



# ***Defect correction as a family of iterative techniques for ODEs***

## ***SCICADE 2003***

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# Contents



- ▶ Problem class: ODE systems (IVP, BVP)
- ▶ Classical IDeC = Iterated Defect Correction
- ▶ IDeC on equidistant and **nonequidistant** grid
- ▶ Collocation = target scheme of IDeC iteration
- ▶ Collocation and (modified) Runge-Kutta
- ▶ **I**QDeC: IDeC with defect **Q**uadrature
- ▶ ——— on nonequidistant grid: **O.K.**
- ▶ ——— Convergence: Sketch of proof
- ▶ **I**PDeC: IDeC with defect inter**P**olation
- ▶ ——— using ‘Radau’ defect
- ▶ ——— symmetric version using ‘Gauss’ defect
- ▶ Conclusion

# Problem class: ODE systems (IVP, BVP)



▶ ODE system:

$$y'(t) = f(t, y(t)), \quad t \in [t_{start}, t_{end}]$$

- with initial or boundary conditions
- exact solution  $y^*(t)$

▶ In this talk:

‘Classical’ smoothness assumptions  
(non-stiff, non-singular)

# Test problem (autonomous, non-stiff)



IVP:

$$y'(t) = f(t, y(t)),$$

$$y(0) = y_{start}$$

with

$$f(t, y) = \begin{pmatrix} -y_2 + y_1(1 - y_1^2 - y_2^2) \\ y_1 + 3y_2(1 - y_1^2 - y_2^2) \end{pmatrix}$$

Solution:  $y^*(t) = (\cos t, \sin t)^T$

# Classical IDeC = Iterated Defect Correction



- ▶  $\eta_h^{[0]}$  ... basic approximation (e.g. by Euler scheme)
- ▶ Iteration  $[\nu] \mapsto [\nu + 1]$ :

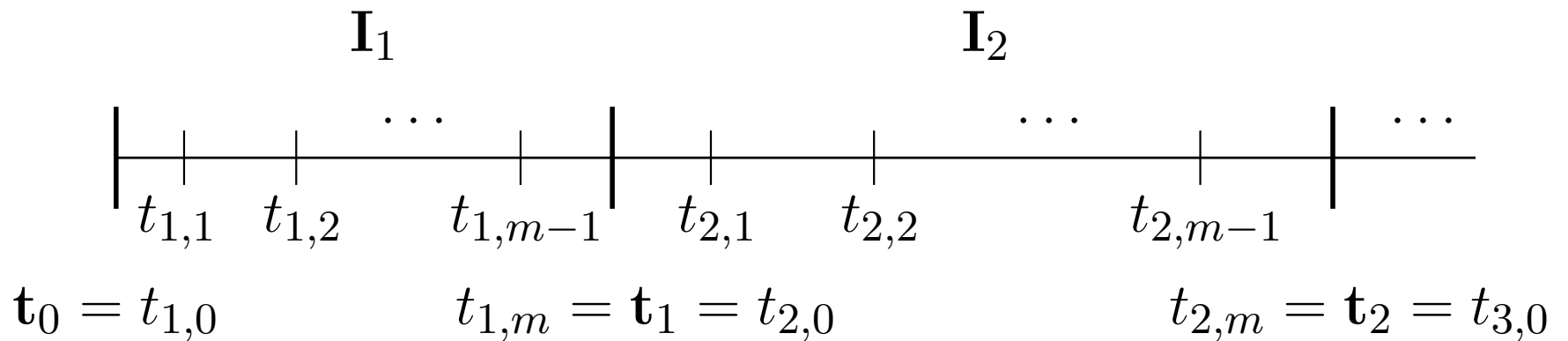
- Interpolate:  $\eta_h^{[\nu]} \mapsto p^{[\nu]}(t)$  (degree  $m$ )
- Compute **defect** of  $p^{[\nu]}(t)$  w.r.t. given ODE:

$$d^{[\nu]}(t) := \frac{d}{dt} p^{[\nu]}(t) - f(t, p^{[\nu]}(t)) \quad (\text{small})$$

