Institut für
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& Wärmeübertragung

Short-to-Long-Scale Interaction in Weakly Viscous Super- to Transcritical Liquid-Layer Flows

Bernhard Scheichl[1,2] Robert I. Bowles[3]

[1]Institute of Fluid Mechanics & Heat Transfer, Technische Universität Wien, Vienna, Austria
[2]AC²T research GmbH, Wiener Neustadt, Austria
[3]Department of Mathematics, UCL

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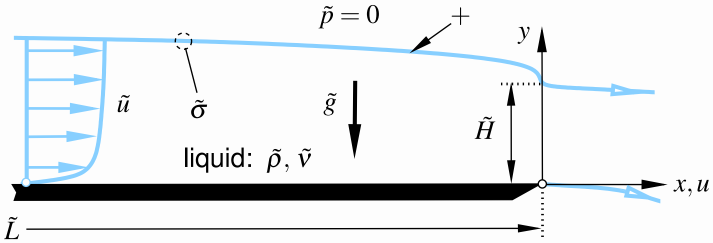
Overview

- Motivation
  - Steady viscous film past horizontal plate: *critical* at trailing edge?
- High-Re limit
  - Free overfall: paradigm of trans- to supercritical free-streamline flows
- Scale separation
  - Novel self-sustained triple deck
- Transcritical long-wave limit
  - Transcritical singularity
  - Hydraulic jump
  - Unsteadiness
- Summary & outlook

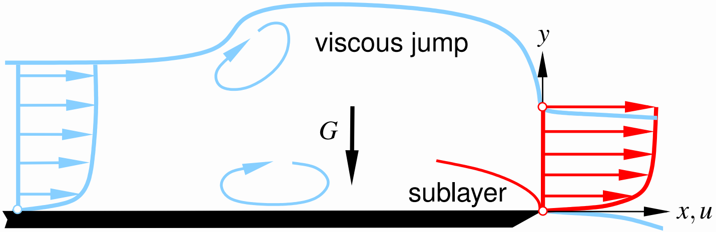


reference state: laminar flow at edge

- $G := Fr^{-2} = \tilde{g}\tilde{H}/\tilde{U}^2, \quad S := We^{-1} = \tilde{\sigma}/(\tilde{\rho}\tilde{H}\tilde{U}^2), \quad Re = \tilde{U}\tilde{H}^2/(\tilde{L}\tilde{\nu}) = O(1)$
- $(x,y) = (\tilde{x},\tilde{y})/\tilde{H}, \quad u = \tilde{u}/\tilde{U}, \quad p = \tilde{p}/(\rho\tilde{U}^2) \sim G(y^+ - y) - S\,\mathrm{d}^2 y^+/\mathrm{d}x^2$

- $G < / = / > 1$: super- / trans- / subcritical in hydraulic limit $Re \gg 1$
- $\varepsilon = \tilde{H}/\tilde{L} \ll 1$

Motivation: steady developed viscous film past horizontal plate



Higuera (JFM 274, 1994) surmounted ill-posedness of shallow-water problem by

- prescribing intrinsic expansive edge singularity
- giving criticality in the sense of the Burns–Lighthill long-wave criterion

$$\int_0^1 \frac{\mathrm{d}y}{(u-c)^2} \sim \frac{1}{G} \sim 1 \quad (c=0), \qquad u \propto y^{0.364\ldots} \quad (y \ll 1)$$

- but not transition towards the important limits

  (1) $G \to 0$, (2) $Re \to \infty$

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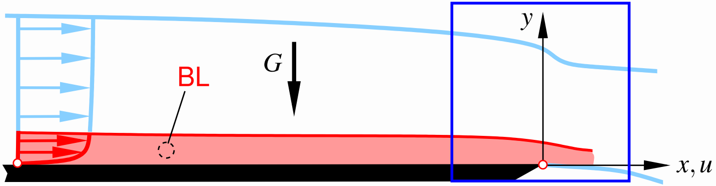


We here

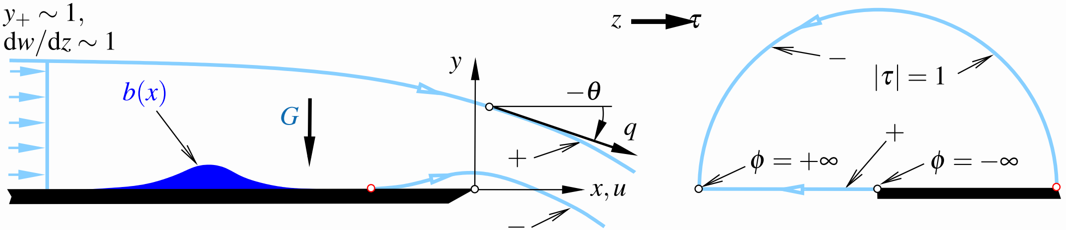
- present preliminary results of an ongoing asymptotic analysis addressing (2)
- therefore start with potential flow in a square edge region
- in the course discover surprising phenomena on this scale upstream.

. . . has attracted surprisingly little attention! $G = Fr^{-2},\ S = We^{-1}$

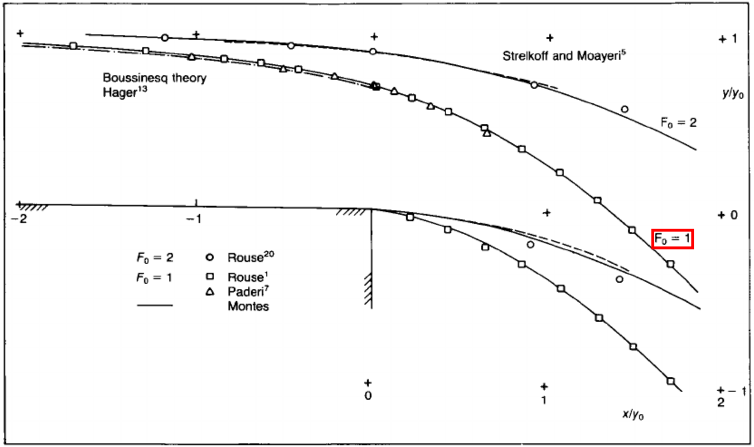
- find $z(w;G,S) = x + \mathrm{i}y,\ z' = \mathrm{i}\theta - \ln q,\ w = \phi + \mathrm{i}\psi$ subject to

$$-\infty < \phi \le 0,\ \psi = 0: \quad y = b(x)$$
$$0 < \phi < \infty,\ \psi = 0: \quad y = y_-(\phi),\ q^2/2 + Sq\,\mathrm{d}\theta/\mathrm{d}\phi + Gy_- = 1/2 + G$$
$$-\infty < \phi < \infty,\ \psi = 1: \quad y = y_+(\phi),\ q^2/2 - Sq\,\mathrm{d}\theta/\mathrm{d}\phi + Gy_+ = 1/2 + G$$
$$\phi \to -\infty:\ z \sim w, \quad \phi \to +\infty:\ y \sim -2Gx^2/(2+G)^2\ \left(q \sim -\sqrt{-2Gy}\right)$$

- by collocation/series truncation: $\mathrm{e}^{-\pi w} = (\tau+1)^2/(4\tau)$ Vanden-Broeck (textbook, 2010)

