Boundary Value Problems with an Essential Singularity

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Contents

- Problem class.
- Singularity of the first kind.
  MATLAB 6 Code SBVP 1.0:
  – Solution by collocation,
  – Error estimate QDeC.
- Singularity of the second kind.
- Error estimate QDeC/BEUL.
- Error estimate QDeC/BOX.
- Error estimate based on mesh halving.
- Conclusions.

Problem class: singular BVPs

$$z'(t) = \underbrace{\frac{M(t)}{t-a} z(t) + f(t, z(t))}_{=:F(t,z(t))}, \qquad t \in (a, b],$$

or

$$z'(t) = \frac{M(t)}{(t-a)^{\alpha}} z(t) + f(t, z(t)), \qquad \alpha > 1,$$

$$B(z(a), z(b)) = 0.$$

Singularity of the first kind

$$z'(t) = \frac{M(t)}{t-a} z(t) + f(t, z(t)), \qquad t \in (a, b],$$
$$B(z(a), z(b)) = 0.$$

Assumptions

- The problem is well-posed $\Downarrow$
  $z(a) \in \operatorname{Ker} M(a) +$ local uniqueness.
- The solution is appropriately smooth.

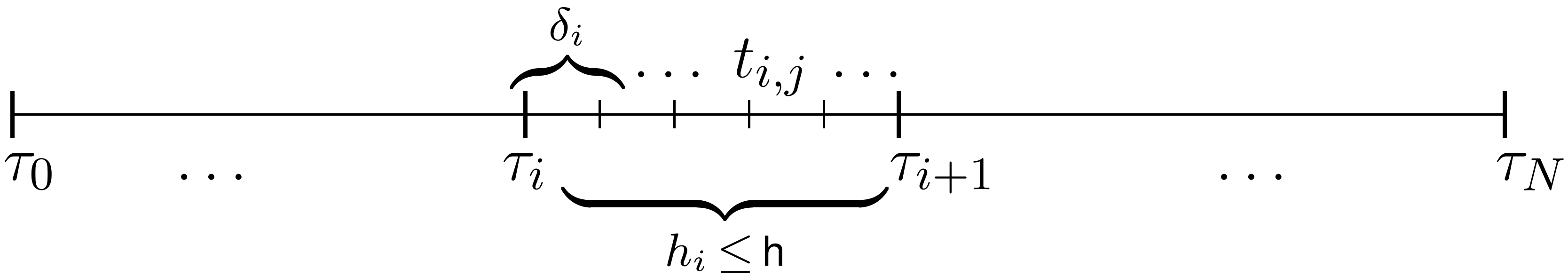
MATLAB Code SBVP 1.0

http://www.math.tuwien.ac.at/~ewa/

- General purpose code written in MATLAB.
- Suitable for a wide range of tolerances $\Downarrow$ variable order method.
- Efficient estimate of the global error.
- Global error estimate made available to the user.
- Global error control as basis for the mesh adaptation strategy $\Downarrow$ reflecting smoothness of the solution only.

Collocation method

$$z'(t) = F(t, z(t)), \quad t \in (a, b],$$
$$B(z(a), z(b)) = 0.$$



$$p(t) := p_i(t), \quad t \in [\tau_i, \tau_{i+1}], \quad p_i \in \mathbf{p}_m$$

Convergence

$$|p(\tau_i) - z(\tau_i)| = O(\mathsf{h}^m), \; i = 0, \dots, N,$$
$$\|p - z\|_\infty = O(\mathsf{h}^m),$$
$$\|p^{(l)} - z^{(l)}\|_\infty = O(\mathsf{h}^{m+1-l}), \; l = 1, \dots, m.$$

Singularity of the first kind: Similar results, no superconvergence in general!

DeC: Global error estimate

Classical idea based on Defect Correction (Zadunaisky, Frank, Stetter, 1975-1978).

- Collocation solution at the mesh points $p(\tau_i),\ i = 0, \dots, N, \Downarrow$
  $q(t)$... piecewise polynomial interpolant of maximal degree $m+1$.
- Use $q(t)$ to define a neighboring problem, NP:

$$z'(t) = F(t, z(t)) + d(t),\ d(t) := q'(t) - F(t, q(t)),$$
$$B(z(a), z(b)) = 0.$$

- Solve OP and NP with an low-order scheme, here Backward Euler scheme (BEUL).

