Fluid mechanics of lubrication I: fundamental aspects of a rigorous theory

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MINILUBES AC^{2}Tion Day

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Main objectives

- existing gap in tribological literature:
  lubrication represented ‘unsatisfactorily accurate’ ⇒
- describing lube flows by adopting first principles of continuum mechanics:
  asymptotic theory of hydromechanical lubrication

Why is this expedient?

- rational estimate of methodical error
- rational extension of classical theory to include e.g.
  EHD, inertia, micro-scale effects (cavitation, surface roughness)

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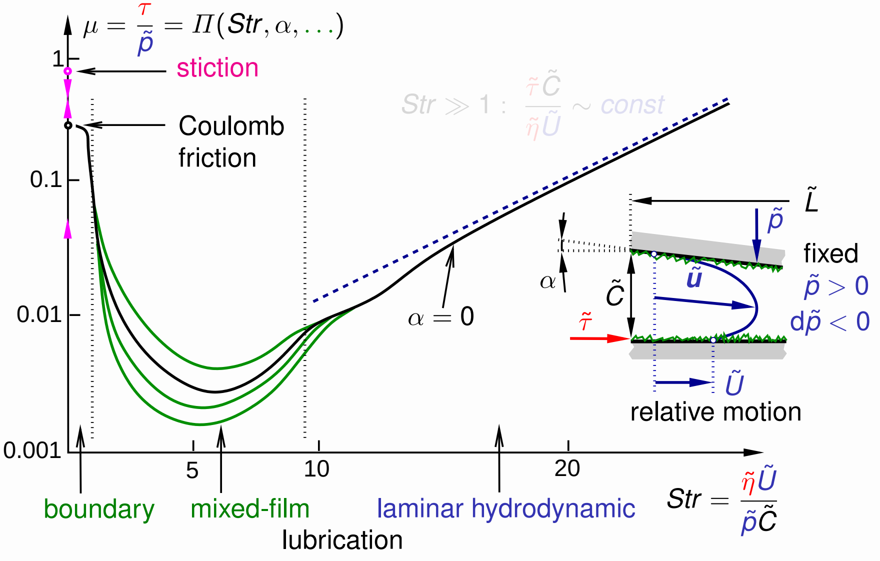
Overview

Phenomenon of lubrication

pressurised counter-sliding (tilted) solid contacts: Stribeck curve



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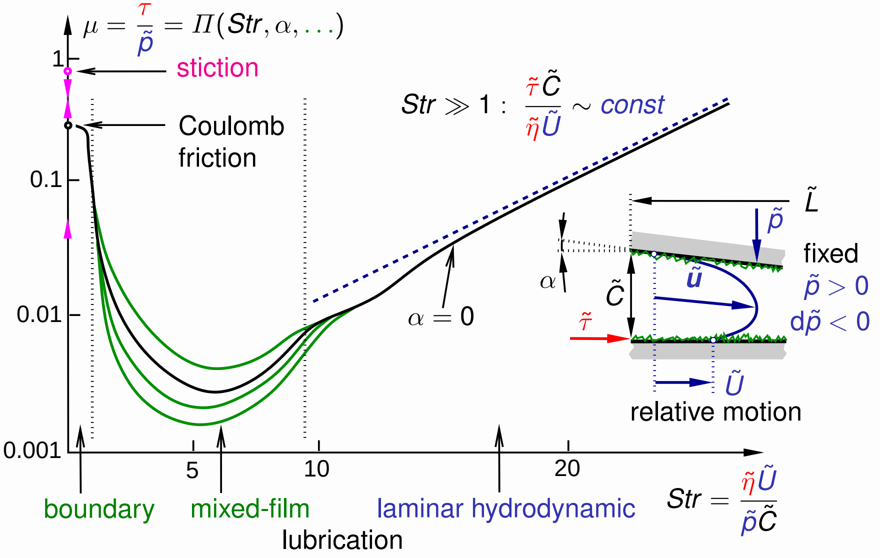