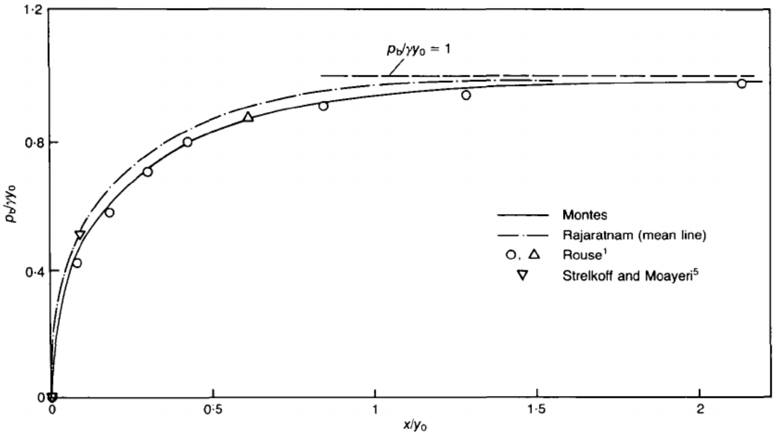


Steady free overfall: earlier results for $S = 0$



- Montes (1992): solutions obtained even for $G = 1/F_0^2 = 1$
- other situations (humps, weirs, ...): $\max(G) < 1$ cf. Dias & Vanden-Broeck (JEM, 2011)
- asymptotic theory for $G \to 0$ Clarke (JFM, 1965)

Steady free overfall: earlier results for $S = 0$



- Montes (1992): base pressure $\propto \sqrt{x/y_0} \ll 1$

Steady free overfall: $S = 0$ implies $G \leq 1$

Considering in the z-plane the

- Cauchy–Riemann relations $q_\phi + q\theta_\psi = 0,\ q_\psi - q\theta_\phi = 0$
- extremal properties of potential $\ln q$
- branching points of isotachs $q = const$,

we can demonstrate

- strictly accelerating flow.

Steady free overfall: $S = 0$ implies $G \leq 1$

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we can demonstrate

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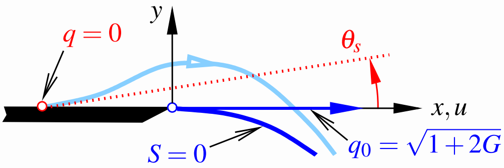
consequences

- forced separation
- $G \leq 1$ only: "waterfall" super- to transcritical

$$x \to -\infty: \quad [u-1, p-G(1-y), y^+ - 1] \sim a_1(G)\exp(\lambda x)[-\lambda\cos(\lambda y), \lambda\cos(\lambda y), \sin y]$$
$$a_1 < 0, \quad 0 < \lambda = G\tan\lambda < \pi/2$$

- $G > 1$: rarefaction wave propagating upstream $\Rightarrow$ supercritical steady state

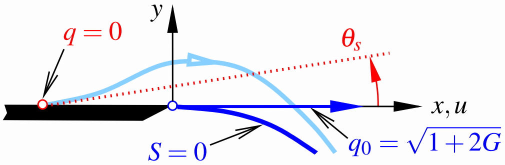


$S > 0$: class of solutions

- free separation at $\theta_s > 0,\ x \geq x_{min}(S)$
- no re-impingement restricts non-uniqueness
- $b \equiv 0$: translational x-invariance

$S = 0$: unique solution

- forced separation at $x = 0$ by gravity



- $S \to 0$: singular perturbation problem

inner Wiener–Hopf-type *dock* problem: $x_{min} = O(S \ln S), \quad \theta_S = O(S^{1/2} \ln S)$

cf. Vanden-Broeck (QJMAM, 1981)

outer solution, $s = q_0 x \to 0_{\mp}$, $c(G) > 0$:

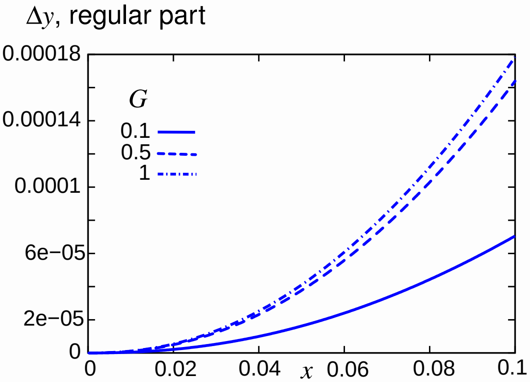
$$\frac{u}{q_0}\bigg|_{y=0} \sim 1 - \frac{3c}{2}(-s)^{1/2} + \frac{15c^2}{8}(-s) + \frac{c\ln(-s)}{\pi q_0}\left[\frac{G}{q_0^2}(-s)^{3/2} + \frac{3S/4}{(-s)^{1/2}}\right] + O\Big((-s)^{3/2}, \frac{S}{(-s)^{1/2}}\Big)$$

$$\frac{q_0 y_-}{c} \sim -s^{3/2} - \frac{\ln s}{\pi q_0}\left[\frac{2G}{5q_0^2}s^{5/2} + \frac{3S}{2}s^{1/2}\right] + O\big(s^{5/2}, S\,s^{1/2}\big)$$

- $S \ll Re^{-1/2}$: Ackerberg singularity at $x = 0$, BL already split upstream — how?

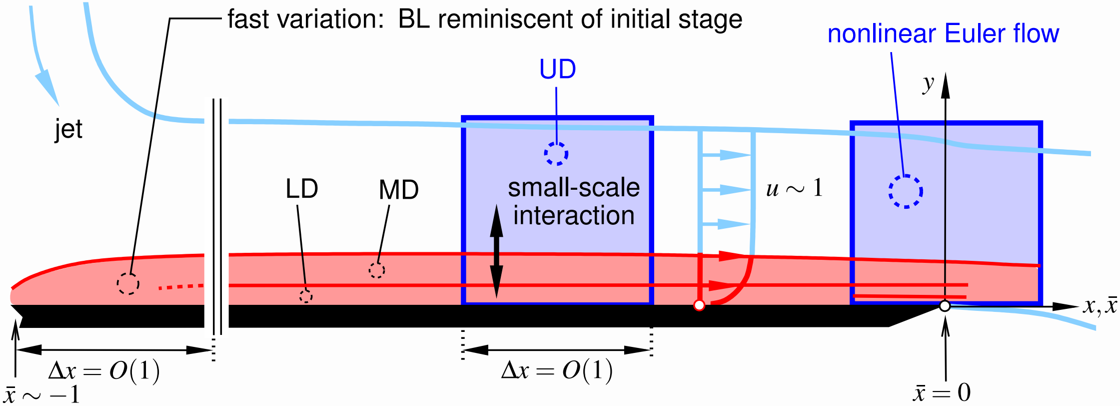
Ackerberg (JFM, 1970)

Steady free overfall: detaching streamline for $S = 0$



$$\frac{q_0 y_-}{c} \sim -s^{3/2} \underbrace{- \frac{\ln s}{\pi q_0}\left[\frac{2G}{5q_0^2}\, s^{5/2} + \frac{3S}{2}\, s^{1/2}\right]}_{\Delta y > 0} + O\left(s^{5/2}, S\, s^{1/2}\right)$$

fast variable x, slow variable $\bar{x} := \varepsilon x$



discrete inviscid small-scale modes

- $[\psi - y, y^+ - 1] \sim \sum_{n=1}^{\infty} a_n^{inv}(G,S)[-\sin(\lambda_n y), \sin\lambda_n]E_n + O(E_1^2), \quad E_n := \exp(\lambda_n \bar{x}/\varepsilon)$
- classical standing-wave dispersion

$$\lambda_n = (G - S\lambda_n^2)\tan\lambda_n \quad \Rightarrow \quad \lambda_n = \mu_n^{inv} > 0 \quad \text{or} \quad \lambda_n = \mathrm{i}k_n \ \ (k_n > 0)$$

Blasius BL (B) singularly perturbed

- $\xi := (1+\bar{x})/\varepsilon, \quad [\eta, f(\bar{x},\eta;G,S,\varepsilon)] := \sqrt{Re/(2\varepsilon\xi)}\,[y, \psi], \quad \beta := f_B''(0) \doteq 0.4696$

MD: $$f \sim f_B(\eta) - a_1\lambda_1 E_1 f_B'(\eta)\left[\frac{\Gamma(\frac{1}{3})(2\lambda_1)^{1/3}}{3^{2/3}\beta^{5/3}}\,\xi^{1/3} - \int_\eta^\infty [f_B(t)^{-2} - 1]\,\mathrm{d}t + \eta + O(\varepsilon^{3/2})\right]$$