DeC: Global error estimate (2)

- Auxiliary scheme for OP: BEUL

$$\frac{\xi_{i+1} - \xi_i}{h_i} = F(t_{i+1}, \xi_{i+1}), \quad + \text{ BC.}$$

- Auxiliary scheme for NP: BEUL

$$\frac{\pi_{i+1} - \pi_i}{h_i} = F(t_{i+1}, \pi_{i+1}) + d_{i+1}, \quad + \text{ BC.}$$

- Error estimate: $p(\tau_i) - z(\tau_i) \approx \pi_i - \xi_i, \; i = 0, \dots, N.$

Drawbacks

QDeC error estimate

QDeC: Based on local defect Quadrature (Auzinger, Koch, Weinmüller, 2001-2002).

- Auxiliary scheme for OP: BEUL

$$\frac{\xi_{i,j} - \xi_{i,j-1}}{\delta_{i,j}} = F(t_{i,j}, \xi_{i,j}) \quad + \quad \text{BC.}$$

- Auxiliary scheme for NP: BEUL

$$\frac{\pi_{i,j} - \pi_{i,j-1}}{\delta_{i,j}} = F(t_{i,j}, \pi_{i,j}) + \bar{d}_{i,j} \quad + \quad \text{BC.}$$

- Error estimate:
  $p(t_{i,j}) - z(t_{i,j}) \approx \pi_{i,j} - \xi_{i,j}, \; i = 0, \dots, N, \; j = 1, \dots m.$

QDeC error estimate (2)

Choice of the modified defect $\bar{d}_{i,j}$

Classical defect: $d(t_{i,j}) = p'(t_{i,j}) - F(t_{i,j}, p(t_{i,j}))$
Modified defect:

- w.r.t. a coll. scheme of order $m+1$

$$\bar{d}_{i,j} := \frac{p(t_{i,j}) - p(t_{i,j-1})}{\delta_i} - \sum_{k=1}^{m+1} \alpha_{j,k} F(t_{i,k}, p(t_{i,k}))$$

- quadrature rule of precision $m+1$

$$\frac{1}{\delta_{i,j}} \int_{t_{i,j-1}}^{t_{i,j}} \varphi(\tau)\, d\tau \approx \sum_{k=1}^{m+1} \alpha_{j,k} \varphi(t_{i,k})$$

QDeC - asympt. corr. (ASCOR)

Consider

$$z'(t) = \frac{M(t)}{t} z(t) + f(t, z(t)), \qquad t \in (0, 1],$$
$$B_a z(0) + B_b z(1) = \beta,$$

- z is an isolated smooth solution,
- spectrum of $M(0)$ is of certain structure.

Then, if h is sufficiently small,

$$\|(z(t_{i,j}) - p(t_{i,j})) - (\xi_{i,j} - \pi_{i,j})\| = O(\mathsf{h}^{m+1}).$$

ASCOR of QDeC - example

Test problem:

$$z'(t) = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} z(t) + t \begin{pmatrix} 0 \\ -\frac{2(t^2+2)+8}{(t^2+2)^2} z_1^2(t) + \frac{8t^2}{(t^2+2)^2} z_1^3(t) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z(0) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} z(1) = \begin{pmatrix} 0 \\ 1/\ln(3) \end{pmatrix}$$

Exact solution

$$z(t) = \left(\frac{1}{\ln(t^2+2)}, -\frac{2t^2}{(t^2+2)\ln^2(t^2+2)} \right)^T .$$

ASCOR of QDeC - example (2)

h	err_{coll}	p_{coll}	$\Delta\text{err}_{\text{est}}$	p_{est}
2^{-2}	1.5763e−04		2.2232e−05	
2^{-3}	9.5865e−06	4.04	6.5978e−07	5.07
2^{-4}	5.9574e−07	4.01	1.7873e−08	5.21
2^{-5}	3.7189e−08	4.00	5.1077e−10	5.13
2^{-6}	2.3237e−09	4.00	1.5205e−11	5.07
2^{-7}	1.4522e−10	4.00	4.6274e−13	5.04
2^{-8}	9.0772e−12	4.00	1.4655e−14	4.98

Singularity of the second kind

$$z'(t) = \frac{M(t)}{(t-a)^{\alpha}} z(t) + f(t, z(t)), \qquad t \in (a, b],$$

$$B(z(a), z(b)) = 0.$$

Assumptions

- The problem is well-posed $\Downarrow$
  $z(a) \in$ Ker $M(a)$ + local uniqueness.
- The solution is appropriately smooth.