Basic (realistic) assumptions

lubricant flow

- ‘simple’ fluid
  excludes multi-phase flow (binary mixture lubricant–air):
  2 (intensive) state variables define local thermodynamic equilibrium
- Newtonian fluid
  lube oils, ionic liquids (vapour pressure very low), H_2O, many gases:
  at normal conditions, even for high pressures & shear rates, not for low temperatures
- laminar
- volume forces (gravity) neglected

bearing geometry

- clearance slender
  compared to typical macro-length (e.g. journal radius)
- perfectly hydrodynamic operation
  ‘hydraulically smooth’ surfaces:
  macroscopic flow description unaffected by mean asperities

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Outline

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continuity

$$\tilde{D}_t\tilde{\rho} + \tilde{\rho}\,\tilde{\nabla}\cdot\tilde{\boldsymbol{u}} = 0$$

momentum

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thermal energy, 1st & 2nd law of thermodynamics

$$\tilde{\rho}\tilde{c}_p\,\tilde{D}_t\tilde{T} = \tilde{\beta}\tilde{T}\,\tilde{D}_t\tilde{p} + \tilde{\Phi} - \tilde{\nabla}\cdot\dot{\tilde{\boldsymbol{q}}}\,, \qquad \tilde{\Phi} = \tilde{\boldsymbol{\Delta}}\cdot\cdot\tilde{\nabla}\tilde{\boldsymbol{u}} > 0$$

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caloric eq of state

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$\tilde{\lambda} = \tilde{\lambda}(\tilde{p}, \tilde{T})$ [W/(m K)]

2nd law of thermodynamics

$\tilde{\eta}\,,\ \tilde{\lambda}\,,\ \tilde{\beta}\,,\ \tilde{c}_p > 0\,,$ seldom $\tilde{\beta} < 0$ (H_2O !)

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Thermodynamic properties of ‘simple’ fluid

caloric eq of state

$$\tilde{h} = \tilde{h}(\tilde{p}, \tilde{T}) \qquad \tilde{c}_p := \left(\frac{\partial \tilde{h}}{\partial \tilde{T}}\right)_{\tilde{p}} \left[\frac{\mathrm{J}}{\mathrm{kg\,K}}\right], \quad \tilde{\beta}\tilde{T} = 1 - \tilde{\rho}\left(\frac{\partial \tilde{h}}{\partial \tilde{p}}\right)_{\tilde{T}}$$

thermal eq of state

$$\tilde{\rho} = \tilde{\rho}(\tilde{p}, \tilde{T}) \qquad \tilde{\beta} := -\frac{1}{\tilde{\rho}}\left(\frac{\partial \tilde{\rho}}{\partial \tilde{T}}\right)_{\tilde{p}} \left[\frac{1}{\mathrm{K}}\right]$$

$\tilde{\eta} = \tilde{\eta}(\tilde{p}, \tilde{T})$ [Pa s]

$\tilde{\lambda} = \tilde{\lambda}(\tilde{p}, \tilde{T})$ [W/(m K)]

2nd law of thermodynamics

$\tilde{\eta}, \tilde{\lambda}, \tilde{\beta}, \tilde{c}_p > 0$, seldom $\tilde{\beta} < 0$ (H_2O !)

Outline

Non-dimensional quantities

kinematic quantities

$$t = \tilde{t}\,\tilde{U}/\tilde{L}\,, \quad \boldsymbol{x} = \tilde{\boldsymbol{x}}/\tilde{L}\,, \quad c = \tilde{c}/\tilde{C}\,, \quad \boldsymbol{u} = \tilde{\boldsymbol{u}}/\tilde{U}$$

reference state

$$p = \tilde{p}/\tilde{p}_r\,, \quad \theta = (\tilde{T} - \tilde{T}_a)/\tilde{T}_r$$

$$\rho = \tilde{\rho}/\tilde{\rho}_r\,, \quad (\eta, \eta') = (\tilde{\eta}, \tilde{\eta}')/\tilde{\eta}_r\,, \quad \lambda = \tilde{\lambda}/\tilde{\lambda}_r\,, \quad \beta = \tilde{\beta}\tilde{T}_a\,, \quad c_p = \tilde{c}_p/\tilde{c}_{p,r}$$