LD: $$f \sim \beta\frac{\zeta^2}{2} - \frac{3^{1/3}\Gamma(\frac{1}{3})}{\beta} a_1\lambda_1 E_1 \int_0^\zeta (\zeta - t)\mathrm{Ai}(t)\,\mathrm{d}t + O(\varepsilon E_1), \ \ \zeta := (2\beta\lambda_1\xi)^{1/3}\eta = O(1)$$

discrete modes subject to BL feedback

- $[\psi - y, y^+ - 1] \sim \sum_n a_n(G,S)[-\sin(\lambda_n y) - \Gamma\cos(\lambda_n y), \sin\lambda_n + \Gamma\cos\lambda_n]\, E_n + c.c.$

- distinguished limit

$$\Lambda := \varepsilon^{1/3} Re^{1/2} = O(1), \quad \Gamma := \underbrace{\frac{2^{5/6}\Gamma(\frac{1}{3})}{3^{2/3}\beta^{5/3}}}_{\doteq 8.0884} \frac{(1+\bar{x})^{5/6}}{\Lambda} \begin{cases} \ll 1: & \text{hierarchical BL} \\ \gg 1: & \text{developed flow} \end{cases}$$

- dispersion detuned & modulated on long scale $\bar{x}$

$$\underbrace{\lambda_n - (G - S\lambda_n^2)\tan\lambda_n}_{\delta_0} = \Gamma\lambda_n^{4/3} \underbrace{(G - S\lambda_n^2 + \lambda_n\tan\lambda_n)}_{\delta_v} \Rightarrow \begin{cases} \mathrm{Re}\,\lambda_n > 0 \\ \mathrm{Im}\,\lambda_n \neq 0: & \text{waves} \end{cases}$$

localised wave crest, continuous spectrum

- linearisation in LD fails for $\bar{x}-\bar{x}_p = O(\varepsilon)$, $\bar{x}_p := 2\varepsilon\ln\varepsilon/(3\mu_1)$, $\mu_n := \mathrm{Re}\,\lambda_n < \mu_{n-1}$
- $[X, Y] := [\mu_1\bar{x}/\varepsilon - (2/3)\ln\varepsilon,\ \zeta|_{\bar{x}=\bar{x}_p} - B(X)] = O(1)$
  $[f, p-G] \sim \varepsilon^{2/3}\beta^{1/3}/(2\mu_1)^{2/3}\left[\Psi(X,Y;G,S,\Lambda),\ \beta\bar{P}(X;*)\right] + O(\varepsilon^{5/3})$

a novel triple-deck problem

LD: $\Psi_Y\Psi_{YX} - \Psi_X\Psi_{YY} = -\mathrm{d}\bar{P}/\mathrm{d}X + \Psi_{YYY}$

$Y=0\colon\ \Psi = \Psi_Y = 0,\quad Y\to\infty\colon\ \Psi \sim [Y+\bar{A}(X;*)+B(X)]^2/2\ \left[+\bar{P}\right],\quad X\to-\infty\colon\ \Psi\sim Y^2/2$

UD: downstream mode absorbed

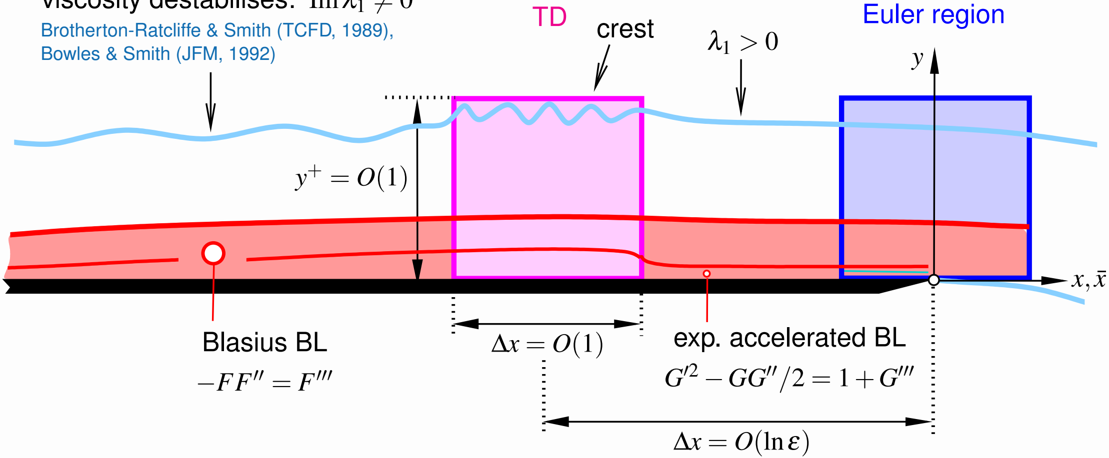
$[A, P] := [\bar{A}, \bar{P}] - \left[\sqrt{-2P^{inv}}, P^{inv}\right],\quad P^{inv} = \beta^{-4/3}a_1^{inv}\mu_1^{inv}(2\mu_1)^{2/3}\exp(\mu_1^{inv}X/\mu_1)$

$-\mathrm{i}k\mathscr{A}(k;*) = \Lambda\dfrac{\beta^{5/3}}{(2\mu_1)^{2/3}}\dfrac{\delta_0(\mathrm{i}k)}{\delta_v(\mathrm{i}k)}\mathscr{P}(k;*),\quad [\mathscr{Q}, Q](s) = \displaystyle\int_{-\infty}^{\infty}\frac{[Q(-t), \mathscr{Q}(t)]}{\sqrt{2\pi}}\exp(\mathrm{i}st)\,\mathrm{d}t$

includes all wavelengths, comparable with Δx

viscosity destabilises: $\mathrm{Im}\,\lambda_1 \neq 0$

Brotherton-Ratcliffe & Smith (TCFD, 1989),
Bowles & Smith (JFM, 1992)



$$k\,\frac{\mathscr{A}(k)}{\mathscr{P}(k)} = \chi\,\frac{(G+Sk^2)\tanh k - k}{G+Sk^2-k\tanh k}, \quad \chi := \Lambda\,\frac{\beta^{5/3}}{(2\mu_1)^{2/3}}$$

$$k\,\frac{\mathscr{A}(k)}{\mathscr{P}(k)} = \chi\,\frac{(G+Sk^2)\tanh k - k}{G+Sk^2 - k\tanh k}, \qquad \chi := \Lambda\,\frac{\beta^{5/3}}{(2\mu_1)^{2/3}}$$

special cases

- highly supercritical flow: $G \ll 1,\ S \ll 1$

$$A'(X) \sim \frac{\chi}{2} \int_{-\infty}^{\infty} P(t) \coth\left[\frac{\pi}{2}(t-X)\right] \mathrm{d}t$$

- highly capillary flow: $S \gg 1$

$$A'(X) \sim \frac{\chi}{2} \int_{-\infty}^{\infty} \frac{P(t)\,\mathrm{d}t}{\sinh[\pi(t-X)/2]}$$

- transcritical flow: $G \to 1_-$

  pole at $k=0$ $\Rightarrow$ algebraic downstream tail $\Rightarrow$ extended theory as $\mu_1^{inv} \to 0$

$$k\,\frac{\mathscr{A}(k)}{\mathscr{P}(k)} = \chi\,\frac{(G+Sk^2)\tanh k - k}{G+Sk^2 - k\tanh k}, \qquad \chi := \Lambda\,\frac{\beta^{5/3}}{(2\mu_1)^{2/3}}$$

long-wave limit

- $G = O(1)$, $S \ll 1$ maintains resonance for $G \to 1_-$

$$A \sim \chi\,\frac{G-1}{G}\,P \quad \checkmark$$

Gajjar & Smith (Mathematatika, 1983)