Convergence of collocation - example

$$z'(t) = \frac{1}{t^2} \begin{pmatrix} -z_2(t) \\ -z_3(t) \\ -z_4(t) \\ 1 - \mathrm{e}^{-z_1(t)/2} \end{pmatrix}, \quad t \in (0, 1],$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} z(0) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} z(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Equidistant grid-points, m=4

Convergence of collocation

h	err$_{\mathrm{grid}}$	p_{grid}	err$_{\mathrm{mesh}}$	p_{mesh}
1/10	2.06e-17		2.06e-17	
1/20	4.26e-18	2.27	4.26e-18	2.27
1/40	1.65e-19	4.69	1.65e-19	4.69
1/80	6.22e-21	4.73	6.22e-21	4.73
1/160	3.84e-22	4.02	3.74e-22	4.06

Reference solution determined with $h = 1/320$.

Gaussian grid-points, m=4

Convergence of collocation

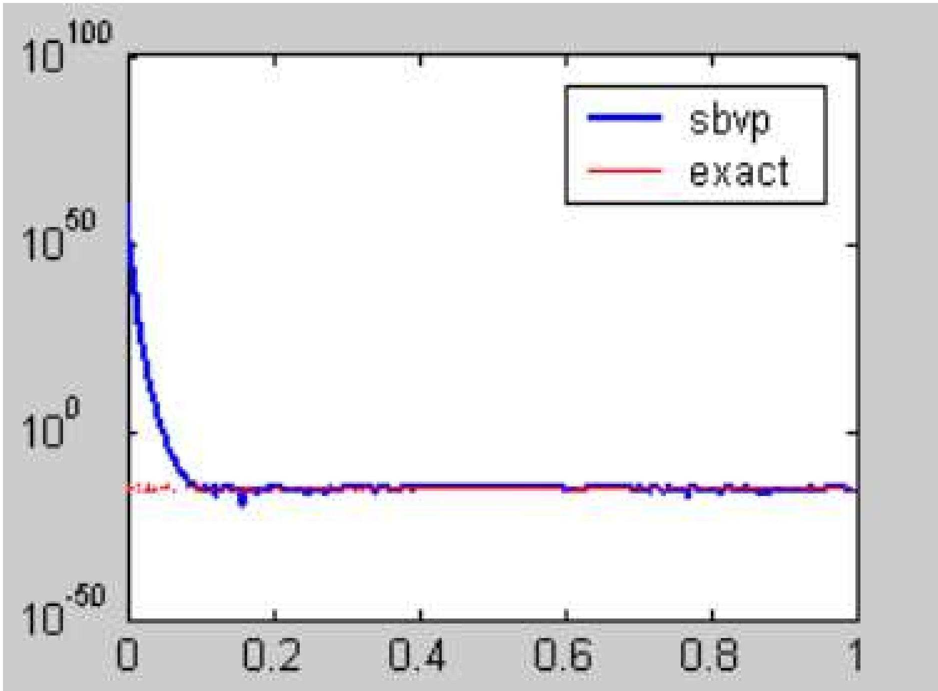
h	err_{grid}	p_{grid}	err_{mesh}	p_{mesh}
1/10	2.12e-09		2.12e-09	
1/20	9.07e-11	4.55	9.07e-11	4.55
1/40	3.92e-12	4.53	3.92e-12	4.53
1/80	1.71e-13	4.52	1.71e-13	4.52
1/160	5.99e-15	4.84	5.99e-15	4.84

Reference solution determined with $h = 1/320$.

Error estimate: QDeC/BEUL

Test problem: $z'(t) = \frac{1}{t^3} z(t) + \mathrm{e}^t - \frac{\mathrm{e}^t}{t^3}, \quad z(1) = \mathrm{e}$

Exact solution $z(t) = \mathrm{e}^t$.