key groups

clearance slenderness $\epsilon := \tilde{C}/\tilde{L}$

temperature ratio $\gamma := \tilde{T}_r/\tilde{T}_a$

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key groups

clearance slenderness	$\epsilon := \tilde{C}/\tilde{L}$
temperature ratio	$\gamma := \tilde{T}_r/\tilde{T}_a$
Reynolds number	$Re := \tilde{U}\tilde{L}\,\tilde{\rho}_r/\tilde{\eta}_r$
Prandtl number	$Pr := \tilde{c}_{p,r}\,\tilde{\eta}_r/\tilde{\lambda}_r$
Péclet number	$Pe := Re\,Pr$

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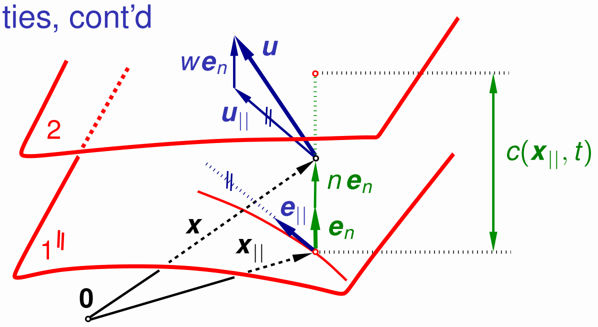
Non-dimensional quantities, cont'd



natural metric

$$\boldsymbol{x} = \boldsymbol{x}_{||} + \epsilon \boldsymbol{e}_n n\,, \quad \boldsymbol{u} = \boldsymbol{u}_{||} + \epsilon \boldsymbol{e}_n w\,, \quad \boldsymbol{u}_{||} = u_{||}\boldsymbol{e}_{||}$$

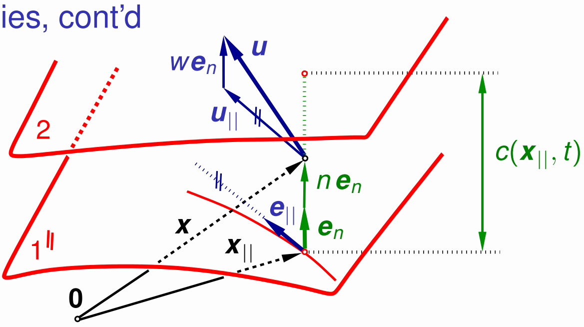
$$\boldsymbol{e}_{||}\cdot\boldsymbol{e}_n = 0\,, \quad \partial_n\,\boldsymbol{e}_{||} = \partial_n\,\boldsymbol{e}_n = \boldsymbol{0}$$

$$\nabla = \tilde{L}\tilde{\nabla} = \nabla_{||} + \epsilon^{-1}\boldsymbol{e}_n\,\partial_n$$

$$\nabla\cdot(\rho\boldsymbol{u}) = \nabla_{||}\cdot(\rho\boldsymbol{u}_{||}) + \underbrace{\boldsymbol{e}_n\cdot\partial_n(\rho\boldsymbol{u}_{||})}_{\boldsymbol{e}_n\cdot\boldsymbol{e}_{||}\partial_n(\rho u_{||}) = 0} + \overbrace{\epsilon\underbrace{\nabla_{||}\cdot(\rho\,\boldsymbol{e}_n w)}_{\rho w\nabla_{||}\cdot\boldsymbol{e}_n}}^{O(\epsilon)} + \underbrace{\boldsymbol{e}_n\cdot\partial_n(\rho\,\boldsymbol{e}_n w)}_{\partial_n(\rho w)}$$

$$D_t = (\tilde{L}/\tilde{U})\tilde{D}_t = \partial_t + \boldsymbol{u}\cdot\nabla = \boldsymbol{u}_{||}\cdot\nabla_{||} + w\,\partial_n$$