- Solve **modified ODE**:  $y' = f(t, y) + d^{[\nu]}(t)$   
(**exact solution**  $p^{[\nu]}(t)$ !) by basic scheme  $\rightarrow \hat{\eta}_h^{[\nu]}$
- New approximation:  $\eta_h^{[\nu+1]} := \eta_h^{[\nu]} - \underbrace{(\hat{\eta}_h^{[\nu]} - \eta_h^{[0]})}_{\text{estimate for } \eta_h^{[\nu]} - y^*}$



# Grid, interpolation intervals



- ▶  $t_{j,l} \dots$  grid points (basic scheme, interpolation),

$$h_{j,l} := t_{j,l} - t_{j,l-1}$$

- ▶  $I_j = [t_{j-1}, t_j] \dots$  interpolation intervals,

$$h_j := t_j - t_{j-1}$$

- ▶ Tables to follow: degree  $m = 4$

# IDeC on equidistant grid: O.K.



Basic scheme: Backward Euler (BEUL)

TARGET = iteration limit ( $\nu \rightarrow \infty$  on fixed grid)

Global error (at  $t = 3$ ), observed order:

h	BEUL	IDeC 1	IDeC 2	IDeC 3	IDeC 4	TARGET
0.200	9.00E-03	5.37E-04	6.74E-05	6.89E-06	1.06E-06	1.45E-06
0.100	4.50E-03	1.42E-04	9.37E-06	4.75E-07	7.92E-08	9.11E-08
0.050	2.25E-03	3.64E-05	1.24E-06	3.10E-08	5.36E-09	5.70E-09
0.025	1.13E-03	9.22E-06	1.59E-07	1.98E-09	3.34E-10	3.56E-10
0.200	1.00	1.92	2.85	3.86	3.74	4.00
0.100	1.00	1.96	2.92	3.94	3.89	4.00
0.050	1.00	1.98	2.96	3.97	4.00	4.00
0.025	1.00	1.98	2.96	3.97	4.00	4.00

$h \downarrow 0$ : order limited by  $\mathcal{O}(h^m)$  ( $m \dots$  polynom. degree)



# IDeC on **nonequidistant** grid: iteration **stalls**

Basic scheme: Backward Euler (BEUL)

Global error (at  $t = 3$ ), observed order:

h	BEUL	IDeC 1	IDeC 2	IDeC 3	IDeC 4	TARGET
0.200	1.19E-02	1.26E-03	4.20E-03	3.31E-03	3.71E-04	1.07E-06
0.100	6.07E-03	3.42E-04	1.36E-03	1.55E-03	2.04E-04	6.68E-08
0.050	3.06E-03	1.03E-04	4.81E-04	7.64E-04	1.04E-04	4.17E-09
0.025	1.54E-03	3.66E-05	1.94E-04	3.83E-04	5.88E-05	2.61E-10
0.200						
0.100	0.98	1.88	1.63	1.10	0.87	4.00
0.050	0.99	1.73	1.50	1.02	0.96	4.00
0.025	0.99	1.49	1.31	1.00	0.83	4.00

(grid points  $t_{j,\ell}$  chosen at random)





# Collocation = target scheme of IDeC iteration

- ▶ Defect  $d(t_{j,\ell}) = p'(t_{j,\ell}) - f(t_{j,\ell}, p(t_{j,\ell})) = 0$  if evaluated for collocation solution  $p(t) = p^{[coll]}(t)$
- ▶  $\Rightarrow p^{[coll]}(t) = \text{fixed point}$  of IDeC iteration
- ▶ I.e.: Collocation is **target scheme** of IDeC
- ▶ Collocation equations:

$$\frac{d}{dt} p^{[coll]}(t_{j,\ell}) = f(t_{j,\ell}, p^{[coll]}(t_{j,\ell}))$$

- ▶ ... equivalent to Runge-Kutta equations:

$$\frac{p_{j,\ell}^{[coll]} - p_{j-1}^{[coll]}}{h_j} = \sum_{\mu=1}^m a_{\ell,\mu} f(t_{j,\mu}, p_{j,\mu}^{[coll]})$$