- highly supercritical flow: $G, P = O(k^2)$

$$(S-1)A'' - GA \sim \chi P \quad \checkmark$$

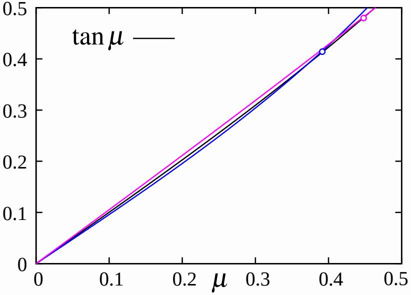
Bowles & Smith (JFM, 1992)

governs upstream edge of hydraulic jump

Transcritical long-wave (KdV) limit: $G = 1 \mp \gamma, \quad \gamma \to 0_+$

- vanishing exponential, here restricted to $|S - 1/3| = O(1)$

$$\mu = (G - S\mu^2)\tan\mu \quad \Rightarrow \quad \mu = \mu_1^{inv} \sim \sqrt{\mp\gamma/(S-1/3)} + O(\gamma^{3/2})$$



$f(\mu) = \mu/(G - S\mu^2) \;\Rightarrow$ slippery sections:

$G = 0.95,\ S = 0.1$

$G = 1.05,\ S = 0.7$

- generic expansive/compressive z^{-1}-singularity (cf. gasdynamics)

$$w - \phi_0 \sim z\,\underbrace{(1+\gamma)}_{\text{detuning}}\; - 4(1/3 - S)z^{-1} + O(z^{-2}, \gamma^2) \quad (x \to -\infty,\ \gamma \to 0)$$

shallow flow matching Euler flow downstream

- shallow-water parameter $\varepsilon \mapsto \sqrt{\gamma}$
- $Re_H := Re/\varepsilon, \ \hat{X} := x\sqrt{\gamma/|S - 1/3|} = O(1)$
- $[u-1, p-(1\mp\gamma)(1-y), y^+ - 1] \sim \gamma[-1, 1, 1]\hat{P}(\hat{X}, v)/3 + O(\gamma^{3/2})$

TD problem $\hat{P}(\hat{X};\nu),\ \hat{\Psi}(\hat{X},\hat{Y};\nu)\ = O(1)$

LD: thickness $\delta = O(\gamma^2)$

$\hat{\Psi}_{\hat{Y}}\hat{\Psi}_{\hat{Y}\hat{X}} - \Psi_{\hat{X}}\Psi_{\hat{Y}\hat{Y}} = -\mathrm{d}\hat{P}/\mathrm{d}\hat{X} + \hat{\Psi}_{\hat{Y}\hat{Y}\hat{Y}}$

$\hat{Y} = 0:\ \ \hat{\Psi} = \hat{\Psi}_Y = 0,\quad \hat{Y} \to \infty:\ \ \hat{\Psi} \sim [\hat{Y} + \hat{A}(X;\nu) + \hat{B}(\hat{X})]^2/2,\quad \bar{X} \to -\infty:\ \ \hat{\Psi} \sim \hat{Y}^2/2$

MD: $\gamma^{3/2} \gtrsim$ thickness $\Delta = \gamma^{13/2} Re \gg \gamma^2$ weak hydraulic jump: Kluwick et al. (Acta Mech, 2010)

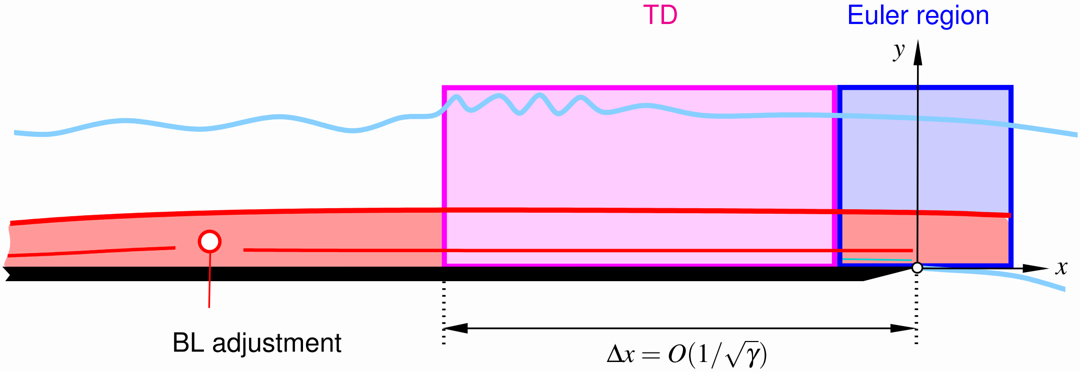
UD: $O(\gamma^2)$-solvability $\Rightarrow$ forced KdV eq.

$$\pm\hat{P} - \hat{P}^2/2 \pm \hat{P}'' = \hat{B} - \nu\hat{A}$$

$+/-\ \ldots$ super-/subcritical upstream (detuning), $S >/< 1/3$ (dispersion)

$\hat{X} \to -\infty:\ \ \hat{P} \sim \sum_{i=1}^{2} \hat{P}_i \exp(q_i\hat{X}) + c.c.,\ \ \pm 1 \pm q^2 = \nu q^{1/3}$

Transcritical long-wave limit



upstream dispersion

$$\pm 1 \pm q^2 = \nu q^{1/3}, \quad \nu \geq 0$$

up to 2 roots with $\operatorname{Re} q \geq 0$

	$+1$, slightly *supercritical:*	-1, slightly *subcritical:*
$-q^2$:	$q_1 = q_2 \leq 1$	no eigensolution
$+q^2$:	$\nu \geq 6/5^{5/6} \Rightarrow q_1 \leq 1/\sqrt{5} \leq q_2$	$q_1 \geq 1$
	$\nu < 6/5^{5/6} \Rightarrow q_2 = q_1^*$	
ν:	*stabilising*	*destabilising*

LD flow linearised (Airy layer, attached) $\Rightarrow$ UD problem decouples

$$\pm\hat{P} - \hat{P}^2/2 \pm \hat{P}'' = \nu \hat{P}^{(1/3)} + \hat{B}(\hat{X}), \qquad f^{(r)}(\hat{X}) := \frac{1}{\Gamma(1-r)} \int_{-\infty}^{\hat{X}} \frac{f'(t)}{(\hat{X}-t)^r}\, \mathrm{d}t \quad (0 < r < 1)$$

cf. acoustic BLs ($r = 1/2$): Smyth (1988), Kluwick (CISM, 1991)

$\hat{X} \to 0_-:$ $\hat{P} \sim \pm 6/\hat{X}^2$ to match compressive/expansive singularity, splits LD

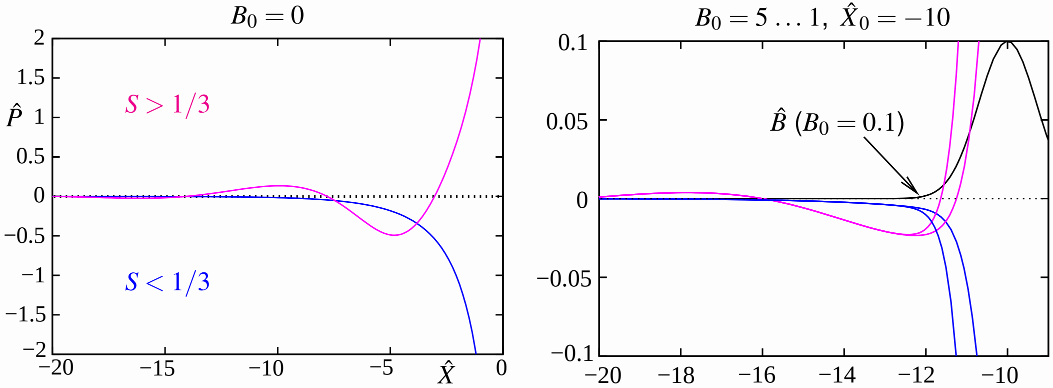
$\hat{X} \to -\infty:$ $\nu > 0$ prevents periodicity of $\hat{P}$