Non-dimensional quantities, cont'd



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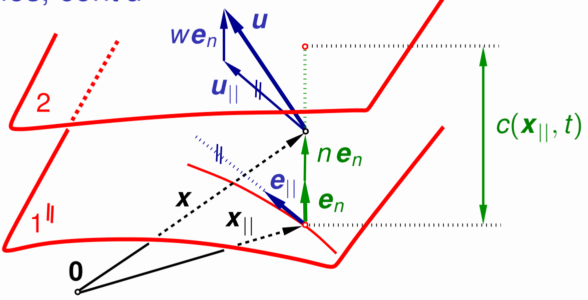

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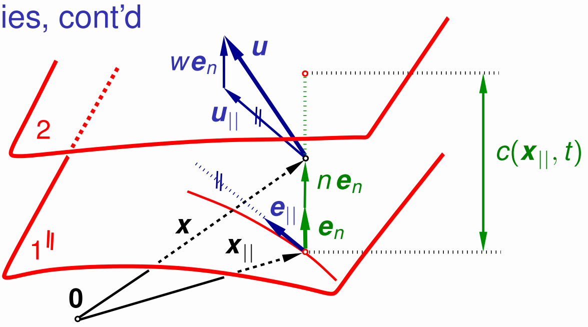
Non-dimensional quantities, cont’d



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Navier–Stokes eqs

$$\tilde{p}_r := \tilde{\eta}_r \tilde{U}\tilde{L}/\tilde{C}^2 \,, \qquad \tilde{T}_r := \tilde{\eta}_r \tilde{U}^2/\tilde{\lambda}_r$$

state $q = q(p, 1 + \gamma\theta) \,, \quad q = \rho \,,\ \eta \,,\ \lambda,\ c_p \qquad \Rightarrow \quad \tilde{\rho}_r$

continuity $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u}) \equiv \partial_t\rho + \nabla_{||}\cdot(\rho\boldsymbol{u}_{||}) + \epsilon\rho w \nabla_{||}\cdot\boldsymbol{e}_n + \partial_n(\rho w) = 0$

momentum $Re\,\epsilon^2\rho\big(\ddot{\boldsymbol{x}}_{ref} + 2\boldsymbol{\Omega}_{ref}\times\boldsymbol{u} + D_t\boldsymbol{u}\big) + \nabla p = \epsilon^2\,\nabla\cdot\boldsymbol{\Delta}$

$$\boldsymbol{\Delta} = \eta\big[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{tr}\big] + (\eta' - \tfrac{2}{3}\eta)(\nabla\cdot\boldsymbol{u})\boldsymbol{I}$$

energy $Pe\,\epsilon^2\rho c_p\,D_t\theta = \beta(1+\gamma\theta)D_t p + \epsilon^2\big[\Phi + \nabla\cdot(\lambda\nabla\theta)\big]$

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Outline

Limit process

classical lubrication approximation

thin film	$\epsilon \ll 1$
quasi-isothermal	$\gamma \ll 1$
inertia neglected	$Re\,\epsilon^2 \ll 1$, laminar flow: $Re \lesssim 10^5$
typical values	$\epsilon \lesssim 10^{-3}$, $Pr_{\text{oil}} \approx \underset{100\ \dots\ 20\,^\circ\mathrm{C}}{70 \dots 10^3}$ $\Rightarrow$ $Pe \lesssim 10^8$, $Pe\,\epsilon^2 \lesssim 10^2$ **!**

$$\nabla\cdot(\rho\boldsymbol{u}) \sim \nabla_{||}\cdot(\rho\boldsymbol{u}_{||}) + \partial_n(\rho w) + O(\epsilon)$$

$$\rho(p, 1 + \gamma\theta) \sim \rho(p, 1) + O(\gamma)$$

expansions

$$\nabla_{||} \sim \nabla^0_{||} + O(\epsilon) \qquad \nabla^0_{||} = \nabla_{||} \quad \text{for} \quad n = 0$$

$$[\boldsymbol{u}_{||}, w, p, \rho, \theta, \eta, \dots](\boldsymbol{x}_{||}, n, t; \epsilon, Re, \gamma, \dots) \sim [\boldsymbol{U}, W, P, \mathcal{Q}, \Theta, \mathcal{N}](\boldsymbol{x}_{||}, n, t) + \cdots$$

$c \sim C(\boldsymbol{x}_{||}, t) + O(\epsilon)$ journal bearing !