# Collocation and (modified) Runge-Kutta



- ▶ ... equivalent to modified Runge-Kutta equations, i.e. Runge-Kutta rewritten as

$$\frac{p_{j,l}^{[coll]} - p_{j,l-1}^{[coll]}}{h_{j,l}} = \sum_{\mu=1}^m \alpha_{l,\mu} \underbrace{f(t_{j,\mu}, p_{j,\mu}^{[coll]})}_{= \frac{d}{dt} p_j^{[coll]}(t_{j,\mu})}$$

- ▶  $\{\alpha_{l,\mu}\}$  ... set of local quadrature coefficients
- ▶ **Important:** Form of the left hand side above is the same as in basic scheme (BEUL):

$$\frac{\eta_{j,l} - \eta_{j,l-1}}{h_{j,l}} = f(t_{j,l}, \eta_{j,l})$$

# IQDeC: IDeC with defect Quadrature



- ▶ Basic scheme: BEUL, iteration as before, **but:**
- ▶ use (modified) **Q**-defect  $\bar{d}_{j,l}^{[\nu]}$  ...  
= residual w.r.t. modified Runge-Kutta equations
- ▶ ... equivalent to defect **Q**uadrature =  
= local weighting with quadrature coefficients  $\alpha_{l,\mu}$  :

$$\begin{aligned}\bar{d}_{j,l}^{[\nu]} &= \frac{\eta_{j,l}^{[\nu]} - \eta_{j,l-1}^{[\nu]}}{h_{j,l}} - \sum_{\mu=1}^m \alpha_{l,\mu} f(t_{j,\mu}, \eta_{j,\mu}^{[\nu]}) = \\ &= \sum_{\mu=1}^m \alpha_{l,\mu} \underbrace{d^{[\nu]}(t_{j,\mu})}_{\uparrow}\end{aligned}$$

classical (pointwise) defect

- ▶ all other algorithmic details as before



# IQDeC on nonequidistant grid:

O.K.

Basic scheme: Backward Euler (BEUL)

Global error (at  $t = 3$ ), observed order:

h	BEUL	IQDeC 1	IQDeC 2	IQDeC 3	IQDeC 4	TARGET
0.200	1.19E-02	2.44E-03	7.31E-05	7.98E-06	1.10E-06	1.07E-06
0.100	6.07E-03	5.99E-04	8.10E-06	4.94E-07	6.66E-08	6.68E-08
0.050	3.06E-03	1.48E-04	9.65E-07	3.07E-08	4.15E-09	4.17E-09
0.025	1.54E-03	3.69E-05	1.18E-07	1.91E-09	2.60E-10	2.61E-10
0.200						
0.100	0.98	2.03	3.17	4.01	4.05	4.00
0.050	0.99	2.01	3.07	4.01	4.01	4.00
0.025	0.99	2.01	3.03	4.01	4.00	4.00

# IQDeC - convergence: Sketch of proof (shorthand notation)



- ▶ Let  $\partial_h \eta_h =$  difference quotient  $\frac{\eta_{j,l} - \eta_{j,l-1}}{h_{j,l}}$
- ▶ Basic scheme = BEUL on *arbitrary* grid:  
$$\partial_h \eta_h^{[0]} = f(\eta_h^{[0]})$$
- ▶ Target scheme (= collocation =)  
= modified Runge-Kutta: 
$$\partial_h p^{[coll]} = Qf(p^{[coll]})$$
- ▶ Q-defect of  $\eta_h^{[0]}$ :  
$$\bar{d}_h^{[0]} := \partial_h \eta_h^{[0]} - Qf(\eta_h^{[0]}) = (f - Qf)(\eta_h^{[0]})$$
- ▶ Neighboring BEUL: 
$$\partial_h \hat{\eta}_h^{[0]} = f(\hat{\eta}_h^{[0]}) + \bar{d}_h^{[0]}$$
- ▶ New approximation: 
$$\eta_h^{[1]} := \eta_h^{[0]} - (\hat{\eta}_h^{[0]} - \eta_h^{[0]})$$

