance in a particular application.

Some recent results on BER performance analysis of DFE's include [4] and [5]. In [4], DFE performance is analyzed in the case of non-zero residual ISI at the DD output (this case is called "biased" DFE, in the sense that in the absence of noise the error does not vanish). In the case of error propagation, an approximative Markov model is used in conjunction with a Fourier-series expansion of the noise distribution in order to derive approximative BER expressions for BPSK and QPSK sources. In [5], a closed-form expression is given for PAM sources in the case of unbiased DFE's for a single-post-echo channel. An iterative method for the BER calculation in the more general case of arbitrary FIR channels is also proposed in [5]. An older but important contribution is contained in [6]: assuming negligible pre-cursor ISI (hence perfect ISI cancellation by the FBF is possible), and white noise at the DD input (which requires an infinite-length FFF), closed-form expressions for the BER of BPSK and QPSK sources are derived. These expressions require, in the general case, the performance of a matrix eigen-decomposition.

In the above-mentioned contributions, residual ISI at the DD input is assumed either (1) present due to pre-cursor ISI and / or the short length of the

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FBF, or (2) negligible due to the use of a long (ideally infinite-length) FFF. However, in [2], [3], it was shown that if the FIR filter lengths and delay parameters are properly chosen, residual ISI can be completely eliminated (for both symbol-spaced and FS FFF's). This amounts essentially to rendering all ISI post-cursor. In this case, the determination of the BER performance is facilitated, as the key factor will be only error propagation. This revives our interest in the work in [6], which treats essentially a special case of unbiased DFE performance. In this contribution, we will attempt to analyze the BER performance of unbiased DFE's, aiming at the derivation of simple mathematical expressions. It will be seen that such simple expressions approximate well (within deviations typically in the order of about 1 dB in SNR) the actual equalizer performance. Moreover, they allow for a number of interpretations and interesting insights on DFE performance.

The rest of the paper is organized as follows. In Section 2 we present our basic model and assumptions. Section 3 contains our main contribution which is the derivation of analytical expressions for the BER of minimally configured unbiased DFE's. In Section 4 we provide a discussion and some interpretations based on the results of section 3. Section 5 contains some computer simulation results in support of our theoretical findings. Finally, Section 6 contains our conclusions.

2 Notation and assumptions

We consider a finite-length DFE receiver, as shown in Figure 1, which depicts the baseband representation of the cascade of an FIR channel (whose output is sampled at symbol rate) and a symbol-rate decision feedback equalizer.

According to the figure, we define the following quantities: $a(k)$ is the transmitted input symbol, $x(k)$ the channel output, $y_f(k)$ the FFF output, $y_b(k)$ the FBF output, $y(k)$ the decision device input, $\hat{a}(k)$ the decision device output, all at time instant k. $H(z)$ is the $z-$ transform of the channel impulse response $\{h(n)\}$, and $F_f(z)$, $F_b(z)$ are the $z-$ transforms of the feed-forward (FFF) and feedback (FBF) filters, respectively. In the above description, all samples are assumed to be taken at the symbol rate. Assuming an FIR symbol-rate channel, our signal model is

$$x(k) = \sum_{i=0}^{N-1} h(i)\, a(k-i)+n(k) = H\bar{A}(k)+n(k) \quad (1)$$

where the symbol-rate sampled channel response $\{h_i,\ i = 0,\ldots,N-1\}$ is assumed to have N non-zero coefficients, $H = [h_0 \ \cdots \ h_{N-1}]$ and $\bar{A}(k) =$