Error estimate: QDeC/BEUL (2)

Convergence of BEUL

h	err_{grid}	p_{grid}	err_{mesh}	p_{mesh}
1/10	7.20e+00		7.20e+00	
1/20	5.93e+01	-3.04	5.93e+01	-3.04
1/40	1.73e+05	-11.5	1.73e+05	-11.5
1/80	8.70e+09	-15.6	8.70e+09	-15.6
1/160	5.99e+17	-26.0	5.99e+17	-26.0

Error estimate: QDeC/BEUL (3)

Matrix of the BEUL scheme

h	cond	p_{cond}	norm	p_{norm}	norminv
1/10	6.99e+05		1.00e+03		6.99e+02
1/20	3.28e+09	-12.2	8.00e+03	-3.00	4.09e+05
1/40	1.55e+15	-18.9	6.40e+04	-3.00	2.42e+10
1/80	9.60e+23	-29.2	5.12e+05	-3.00	1.88e+18
1/160	5.10e+37	-45.6	4.10e+06	-3.00	1.25e+31

Error estimate: QDeC/BOX

- Box scheme for OP:

$$\frac{\xi_{i,j}-\xi_{i,j-1}}{\delta_{i,j}} = F(t_{i,j-\frac{1}{2}}, \frac{\xi_{i,j}+\xi_{i,j-1}}{2}) \quad + \quad \text{BC}.$$

- Box scheme for NP:

$$\frac{\pi_{i,j}-\pi_{i,j-1}}{\delta_{i,j}} = F(t_{i,j-\frac{1}{2}}, \frac{\pi_{i,j}+\pi_{i,j-1}}{2}) + \frac{\bar{d}_{i,j}+\bar{d}_{i,j-1}}{2} \quad + \quad \text{BC}.$$

- Error estimate:

$$p(t_{i,j}) - z(t_{i,j}) \approx \pi_{i,j} - \xi_{i,j}, \; i = 0, \ldots, N, \; j = 1, \ldots m.$$

h	Δerr$_{\text{mesh}}$	err$_{\text{est}}$	p_{est}	cond	p_{cond}	norm	norminv
1/10	1.11e−08	5.35e−09		2.28e+03		8.00e+03	2.85e−01
1/20	7.02e−10	2.26e−10	4.47	1.68e+04	−2.88	6.40e+04	2.62e−01
1/40	4.38e−11	1.02e−11	4.55	1.28e+05	−2.93	5.12e+05	2.50e−01
1/80	2.73e−12	4.42e−13	4.53	9.98e+05	−2.96	4.10e+06	2.44e−01
1/160	1.72e−13	1.90e−14	4.54	7.89e+06	−2.98	3.28e+07	2.41e−01

Error estimate: mesh halving

- Collocation on mesh h: p_h.
- Collocation on mesh $h/2$: $p_{h/2}$.
- Error structure: $\delta(t) = e(t)h^m + O(h^{m+1})$
  Error estimate: $\mathcal{E}(t) := \frac{2^m}{1-2^m}(p_{h/2}(t) - p_h(t))$

h	err_{grid}	err_{mesh}	$\Delta\text{errest}_{\text{grid}}$	p_{grid}	$\Delta\text{errest}_{\text{mesh}}$	p_{mesh}
1/2	8.80e−06	8.80e−06	1.61e−07		1.61e−07	
1/4	5.41e−07	4.65e−07	6.81e−09	4.56	1.16e−08	3.80
1/8	3.02e−08	2.77e−08	2.11e−10	5.01	3.00e−11	8.59
1/16	1.82e−09	1.71e−09	7.30e−12	4.86	4.76e−13	5.98
1/32	1.11e−10	1.07e−10	2.26e−13	5.01	6.50e−15	6.19

Conclusions

(1) Singularity of the first kind:

- ASCOR of the error estimate QDeC ✓

(2) Singularity of the second kind:

- Convergence of the collocation scheme ✓
- Error estimate QDeC/BEUL FAILS!
- Error estimate QDeC/BOX $\frac{1}{2}$✓
- Error estimate based on mesh halving ✓
- To do: Theory for
  - convergence of the collocation scheme,
  - ascor of estimate based on mesh halving.