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typical values	$\epsilon \lesssim 10^{-3}$, $Pr_{\text{oil}} \approx \underset{100\ \ldots\ 20\,^\circ\mathrm{C}}{70 \ldots 10^3}$ $\Rightarrow$ $Pe \lesssim 10^8$, $Pe\,\epsilon^2 \lesssim 10^2$ **!**

$$\nabla\cdot(\rho\boldsymbol{u}) \sim \nabla_{||}\cdot(\rho\boldsymbol{u}_{||}) + \partial_n(\rho w) + O(\epsilon)$$

$$\rho(p, 1 + \gamma\theta) \sim \rho(p, 1) + O(\gamma)$$

expansions

$$\nabla_{||} \sim \nabla^0_{||} + O(\epsilon) \qquad \nabla^0_{||} = \nabla_{||} \quad \text{for} \quad n = 0$$

$$[\boldsymbol{u}_{||}, w, p, \rho, \theta, \eta, \ldots](\boldsymbol{x}_{||}, n, t; \epsilon, Re, \gamma, \ldots) \sim [\boldsymbol{U}, W, P, \mathcal{Q}, \Theta, \mathcal{N}](\boldsymbol{x}_{||}, n, t) + \cdots$$

$c \sim C(\boldsymbol{x}_{||}, t) + O(\epsilon)$ journal bearing !

Limit process

classical lubrication approximation

thin film	$\epsilon \ll 1$
quasi-isothermal	$\gamma \ll 1$
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Leading-order eqs

state & energy $\quad Q = Q(P, 1)\,, \quad Q = \mathcal{Q}\,,\ \mathcal{N}$

continuity $\quad \partial_t \mathcal{Q} + \nabla_{||}^0 \cdot (\mathcal{Q}\boldsymbol{U}) + \partial_N(\mathcal{Q}W) = 0 \qquad (1)$

momentum $\quad \nabla_{||}^0 P = \partial_n(\mathcal{N}\,\partial_n \boldsymbol{U})\,, \quad \partial_n P = 0 \quad \Rightarrow \quad \partial_n \mathcal{Q} = \partial_n \mathcal{N} = 0 \qquad (2)$

kinematic BCs

$$n = 0: \qquad \boldsymbol{U} = \boldsymbol{U}_1(\boldsymbol{x}_{||}, t)\,, \quad W = W_{p,1}(\boldsymbol{x}_{||}, t) \qquad (3)$$

$$n = C(\boldsymbol{x}_{||}, t): \quad \boldsymbol{U} = \boldsymbol{U}_2(\boldsymbol{x}_{||}, t)\,, \quad W = \partial_t C + \boldsymbol{U}_2 \cdot \nabla_{||}^0 C + W_{p,2}(\boldsymbol{x}_{||}, t) \qquad (4)$$

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$$(2), (3), (4) \quad \Rightarrow \quad \boldsymbol{U} = \underbrace{\frac{\nabla_{||}^0 P}{2\mathcal{N}(P)} n(n - C)}_{\text{Hagen–Poisseuille}} + \underbrace{\frac{n}{C}(\boldsymbol{U}_2 - \boldsymbol{U}_1)}_{\text{Couette}}$$