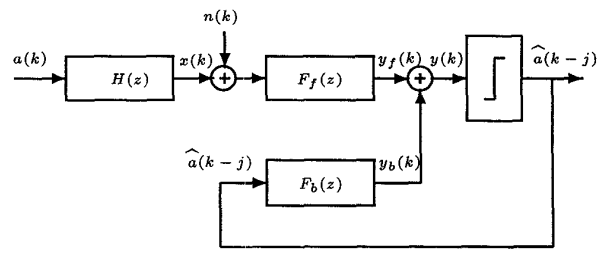

Figure 1: Schematic diagram of a symbol-rate DFE

$[a(k) \ \cdots \ a(k-N+1)]^T$, $n(k)$ is the additive noise sample at the channel output, and T denotes matrix transpose. In the sequel we will assume for simplicity that all h_i are real. We also define the FFF and FBF vectors as

$$\begin{aligned} F_f &= [F_f^0 \ \cdots F_f^{L_f-1}] \\ F_b &= [F_b^0 \ \cdots F_b^{L_b-1}] \end{aligned} \quad (2)$$

where we have assumed that they contain L_f and L_b non-zero coefficients, respectively. The input to the decision device (DD) $y(k)$ can then be written as:

$$\begin{aligned} y(k) &= y_f(k)+y_b(k) = X^T(k)F_f(k)+\widehat{A}^T(k)F_b(k) \\ &= \widetilde{X}^T(k)F(k) \end{aligned} \quad (3)$$

where the regressors $X(k)$, $\widehat{A}(k)$ and $\widetilde{X}(k)$ are defined as

$$\begin{aligned} X^T(k) &= [x(k) \ \cdots x(k-L_f+1)] \\ \widehat{A}^T(k) &= [\hat{a}(k-j)\cdots\hat{a}(k-j-L_b+1)] \\ \widetilde{X}(k) &= [X^T(k) \ \ \widehat{A}^T(k)]^T \end{aligned} \quad (4)$$

j is a positive integer and $F(k)$ is defined as:

$$F(k) = [F_f^T(k) \ \ F_b^T(k)]^T \quad (5)$$

The DD output $\hat{a}(k)$ is defined as

$$\hat{a}(k-j) = \mathrm{dec}(y(k)) \quad (6)$$

where $\mathrm{dec}(y(k))$ denotes the closest constellation symbol to $y(k)$, and is assumed to be an estimate of the transmitted symbol $a(k-j)$. Notice that, according to (6), the DD output is delayed with respect to the channel input. This delay j is positive since the receiver is a causal system. Moreover, the choice of this delay is important for the performance of the receiver.

In [2], [3] it was shown that the following theorem holds for symbol-rate finite-length DFE receivers:

<u>Theorem I:</u> Perfect zero-forcing FIR decision feedback

equalizers exist in the absence of noise if $L_f+L_b \geq M$, where L_f, L_b are the FFF and FBF lengths, respectively, and $M = \max(L_f+N-1, j+L_b-1)$.

A similar theorem was shown in [3] for fractionally-spaced DFE's. Assuming that we choose indeed the lengths of the FFF and FBF filters to satisfy the condition of Theorem I, we will now analyze the performance of the equalizer in terms of BER.

3 Performance analysis

In a first stage, we will consider the minimal configuration that guarantees, according to Theorem I, the complete elimination of residual ISI, i.e. $L_f = 1$, $L_b = N-1$. It can then be shown that, for $j = 1$ (and assuming that both $\{x(k)\}$ and $\{\hat{a}(k-j)\}$ are further delayed by 1 symbol period before entering the FFF and FBF, respectively), the DD input will be given by:

$$y(k) = a(k-1) + \frac{1}{h_0}\left(\sum_{i=1}^{N-1} \Delta a(k-i-1) + n(k-1)\right) \quad (7)$$

where $\Delta a(k-l) = a(k-l) - \hat{a}(k-l)$. Notice that the expression (7) is almost identical in form to the expression for the DD input considered in [6] (where the normalization $h_0 = 1$ was used). In the sequel we will study the BER performance of the process $y(k)$ described in (7), assuming the input $a(k)$ to be BPSK, and $\{n(k)\}$ to be zero-mean AWGN of variance σ^2.