# IQDeC - convergence: Sketch of proof (2)



- ▶ Error w.r.t. target  $p^{[coll]}$ :

$$\varepsilon_h^{[0]} = \mathcal{O}(h), \quad \partial_h \varepsilon_h^{[0]} = \mathcal{O}(h); \quad \underbrace{\varepsilon_h^{[1]}}_{new} = \underbrace{\varepsilon_h^{[0]}}_{old} - \underbrace{\hat{\varepsilon}_h^{[0]}}_{old},$$

- ▶ Difference equation for  $\varepsilon_h^{[0]} = \eta_h^{[0]} - p^{[coll]}$ :

$$\begin{aligned} \partial_h \varepsilon_h^{[0]} &= f(\eta_h^{[0]}) - f(p^{[coll]}) + (f - Qf)(p^{[coll]}) \\ &= Df \cdot \varepsilon_h^{[0]} + (f - Qf)(p^{[coll]}) \end{aligned}$$

- ▶ Difference equation for  $\hat{\varepsilon}_h^{[0]} = \hat{\eta}_h^{[0]} - \eta_h^{[0]}$ :

$$\begin{aligned} \partial_h \hat{\varepsilon}_h^{[0]} &= f(\hat{\eta}_h^{[0]}) - f(\eta_h^{[0]}) + (f - Qf)(\eta_h^{[0]}) \\ &= \hat{D}f \cdot \hat{\varepsilon}_h^{[0]} + (f - Qf)(\eta_h^{[0]}) \end{aligned}$$

# IQDeC - convergence: Sketch of proof (3)



- ▶  $\Rightarrow$  Difference equation for  $\varepsilon_h^{[1]} = \eta_h^{[1]} - p^{[coll]}$  :

$$\begin{aligned}\partial_h \varepsilon_h^{[1]} &= \partial_h \varepsilon_h^{[0]} - \partial_h \hat{\varepsilon}_h^{[0]} \\ &= Df \cdot \varepsilon_h^{[1]} + (f - Qf)(\varepsilon_h^{[0]}) + \mathcal{O}(\mathbf{h}^2)\end{aligned}$$

- ▶ Estimate of inhomogeneous term:

$$\| (f - Qf)(\varepsilon_h^{[0]}) \| \leq \mathcal{C} \cdot \mathbf{h} \cdot \underbrace{\| \partial_h \varepsilon_h^{[0]} \|}_{\mathcal{O}(\mathbf{h})} + \mathcal{O}(\mathbf{h}^2) = \mathcal{O}(\mathbf{h}^2)$$

- ▶ Stability of BEUL  $\Rightarrow$  estimate for  $\varepsilon_h^{[1]}$  and  $\partial_h \varepsilon_h^{[1]}$  :

$$\varepsilon_h^{[1]} = \mathcal{O}(\mathbf{h}^2), \quad \partial_h \varepsilon_h^{[1]} = \mathcal{O}(\mathbf{h}^2)$$

- ▶ Induction  $\Rightarrow$

$$\varepsilon_h^{[\nu]} = \mathcal{O}(\mathbf{h}^{\nu+1}), \quad \partial_h \varepsilon_h^{[\nu]} = \mathcal{O}(\mathbf{h}^{\nu+1}), \quad \nu = 0, 1, 2, \dots$$