$\hat{X} \to +\infty:$ $P \to 0$ or $\hat{P} \sim 2 - 2\nu \hat{X}^{-1/3}/\Gamma(\frac{2}{3}) + O(\hat{X}^{-2/3})$ (jump + upstream history)

implicit solution involves

- history integral splitted 3-part (asymptotic, past, current), no lagging of $\bar{X}$
- higher-order Runge–Kutta scheme for current step $\hat{X} - \Delta\hat{X} \le t$, $\Delta\hat{X}$ adaptive

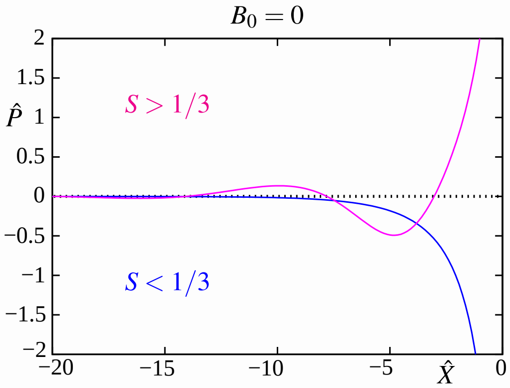
$\hat{X} \to -\infty$: $\hat{P} \to 0_-$

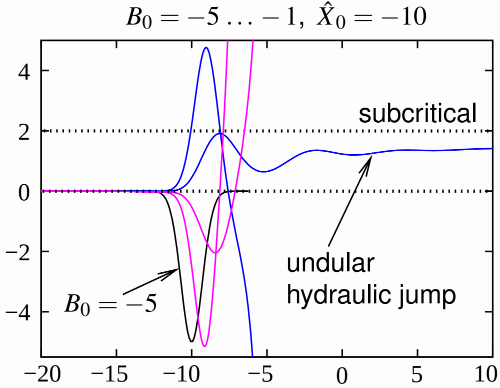


hump triggers singularity spontaneously!

Slightly supercritical flow: $\nu = 1, \quad \hat{B}(\hat{X}) = B_0 \exp\left[-(\hat{X} - \hat{X}_0)^2\right] \le 0$

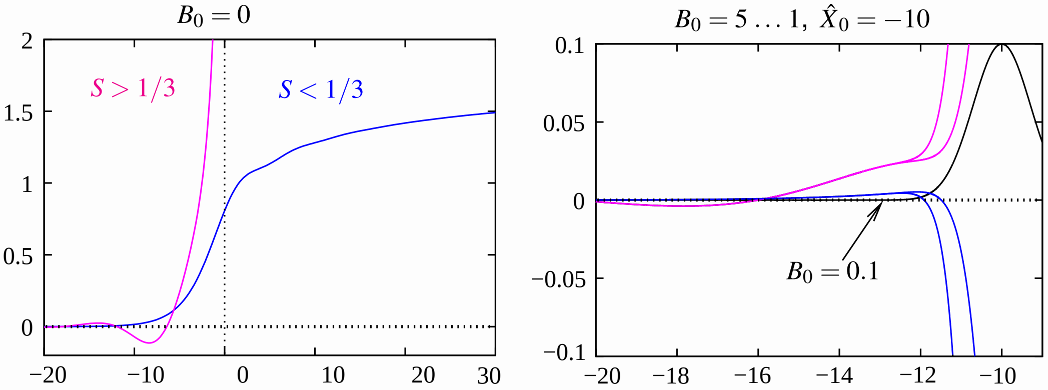
$\hat{X} \to -\infty: \quad \hat{P} \to 0_-$





slight dent triggers hydraulic jump!

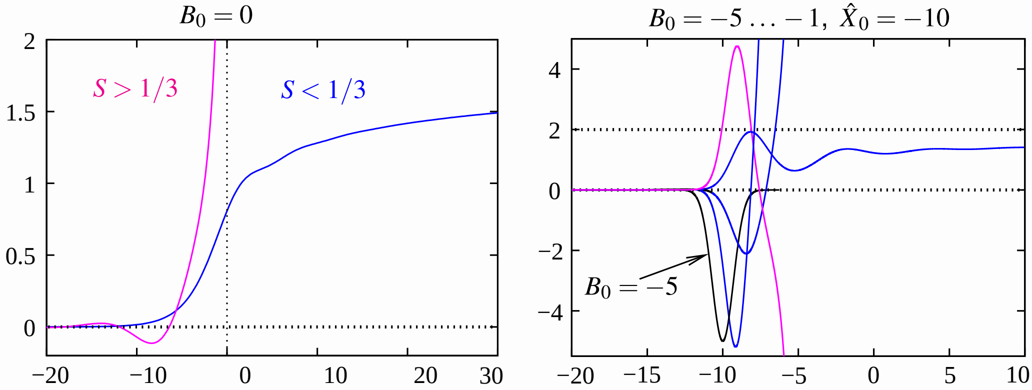
$\hat{X} \to -\infty: \quad \hat{P} \to 0_+$



$S < 1/3$: singularity requires topography!

Slightly supercritical flow: $\nu = 1, \quad \hat{B}(\hat{X}) = B_0 \exp\left[-(\hat{X} - \hat{X}_0)^2\right] \leq 0$

$\hat{X} \to -\infty: \quad \hat{P} \to 0_+$



slight dent triggers hydraulic jump!

Steady overfall — unsteady transcritical flow upstream

$$c(\text{surface waves}) \gg c(\text{TS waves})$$

- LD quasi-steady
- UD unsteady

$$\hat{P}_T \pm \hat{P}_{\hat{X}} - \hat{P}\hat{P}_{\hat{X}} \pm \hat{P}_{\hat{X}\hat{X}\hat{X}} = \nu \partial_{\hat{X}}{}^{1/3} \hat{P}_{\hat{X}} + B_{\hat{X}}(\hat{X}, T)$$

linear waves: $P \sim P_0(X)$

- no self-sustained eigenmodes $\propto \exp[\mathrm{i}(kX - \omega T)]$
- thus locally absolutely stable for $|k| \to \infty$

$$-\mathrm{i}\omega \pm \mathrm{i}k - \mathrm{i}\hat{P}_0(\hat{X}_0)k \mp \mathrm{i}k^3 \sim \nu(\mathrm{i}k)^{4/3}$$

- global mode for $T \gg 1$?
- but convectively unstable near $\hat{X} = 0$!

nonlinear waves reflected by singularity

- If $S = 1/3$, or on a longer length scale and with $Fr > 1$ closer to unity,

$$\hat{P}_{\hat{X}} - \hat{P}\hat{P}_{\hat{X}} = \nu \partial_{\hat{X}}{}^{1/3}\hat{P}_{\hat{X}} + B_{\hat{X}}(\hat{X}).$$

- If $S = 1/3$, or on a longer length scale and with $Fr > 1$ closer to unity,

$$\hat{P}_{\hat{X}} - \hat{P}\hat{P}_{\hat{X}} = \nu \partial_{\hat{X}}{}^{1/3} \hat{P}_{\hat{X}} + B_{\hat{X}}(\hat{X}).$$

- Integrating with supercritical flow, $\hat{P} = 0$, upstream

$$\hat{P} - \hat{P}^2/2 = \nu \partial_{\hat{X}}{}^{1/3} \hat{P} + B(\hat{X}).$$

- If $S = 1/3$, or on a longer length scale and with $Fr > 1$ closer to unity,

$$\hat{P}_{\hat{X}} - \hat{P}\hat{P}_{\hat{X}} = \nu \partial_{\hat{X}}{}^{1/3}\hat{P}_{\hat{X}} + B_{\hat{X}}(\hat{X}).$$