3.1 No error propagation

Assuming correct decisions, $\Delta a(k) = 0$, $\forall\ k$, and (7) reduces to

$$y(k) = a(k-j) + \frac{1}{h(0)} n(k-1) \quad (8)$$

which corresponds to an additive noise channel: the noise in the model (8) is only scaled. Notice that, as opposed to [6], the noise is perfectly white in our model, whereas in [6], it is assumed white as an approximation (which would require an infinite-length FFF). The BER in this trivial case is straightforward to compute: assuming binary antipodal signaling ($a(k) = \pm 1$ equally likely), the uncoded bit error probability is given by:

$$P_0 = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{|h_0|^2}{2\sigma^2}}\right) \quad (9)$$

It is expected, of course, that, as the assumption of correct decisions will not be valid in practice, the expression in (9) will be optimistic. It can hence be considered only as a performance bound for this case, which should be approached only for high SNR's.

3.2 Error propagation

According to Eq. (7), the vector $\mathbf{\Delta} a(k) = [\Delta a(k-2) \cdots \Delta a(k-N)]$ is a first-order Markov process. Assuming $\{a(k)\}$ to be BPSK, $\mathbf{\Delta} a(k)$ has 2^{N-1} states, stemming from the correctness or not of each of the $N-1$ corresponding symbols (more rigorously, each component of $\mathbf{\Delta} a(k)$ can be either $\Delta a(k-i-1) = 0$, or $\Delta a(k-i-1) = 2a(k-i-1) = \pm 2$). Each of the 2^{N-1} states will occur with different probability. Assuming that all $\Delta a(k)$ are i.i.d, then these probabilities are given in the following table:

Probability	Non-zero elements in $\mathbf{\Delta} a(k)$
$(1-P_e)^{N-1}$	0
$P_e(1-P_e)^{N-2}$	1
$\vdots$	$\vdots$
P_e^{N-1}	$N-1$

In other words, the probability of $\mathbf{\Delta} a(k)$ containing j errors is $P_e^j(1-P_e)^{N-1-j}$ ($j \in [0, N-1]$). The error probability can now be expressed as:

$$\begin{aligned} P_e &= Pr\left[y(k) > 0 | a(k-1) = -1\right] = \\ &(1-P_e)^{N-1} Pr\left[\frac{1}{h_0} n(k-1) > 1\right] \\ &+ P_e(1-P_e)^{N-2} \sum_{i=1}^{N-1} Pr\left[\frac{2h_i}{h_0} a(k-i-1) + \frac{n(k-1)}{h_0} > 1\right] \\ &+ \cdots + \\ &+ P_e^{N-1} Pr\left[\frac{2}{h_0}\sum_{i=1}^{N-1} h_i a(k-i-1) + \frac{n(k-1)}{h_0} > 1\right] = \\ &(1-P_e)^{N-1}P_0 + P_e(1-P_e)^{N-2}P_1 + \cdots + P_e^{N-1}P_{N-1} \end{aligned} \quad (10)$$

In (10), each P_j comprises of $\binom{N-1}{j}$ probability terms. An exact solution of the polynomial equation (10) would require the solution of an equivalent eigenvalue problem (see [6]). Here instead, we will use an approximation of (10) by keeping terms only up to second order in P_e, and neglecting higher order terms. Especially in high SNR regions (which are of most importance since they require many simulation runs to determine P_e), this approximation is expected to yield minimal penalty. Eq. (10) then is approximated by:

$$\begin{aligned} &P_e^2\left[P_0(N-1)(N-2)/2 - P_1(N-2) + P_2\right] + \\ &P_e\left[P_1 - (N-1)P_0 - 1\right] + P_0 = 0 \end{aligned} \quad (11)$$

The error probability can thus be approximated (for $N > 2$) by the solution to the quadratic equation (11):

$$\begin{aligned} &P_e \simeq 1/\left(2[(N-1)(N-2)P_0/2 - (N-2)P_1 + P_2]\right) \times \\ &[(-P_1 + (N-1)P_0 + 1) \pm \\ &\sqrt{[P_1-(N-1)P_0-1]^2 - 4P_0[(N-1)(N-2)/2P_0 - (N-2)P_1 + P_2]}\Big] \end{aligned} \quad (12)$$

with the constraint $0 \leq P_e \leq 1$. Interestingly, only three probabilities are involved in (12), namely, P_0, P_1, and P_2. P_0 is given in (9), whereas P_1 and P_2 are given below:

$$P_1 = \frac{1}{4}\sum_{i=1}^{N-1}\left(\text{erfc}\left[\frac{1-2\left(\frac{h_i}{h_0}\right)}{\sqrt{2\sigma^2/|h_0|^2}}\right]+\text{erfc}\left[\frac{1+2\left(\frac{h_i}{h_0}\right)}{\sqrt{2\sigma^2/|h_0|^2}}\right]\right) \tag{13}$$

$$P_2 = \frac{1}{8}\sum_{j<l}^{N-1} P_{++}(j,l)+P_{+-}(j,l)+P_{-+}(j,l)+P_{--}(j,l) \tag{14}$$

where

$$P_{\pm\pm}(j,l) = \frac{1}{2}\text{erfc}\left[\frac{1-2\left(\frac{\pm h_j \pm h_l}{h_0}\right)}{\sqrt{2\sigma^2/|h_0|^2}}\right] \tag{15}$$

Note that the expression in (12), while expected to be accurate due to the inclusion of second order terms, remains computationally very simple. It involves no Fourier-series expansion or eigenvalue decomposition, and only requires the computation of three error probability terms. It is thus much simpler than previously reported methods for DFE performance evaluation.

4 Discussion

In the trivial case N=2, only linear terms are contained in (10), and the BER is given by the simple formula

$$P_e = P_0/\left(1+P_0-P_1\right) \tag{16}$$

which coincides with the result in [6]. Notice also that for N=3, the formula (12) is exact, as only up to quadratic terms exist in this case in (10). An even simpler approximation would take into account only the linear terms in (10), yielding:

$$P_e \simeq P_0/\left(1+(N-1)P_0-P_1\right) \tag{17}$$

Even though less rigorous than (11), Eq. (17) allows for some interesting observations regarding the performance of the considered minimal DFE setup.

Consider first the case $N = 3$. If the arguments of the two components of each of the terms in (13) have opposite signs, then for vanishing noise, $P_1 = 1$ and (17) gives $P_e \simeq 1/2$. Hence, in this case, if both $h_1^2/h_0^2 > 1/4$ and $h_2^2/h_0^2 > 1/4$, the DFE will have an erratic behavior and act as an oscillator! Therefore it is important for the good operation of the equalizer to have h_1^2 and h_2^2 as small as possible, compared to h_0^2. This appears intuitively correct, as one expects that the minimal unbiased configuration $N_f = 1$, $N_b = N-1$ which depends heavily on $1/h_0$, will be suitable for a channel as causal as possible. Using similar arguments for $N > 3$, it is clear that, for the configuration studied, the strong non-causality of the channel may result in the instability of the DFE receiver. On the other hand, good behavior is expected, if, for each $1 \leq i \leq N-1$, the following holds

$$h_i^2/h_0^2 < 1/4 \tag{18}$$

By observing (17), it appears also that, at some SNR's, the denominator may be close to zero, thus leading to "notches" in the BER expressions as a function of SNR. This fact justifies the use of the quadratic expression (12), which tends to be more insensitive to this problem.