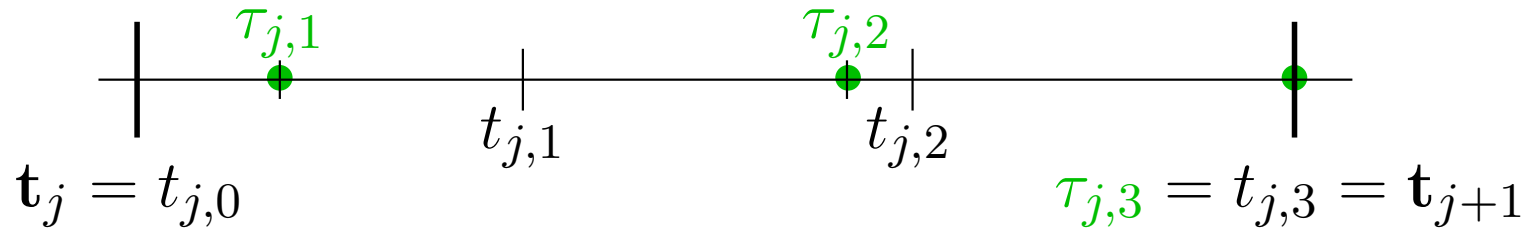
# IPDeC: IDeC with defect interpolation



- ▶ **Wanted:** Algorithmic combination of equidistant nonequidistant grids, such that
  - basic scheme works with constant step size  $h$  (most desirable for implicit schemes, IVP case)
  - target scheme defined w.r.t. nonequidistant grid (superconvergent collocation)
- ▶ **Realization:**
  - basic scheme (e.g. BEUL) as before, on piecewise equidistant grid  $\{t_{j,l}\}$
  - **Interpolate** defect at nonequidistant nodes  $\{\tau_{j,k}\}$
  - use **interpolated defect**  $\tilde{d}(t)$  instead of  $d(t)$
  - $\Rightarrow$  **target scheme = collocation at the  $\{\tau_{j,k}\}$  !**



# Example: equidistant / **Radau** nodes



- ▶  $t_{j,l}$  ... grid points (for basic scheme)
- ▶  $\tau_{j,k}$  ... nodes for **defect interpolation**
- ▶ Interpolated defect  $\tilde{d}(t) \equiv 0$  for Radau collocation
- ▶  $\Rightarrow$  Radau is target scheme
- ▶ Convergence: ✓      Proof: ✓



# IPDeC using 'Radau' defect

Basic scheme: Backward Euler (BEUL)

Target scheme: Radau IIa collocation, degree  $m = 3$

Global error (at  $t = 3$ ) ... superconvergence  $2m - 1$ :

h	BEUL	IPDeC 1	IPDeC 2	IPDeC 3	IPDeC 4	RADAU
0.200	1.20E-02	9.13E-04	1.62E-04	1.50E-05	1.84E-06	1.22E-07
0.100	6.00E-03	2.47E-04	2.25E-05	1.14E-06	6.79E-08	3.86E-09
0.050	3.00E-03	6.41E-05	2.96E-06	7.82E-08	2.30E-09	1.21E-10
0.025	1.50E-03	1.63E-05	3.79E-07	5.10E-09	7.47E-11	3.78E-12
0.200						
0.100	1.00	1.89	2.85	3.71	4.76	4.99
0.050	1.00	1.95	2.93	3.87	4.88	5.00
0.025	1.00	1.97	2.96	3.94	4.94	5.00

# IPDeC: symmetric version using 'Gauss' defect



Basic scheme: Implicit Trapezoidal Rule (ITR)

Target scheme: Gauss collocation, degree  $m = 3$

Global error (at  $t = 3$ ) ... superconvergence  $2m$  :

h	ITR	IPDeC 1	IPDeC 2	IPDeC 3	GAUSS
0.200	1.11E-03	1.29E-06	2.07E-08	1.75E-09	1.79E-09
0.100	2.78E-04	8.06E-08	3.26E-10	2.87E-11	2.88E-11
0.050	6.94E-05	5.04E-09	5.10E-12	4.53E-13	4.54E-13
0.025	1.74E-05	3.15E-10	7.99E-14	5.59E-15	6.57E-15
0.200	2.00	4.00	5.99	5.93	5.96
0.100	2.00	4.00	6.00	5.99	5.99
0.050	2.00	4.00	6.00	6.34	6.11
0.025	2.00	4.00	6.00	6.34	6.11

# Conclusion



- ▶ General template for computing defects:  
by preconditioning (weighting), i.e.

$$d_{j,\ell}^{modified} = \sum_{\mu=1}^m \omega_{\ell,\mu} \underbrace{d^{[\nu]}(t_{j,\mu})}_{\uparrow}$$

classical (pointwise) defect

- ▶ In contrast to classical IDeC:
  - Full grid flexibility
  - Superconvergence
- ▶ Stiff problems  $\Rightarrow$  talk by H. Hofstätter
- ▶ Related material  $\Rightarrow$  talks by O. Koch, E. Weinmüller