- Integrating with supercritical flow, $\hat{P} = 0$, upstream

$$\hat{P} - \hat{P}^2/2 = \nu \partial_{\hat{X}}{}^{1/3}\hat{P} + B(\hat{X}).$$

- With $\nu = 0$

$$\hat{P} - \hat{P}^2/2 - B = 0, \quad \begin{cases} \hat{P} = 1 - \sqrt{1-2B}, & \text{supercritical flow} \\ \hat{P} = 1 + \sqrt{1-2B}, & \text{subcritical flow} \end{cases}$$

- If $S = 1/3$, or on a longer length scale and with $Fr > 1$ closer to unity,

$$\hat{P}_{\hat{X}} - \hat{P}\hat{P}_{\hat{X}} = \nu {\partial_{\hat{X}}}^{1/3}\hat{P}_{\hat{X}} + B_{\hat{X}}(\hat{X}).$$

- Integrating with supercritical flow, $\hat{P} = 0$, upstream

$$\hat{P} - \hat{P}^2/2 = \nu {\partial_{\hat{X}}}^{1/3}\hat{P} + B(\hat{X}).$$

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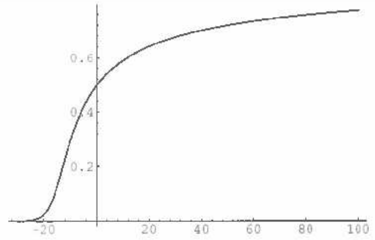
- Transcritical flow at bump crest if $B = 1/2$, $\hat{P} = 1$.

Jump solution connecting supercritical ($\hat{P} = 0$) to subcritical ($\hat{P} = 2$) flow



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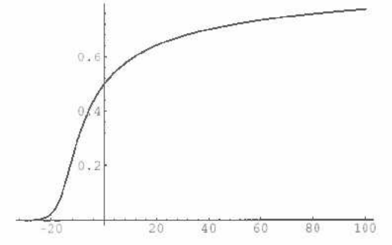
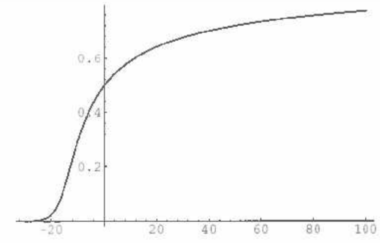


As $\hat{X} \to -\infty$,

$$\hat{P} \sim \exp(\hat{X}/\nu^3)$$

- exponential departure from supercritical flow upstream

Jump solution connecting supercritical ($\hat{P} = 0$) to subcritical ($\hat{P} = 2$) flow

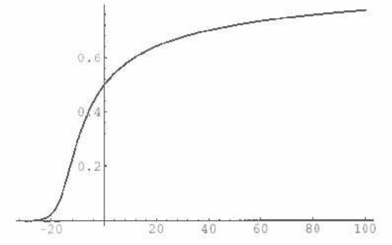


As $\hat{X} \to \infty$,

$$\hat{P} \sim 2 - \frac{2N}{\hat{X}^{1/3}}, \quad N = \frac{\nu}{\Gamma(1/3)}$$

- viscous-driven algebraic decay towards subcritical downstream

Jump solution connecting supercritical ($\hat{P} = 0$) to subcritical ($\hat{P} = 2$) flow



Length scale is of $O(N^3)$

Write

$$\hat{P} = \hat{P}_0 + N\hat{P}_1 + \cdots, \qquad B = ag(\hat{X}), \qquad a = 1/2 + N\beta + \cdots$$

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$$\hat{P}_0 = \hat{P}^{\mathrm{sup}} = 1 - \sqrt{1-g}, \quad \hat{P} = \hat{P}^{\mathrm{sup}} + N\frac{1}{1-\hat{P}^{\mathrm{sup}}}\left(\beta g + \int_{-\infty}^{\hat{X}} \frac{\hat{P}^{\mathrm{sup}}{}_{\xi}}{(\hat{X}-\xi)^{1/3}}\mathrm{d}\xi\right) + \cdots$$

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Solution regular at $\hat{X} = 0$ gives $\beta = -W_0$

$$\int_{-\infty}^{0} \frac{\hat{P}^{\mathrm{sup}}_{\xi}}{(-\xi)^{1/3}}\mathrm{d}\xi.$$

Write

$$\hat{P} = \hat{P}_0 + N\hat{P}_1 + \cdots, \qquad B = ag(\hat{X}), \qquad a = 1/2 + N\beta + \cdots$$

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$$\int_{-\infty}^{0} \frac{\hat{P}^{\text{sup}}_{\xi}}{(-\xi)^{1/3}}\mathrm{d}\xi.$$

Flow is critical at bump crest at height

$$1/2 - NW_0.$$

As $\hat{X} \to 0$ we have

$$\hat{P} = \left(1 - \gamma|\hat{X}| + \cdots\right) + N\left(\frac{W_1 \operatorname{sgn}\hat{X}}{\gamma} - 3\frac{\mathrm{H}(\hat{X})}{\hat{X}^{1/3}} + \cdots\right) + \cdots$$

$$g(\hat{X}) \sim 1 - \gamma^2\hat{X}^2 + \cdots, \quad W_1 = \int_{-\infty}^{0} \frac{\hat{P}^{\text{sup}}{}_{\xi\xi}}{(-\xi)^{1/3}} \mathrm{d}\xi$$

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We have a near-wake term due to the viscous response to the jump in derivative of $\hat{P}^{\text{sup}}$

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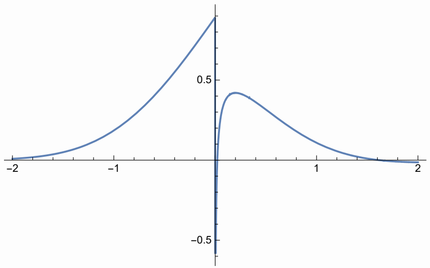

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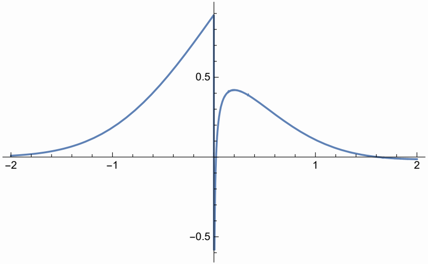

$$\hat{X} = O(N^{3/4})$$

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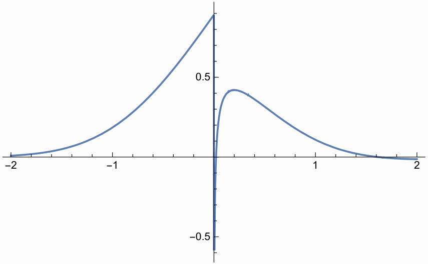

$$\hat{X} = O(N^{3/4})$$
$$\hat{P} = 1 + N^{3/4}\gamma^{1/4}2^{3/4}\Pi(Z) + \cdots,$$
$$\hat{X} = N^{3/4}\gamma^{-3/4}2^{3/4}Z,$$
$$a = 1/2 + NW_0 + N^{3/2}\gamma^{1/2}2^{1/2}S + \cdots$$

Smoothing the singularity — inner region

$$-\Pi^2 + Z^2 - S = \int_{-\infty}^{Z} \frac{\Pi_\xi}{(Z-\xi)^{1/3}}\,\mathrm{d}\xi,$$

$$\Pi \sim -|Z| - \frac{3\,\mathrm{H}(Z)}{2Z^{1/3}}, \quad |Z| \to \infty.$$

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For $S = O(1)$ full viscous-inviscid interaction at bump crest

Smoothing the singularity — inner region

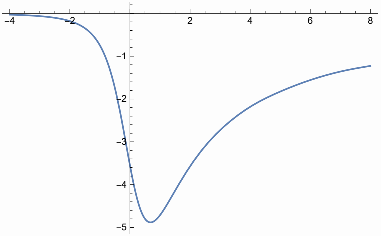
$$-\Pi^2 + Z^2 - S = \int_{-\infty}^{Z} \frac{\Pi_\xi}{(Z-\xi)^{1/3}} \mathrm{d}\xi,$$