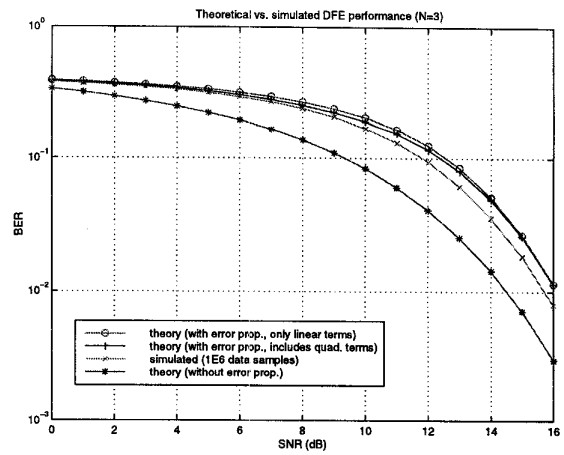

Figure 2: A comparison of theoretical vs. simulated performance

5 Simulation results

Figure 2 shows the actual (simulated based on 10^6 data samples) and predicted (using (9)– no error propagation, (12), or (17)) BER performance for a 3-tap channel with impulse response $H = [1 \;\; 2 \;\; 0.5]$. Notice that (12) gives a prediction that is within 1 dB of the actual performance (about 2/3 of a dB at the 10^{-1} BER level). This small, albeit finite mismatch is probably due to the assumption that the components of the error vector $\Delta a(k)$ are assumed to be i.i.d., whereas this may not necessarily be the case in practice. Notice also that the linear approximation in (17) is, as expected, very close to the quadratic solution in (12) – in this case, as mentioned above, the quadratic solution is exact –. On the other hand, the expression in (9), as expected, underestimates the BER (espe-

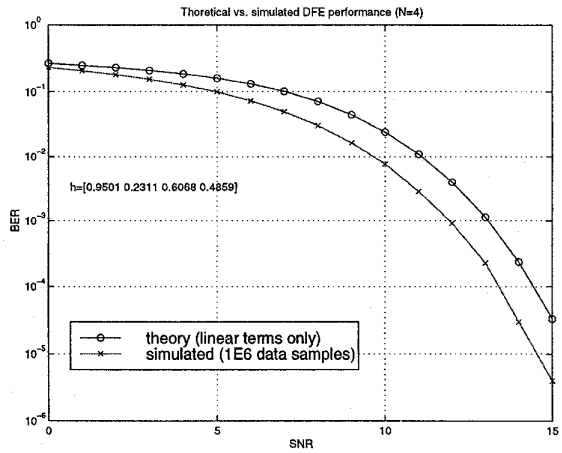

Figure 3: Linear approximation for $N = 4$

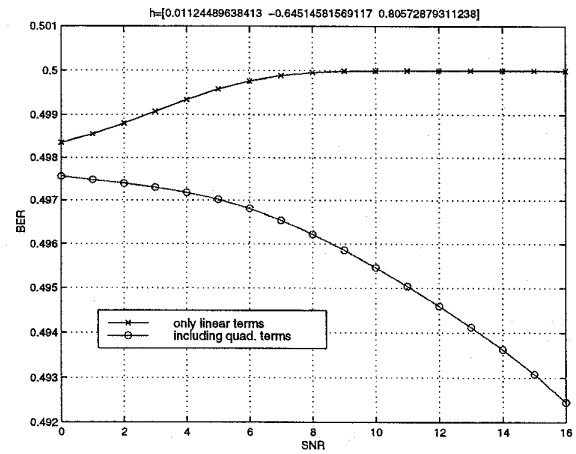

Figure 4: The effect of a strongly non-causal channel

cially at low-to-moderate SNR's), as it does not take into account error propagation.

A similar example can be seen in Figure 3, where we evaluate only the linear BER approximation given by (17) for the case $N = 4$. The considered channel impulse response is $H = [0.9501\ 0.2311\ 0.6068\ 0.4859]$. Again we observe that, even the linear approximation, is about only 1 dB away from the actually simulated DFE performance. This reinforces the value of the simple expression (17).

In order to demonstrate the "oscillation" effect mentioned above, we consider a strong non-causal example, by choosing $H = [0.0112\ -0.645\ 0.807]$. Figure 4 shows the BER performance predicted through both (12) and (17). The plot shows that, in full agreement with our analysis is Section 4, the BER predicted by (17) converges to 1/2 for high SNR values. This is due to the strong non-causality of the channel. Notice also that, this behavior is not far from the behavior of the exact solution (12), according to which, the BER again remains very close to 1/2 (notice the very small scale in Figure 4). However, contrarily to the expression (17), the expression (12) captures properly the fact that, with vanishing noise, the performance should always improve.

6 Conclusions

We have studied theoretically the BER performance of unbiased DFE receivers. The absence of residual ISI in this case has allowed us to derive computationally simple formulas that approximate well the actual DFE performance. Moreover, the derived expressions have allowed to describe mathematically the problem of instability and erratic behavior that sometimes may be observed in DFE's. Future work will be targeted to the performance analysis of arbitrary-length (as opposed to minimal) unbiased DFE's.

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