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Integral mass balance

$$\int_0^C \boldsymbol{U}\,\mathrm{d}n = \boldsymbol{Q} + C\,\boldsymbol{U}_m\,, \qquad \boldsymbol{Q} := -\frac{C^3\,\nabla_{||}^0 P}{12\,\mathcal{N}}\,, \quad \boldsymbol{U}_m := \frac{\boldsymbol{U}_1 + \boldsymbol{U}_2}{2}$$

generalised Reynolds eq O. Reynolds (1886), A. Sommerfeld (1904), L. Prandtl (1937)

$$\nabla_{||}^0\cdot(-\mathcal{Q}\boldsymbol{Q}) = (\underbrace{\partial_t}_{\text{'squeeze'}} + \underbrace{\boldsymbol{U}_m\cdot\nabla_{||}^0}_{\text{Couette}+\text{sliding}=\text{'wedge'}})(\mathcal{Q}C) + \mathcal{Q}C\,\nabla_{||}^0\cdot\boldsymbol{U}_m + \mathcal{Q}\underbrace{(W_{p,2} - W_{p,1})}_{\text{permeability}}$$

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elliptic 2nd-order PDE for $P(\boldsymbol{x}_{||}, t)$ and given $C(\boldsymbol{x}_{||}, t)$, $\boldsymbol{U}_m(\boldsymbol{x}_{||}, t)$

kinematic wave operator $\partial_t + \boldsymbol{U}_m\cdot\nabla_{||}^0$ most relevant for gas bearings

linear for incompressible lubricant with constant properties $(\mathcal{Q} \equiv \mathcal{N} \equiv 1)$

in general to be solved numerically

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generalised Reynolds eq — some important properties

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rigid contacts, no Navier slip $\quad \nabla^0_{||}\boldsymbol{\cdot}[\boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_m] = 0$

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Validation of tribo-systems

typically find

- $P(\boldsymbol{x}_{||}, t)\,, \quad \boldsymbol{x}_{||} \in \boldsymbol{\Omega}$ subject to $P(\partial\boldsymbol{\Omega}, t) = P_a$
- load-bearing capacity $\boldsymbol{F}(t) = \int_{\Omega} P\boldsymbol{e}_n \,\mathrm{d}\Omega$

clearance $C(\boldsymbol{x}_{||}, t)$ is

- prescribed
- found from fluid–structure interaction

  machinery (e.g. shaft) dynamics $\Rightarrow$ $\boldsymbol{F} = \boldsymbol{F}(\ddot{C}, \dot{C}, C)$

  EHL $\Rightarrow$ $P = P(C)$

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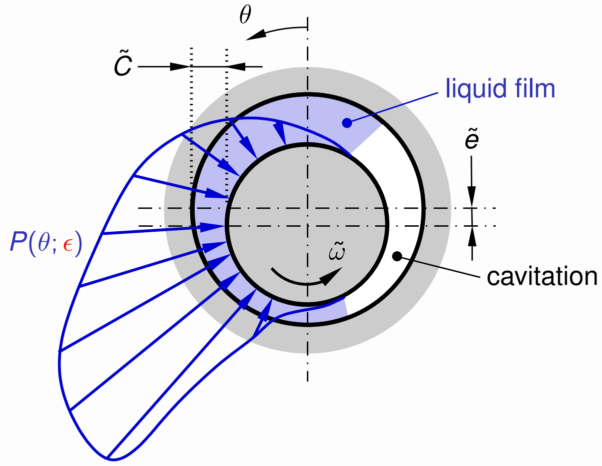
Classical application: journal bearing



reference quantities $\tilde{U}_m = \tilde{\omega}\tilde{R}_i\,, \quad \tilde{p}_r = \tilde{\eta}_r\tilde{\omega}\tilde{R}_i^2/\tilde{C}^2$

geometrical parameters $\epsilon = \tilde{C}/\tilde{R}_i \ll 1\,,$ eccentricity $\varepsilon = \tilde{e}/(\tilde{R}_a - \tilde{R}_i)$

non-dimensional quantities $C = 1 + \varepsilon\cos\theta + O(\epsilon^2)\,, \quad U_m\,(= \mathcal{N} = \mathcal{Q}) = 1$

Further outlook

include

- EHL
- inertia ($Re\,\epsilon^2 \sim 1$, start-up, high-speed rotors, rapid load cycles)
- turbulence
- film rupture & cavitation (surface tension)
- effects acting on micro-scale $\ll \epsilon$ (surface roughness, mixed friction)

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- multiple scales, matched asymptotic expansions
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Thank you for your attention !