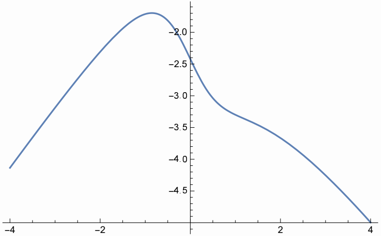
As $S \to -\infty$,

$$\Pi \sim |S|^{1/2}, \quad Z \sim |S|^{1/2}.$$

Smoothing the singularity — inner region

As $S \to -\infty$





$$\Pi(Z) \sim -\sqrt{|S| + Z^2}$$

$$-\frac{1}{2} \frac{1}{\sqrt{|S|+Z^2}} \int_{-\infty}^{Z} \frac{\xi}{\sqrt{|S|+\xi^2}} \frac{\mathrm{d}\xi}{(Z-\xi)^{1/3}} + \cdots$$

Near-wake is smoothed.

$$-\Pi^2 + Z^2 - S = \int_{-\infty}^{Z} \frac{\Pi_\xi}{(Z-\xi)^{1/3}} \mathrm{d}\xi,$$

As $S \to \infty$

$$\Pi = S^{3/2}\hat{\Pi}, \quad Z = S^{3/2}\hat{Z}$$

giving

$$-\hat{\Pi}^2 + \hat{Z}^2 - \frac{1}{S^2} = \frac{1}{S^2}\int_{-\infty}^{\hat{Z}} \frac{\hat{\Pi}_\xi}{(\hat{Z}-\xi)^{1/3}} \mathrm{d}\xi,$$

$$-\hat{\Pi}^2+\hat{Z}^2-\frac{1}{S^2}=\frac{1}{S^2}\int_{-\infty}^{\hat{Z}}\frac{\hat{\Pi}_\xi}{(\hat{Z}-\xi)^{1/3}}\mathrm{d}\xi,\quad S\to\infty.$$

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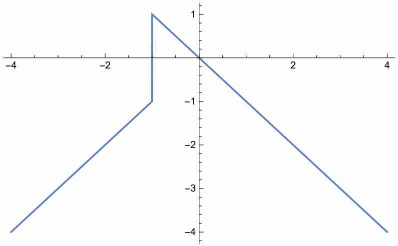
We have $\hat{\Pi} = \pm\hat{Z}$

$$-\hat{\Pi}^2 + \hat{Z}^2 - \frac{1}{S^2} = \frac{1}{S^2}\int_{-\infty}^{\hat{Z}} \frac{\hat{\Pi}_\xi}{(\hat{Z}-\xi)^{1/3}}\mathrm{d}\xi, \quad S \to \infty.$$



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But also, about any $\hat{Z} = -D$, branching from supercritical flow

$$\hat{Z} = -D + S^{-6}q, \qquad \hat{\Pi} = -D + D\theta(q),$$

$$2\theta - \theta^2 = \int_{-\infty}^{q} \frac{\theta_\xi}{(q-\xi)^{1/3}}\mathrm{d}\xi$$

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Incipient jumps in the inner region

$$-\hat{\Pi}^2+\hat{Z}^2-\frac{1}{S^2}=\frac{1}{S^2}\int_{-\infty}^{\hat{Z}}\frac{\hat{\Pi}_\xi}{(\hat{Z}-\xi)^{1/3}}\mathrm{d}\xi, \quad S\to\infty.$$

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- exponential departure from $\theta=0$ upstream

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- algebraic decay towards $\theta=2$ downstream

$$\theta\sim 2-\frac{1}{q^{1/3}}.$$

- How do we fix D?

The position of the jump

$$2\theta - \theta^2 = \int_{-\infty}^{q} \frac{\theta_\xi}{(q-\xi)^{1/3}} \mathrm{d}\xi + O(S^{-2}), \quad \theta \sim 2 - \frac{1}{q^{1/3}}.$$

The far-wake decay of the jump solution is of order of the detuning/bump height when

$$q = O(S^6),$$

i.e.

$$\hat{Z} = O(1).$$

Write

$$\hat{P} = \hat{P}_0 + S^{-2}\hat{P}_1 + \cdots$$

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$$\hat{P}_0 = \begin{cases} \hat{Z}, & \hat{Z} < -D, \\ -\hat{Z} & \hat{Z} > -D \end{cases}$$

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Now

$$\hat{P}_{0\hat{Z}} = 1 + 2D\delta(\hat{Z}+D) - \ 2\,\mathrm{H}(\hat{Z}+D)$$

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Now

$$\hat{P}_{0\hat{Z}} = 1 + \underbrace{2D\delta(\hat{Z} + D)}_{\text{jump}} - \underbrace{2\,\mathrm{H}(\hat{Z} + D)}_{\text{subcritical not supercritical flow}}$$

$$\hat{P}_1 = \begin{cases} 0, & \hat{Z} < -D, \\ \dfrac{1}{2\hat{Z}}\left(1 + \displaystyle\int_{-\infty}^{\hat{Z}} \dfrac{\hat{P}_{0\xi}}{(\hat{Z}-\xi)^{1/3}}\mathrm{d}\xi\right) & \hat{Z} > -D \end{cases}$$

So

$$\hat{P}_1 = \frac{1}{2\hat{Z}}\left(\underbrace{1}_{\text{Bump height}} + \underbrace{\frac{2D}{(\hat{Z}+D)^{1/3}}}_{\text{Viscous wake of jump}} - \underbrace{3(\hat{Z}+D)^{2/3}}_{\text{reaction of pressure to boundary layer in subcritical as opposed to supercritical flow}}\right), \quad \hat{Z} > -D$$

Regularity at $\hat{Z}=0$ gives $D=1$

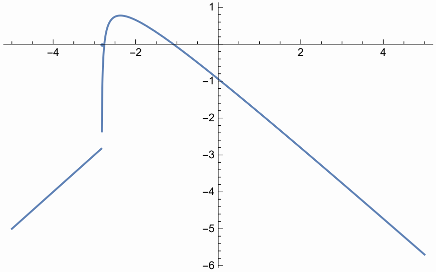
$$\hat{P}=-\hat{Z}+\frac{1}{S^2}\frac{1}{2\hat{Z}}\left(1+\frac{2}{(\hat{Z}+1)^{1/3}}-3(\hat{Z}+1)^{1/3}\right),\quad \hat{Z}>-1$$

$$\hat{P}\sim-\hat{Z}-\frac{1}{S^2}\frac{3}{2\hat{Z}^{1/3}},\quad \hat{Z}\to\infty,$$

Hence matching with outer flow.

Regularity at $\hat{Z} = 0$ gives $D = 1$

$$\hat{P} = -\hat{Z} + \frac{1}{S^2}\frac{1}{2\hat{Z}}\left(1 + \frac{2}{(\hat{Z}+1)^{1/3}} - 3(\hat{Z}+1)^{1/3}\right), \quad \hat{Z} > -1$$



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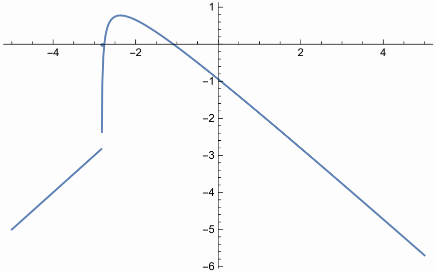

Jump at $Z = -S^{3/2}$ of magnitude $2S^{3/2}$ and thickness $Z \sim O(S^{3/2}S^{-6}) = O(S^{9/2})$.

When $S \sim O(N^{-1/2})$, jump is pushed to an $\hat{X} = O(1)$ distance upstream of crest. It becomes narrower.

Jump upstream of crest

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Supercritical flow for $\hat{X} < \hat{X}_J$, subcritical for $\hat{X}_J < \hat{X} < 0$ and supercritical again for $\hat{X} > 0$.

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This solution is regular at the obstacle crest if

$$\beta \quad + \quad W_0 \quad + \frac{2-\hat{P}_J}{(-\hat{X}_J)^{1/3}} + \int_{\hat{X}_J}^{0} \frac{2(\sqrt{1-g})_\xi}{(-\xi)^{1/3}} \mathrm{d}\xi = 0$$

$\hat{P}_J$ is pressure just upstream of jump. Strength of jump is $2-\hat{P}_J$.

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This gives an expression for the jump position as a function of obstacle shape, g, and height, β.

This solution is regular at the obstacle crest if

$$\underbrace{\beta}_{\text{height}} + \underbrace{W_0}_{\substack{\text{supercritical}\\ \text{flow}}} + \underbrace{\frac{2-\hat{P}_J}{(-\hat{X}_J)^{1/3}}}_{\substack{\text{wake of}\\ \text{jump}}} + \underbrace{\int_{\hat{X}_J}^{0} \frac{2(\sqrt{1-g})_\xi}{(-\xi)^{1/3}} \mathrm{d}\xi}_{\substack{\text{subcritical not}\\ \text{supercritical flow}}} = 0$$

As β increases jump is pushed ahead of the obstacle so $\hat{P}_J = 0$

$$\beta + W_0 + \frac{2}{(-\hat{X}_J)^{1/3}} + (-2W_0) = 0$$

$$W_0 = -\int_{-\infty}^{0} \frac{(\sqrt{1-g})_\xi}{(-\xi)^{1/3}} \mathrm{d}\xi > 0,$$

As β increases jump is pushed ahead of the obstacle so $\hat{P}_J = 0$

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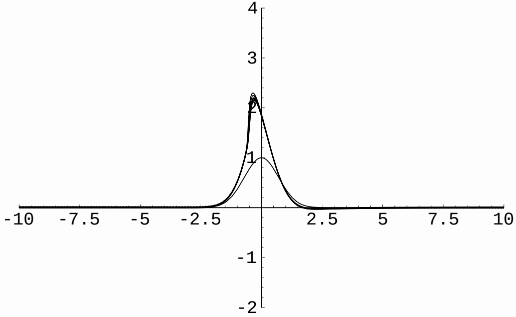
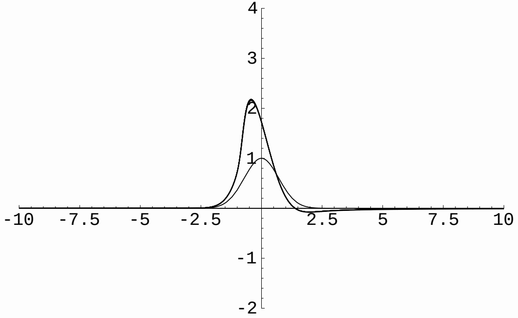
Jump is pushed to $\hat{X}_J = -\infty$ when $\beta = W_0$ and the flow is blocked with subcritical flow upstream.

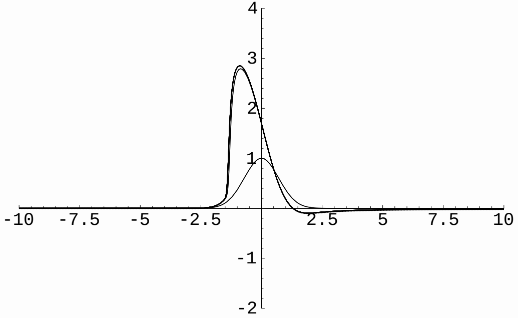
Runge-Kutta for time marching and FFT to evaluate fractional derivative



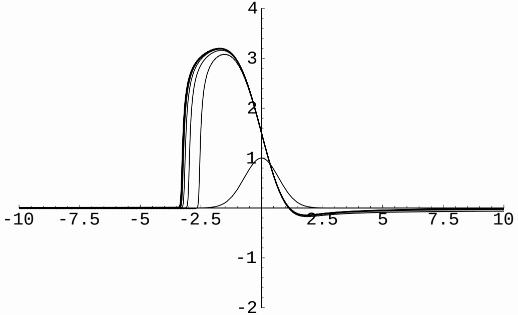
Time marching to a steady solution



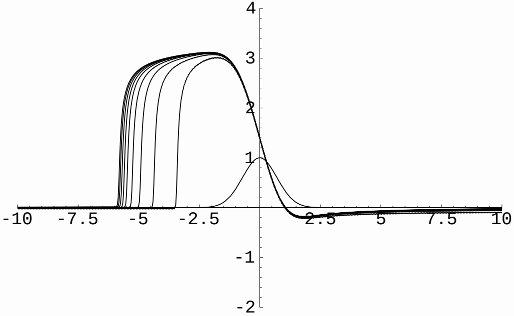
Time marching to a steady solution



Time marching to a steady solution



Time marching to a steady solution



With $\sigma = -\hat{X}_J/\hat{X}$, as $\hat{X} \to \infty$,

$$\hat{P}_1 \sim \frac{2(1-\hat{P}_J)}{\hat{X}^{1/3}}\left(\frac{1}{(1+\sigma)^{1/3}} - 1\right) + \frac{1}{3\hat{X}^{4/3}} 2(1-\hat{P}_J)\sigma\hat{X} + \frac{1}{3\hat{X}^{4/3}}\left(2\int_{\hat{X}_J}^{0} \sqrt{1-g}\,\mathrm{d}\xi - \int_{-\infty}^{\infty} 1 - \sqrt{1-g}\,\mathrm{d}\xi\right)$$

Far-wake behaviour

With $\sigma = -\hat{X}_J/\hat{X}$, as $\hat{X} \to \infty$,

$$\hat{P}_1 \sim \frac{2(1-\hat{P}_J)}{\hat{X}^{1/3}} \left(\frac{1}{(1+\sigma)^{1/3}} - 1 \right) + \frac{1}{3\hat{X}^{4/3}} 2(1-\hat{P}_J)\sigma\hat{X} + \frac{1}{3\hat{X}^{4/3}} \left(2\int_{\hat{X}_J}^{0} \sqrt{1-g}\, \mathrm{d}\xi - \int_{-\infty}^{\infty} 1 - \sqrt{1-g}\, \mathrm{d}\xi \right)$$

As $\sigma \to 0$, so no jump,

$$\hat{P}_1 \sim -\frac{1}{3\hat{X}^{4/3}} \int_{-\infty}^{\infty} (1 - \sqrt{1-g})\, \mathrm{d}\xi$$

With $\sigma = -\hat{X}_J/\hat{X}$, as $\hat{X} \to \infty$,

$$\hat{P}_1 \sim \frac{2(1-\hat{P}_J)}{\hat{X}^{1/3}}\left(\frac{1}{(1+\sigma)^{1/3}} - 1\right) + \frac{1}{3\hat{X}^{4/3}} 2(1-\hat{P}_J)\sigma\hat{X} + \frac{1}{3\hat{X}^{4/3}}\left(2\int_{\hat{X}_J}^{0} \sqrt{1-g}\,\mathrm{d}\xi - \int_{-\infty}^{\infty} 1 - \sqrt{1-g}\,\mathrm{d}\xi\right)$$

$\sigma \to \infty$, so blocked flow

$$\hat{P}_1 \sim -\frac{2}{\hat{X}^{1/3}}.$$

Summary & outlook

novelties

- steady inviscid free-surface flows exist only for $G \leq 1$
- theory of viscous–inviscid interaction therein beyond shallow-water limit
- self-excited TD separated from Euler (downfall) region, serves as mode filter
- $G \to 1$: KdV-limit describes UD, copes with transcritical singularity
- Higuera's singularity over-restrictive?

coming next

- transcritical flow with full LD
- solving TD problem
  - combining Veldman's indirect with pseudo-spectral method
- weakly 3D hump & UD

Thank you for your attention!