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NONLINEAR MODELLING OF DIELECTRIC ELASTOMER SINGLE LAYER PLATES USING A MULTIPLICATIVE DECOMPOSITION OF THE DEFORMATION GRADIENT TO ACCOUNT FOR ELECTROSTRICTION

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Abstract. In this paper we study dielectric elastomers accounting for constitutive coupling by means of electrostriction using a multiplicative decomposition of the deformation gradient tensor. The resulting constitutive relations are reduced to the special case of thin single layer plates made of incompressible dielectric elastomers. As an example problem we study the electro-mechanically coupled behavior of such a single layer plate in the absence of mechanical forces with special emphasis on the effect of electrostriction on the breakdown instability.

1 Three-dimensional constitutive relations

This section discusses relations involving certain physical entities, which are the non-symmetric Cauchy stress tensor $\boldsymbol{\sigma}$ and the spatial polarization vector $\mathbf{p}$, as well as their material counterparts, the second Piola-Kirchhoff stress tensor $\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T}$ and the material electric polarization vector $\boldsymbol{\mathcal{P}} = J\mathbf{F}^{-1} \cdot \mathbf{p}$. $\mathbf{F}$ is the deformation gradient tensor, $J = \det\mathbf{F}$ its determinant and $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ is the symmetric right Cauchy-Green tensor. Moreover, we have the spatial electric field vector $\mathbf{e}$ and the material one $\boldsymbol{\mathcal{E}} = \mathbf{e} \cdot \mathbf{F}$. With the free energy per unit mass $\Psi = \Psi(\mathbf{C}, \boldsymbol{\mathcal{E}})$ and the mass density ρ_0 in the reference configuration we write the thermodynamic relation

$$\rho_0 \dot{\Psi} = \left(\mathbf{S} + \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1}\right) \cdot\cdot \frac{1}{2}\dot{\mathbf{C}} - \boldsymbol{\mathcal{P}} \cdot \dot{\boldsymbol{\mathcal{E}}}, \tag{1}$$

in which $\mathbf{S}^{pol} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1}$ is the so-called second Piola-Kirchhoff polarization stress tensor. We also note that the skewsymmetric part of the bracketed term must vanish due to the local balance of moment of momentum, which reads

$$\text{skew}\left(\mathbf{S} + \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1}\right) = \mathbf{0}; \tag{2}$$

$\mathbf{S}^S = \mathbf{S} + \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1}$ is denoted as the symmetric stress tensor in the following. The above thermodynamic relation is a well known form of the time derivative of the free energy as it has been frequently reported in the literature, see e.g. [1], [2].

1.1 Multiplicative decomposition

Following the approach proposed in [3] we introduce a multiplicative decomposition of the deformation gradient tensor as

$$\mathbf{F} = \mathbf{F}_{me} \cdot \mathbf{F}_{el}, \tag{3}$$

with a mechanical part $\mathbf{F}_{me}$ and an electrical part $\mathbf{F}_{el} = \mathbf{F}_{el}(\boldsymbol{\mathcal{E}})$, such that the right Cauchy-Green tensor is $\mathbf{C} = \mathbf{F}_{el}^T \cdot \mathbf{C}_{me} \cdot \mathbf{F}_{el}$ with the mechanical part $\mathbf{C}_{me} = \mathbf{F}_{me}^T \cdot \mathbf{F}_{me}$. Now, the free energy is assumed as the sum of a purely mechanical part depending only on $\mathbf{C}_{me}$ and an electrical part; hence, we write

$$\Psi = \Psi_{me}(\mathbf{C}_{me}) + \Psi_{el}(\mathbf{C}, \boldsymbol{\mathcal{E}}), \tag{4}$$

and compute the time derivative to

$$\dot{\Psi} = \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot\cdot \dot{\mathbf{C}}_{me} + \frac{\partial \Psi_{el}}{\partial \mathbf{C}} \cdot\cdot \dot{\mathbf{C}} + \frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}} \cdot \dot{\boldsymbol{\mathcal{E}}}. \tag{5}$$

All second rank tensors in this relation are symmetric. It remains to compute the time derivative $\dot{\mathbf{C}}_{me}$. Using $\mathbf{C}_{me} = \mathbf{F}_{el}^{-T} \cdot \mathbf{C} \cdot \mathbf{F}_{el}^{-1}$ we find

$$\dot{\mathbf{C}}_{me} = \mathbf{F}_{el}^{-T} \cdot \dot{\mathbf{C}} \cdot \mathbf{F}_{el}^{-1} - 2\text{sym}\left(\mathbf{C}_{me} \cdot \dot{\mathbf{F}}_{el} \cdot \mathbf{F}_{el}^{-1}\right). \tag{6}$$

With the symmetry of $\partial \Psi_{me} / \partial \mathbf{C}_{me}$ and $\mathbf{F}_{el} = \mathbf{F}_{el}(\boldsymbol{\mathcal{E}})$ the first term in the above relation for $\dot{\Psi}$ becomes

$$\frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot\cdot \dot{\mathbf{C}}_{me} = \mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{F}_{el}^{-T} \cdot\cdot \dot{\mathbf{C}} - \left(2\mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{C}_{me} \cdot\cdot \frac{\partial \mathbf{F}_{el}}{\partial \boldsymbol{\mathcal{E}}}\right) \cdot \dot{\boldsymbol{\mathcal{E}}}, \tag{7}$$

and eventually we find

$$\dot{\Psi} = \left(\mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{F}_{el}^{-T} + \frac{\partial \Psi_{el}}{\partial \mathbf{C}} \right) \cdot \cdot \dot{\mathbf{C}} - \left(2\mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{C}_{me} \cdot \cdot \frac{\partial \mathbf{F}_{el}}{\partial \boldsymbol{\mathcal{E}}} - \frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}} \right) \cdot \dot{\boldsymbol{\mathcal{E}}}. \quad (8)$$

Comparing the two relations for $\dot{\Psi}$ renders two constitutive relations, one for the symmetric second Piola-Kirchhoff stress tensor and one for the material polarization vector,

$$\begin{aligned} \mathbf{S}^S = \mathbf{S} + \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} &= 2\rho_0 \mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{F}_{el}^{-T} + 2\rho_0 \frac{\partial \Psi_{el}}{\partial \mathbf{C}}, \\ \boldsymbol{\mathcal{P}} &= 2\rho_0 \mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{C}_{me} \cdot \cdot \frac{\partial \mathbf{F}_{el}}{\partial \boldsymbol{\mathcal{E}}} - \rho_0 \frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}}. \end{aligned} \quad (9)$$

Here, the material polarization is composed of two parts, an electrical one $\boldsymbol{\mathcal{P}}_{el}$ and $\boldsymbol{\mathcal{P}}_{coup}$ that accounts for the constitutive coupling; these two are

$$\boldsymbol{\mathcal{P}}_{el} = -\rho_0 \frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}} \quad \text{and} \quad \boldsymbol{\mathcal{P}}_{coup} = 2\rho_0 \mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{C}_{me} \cdot \cdot \frac{\partial \mathbf{F}_{el}}{\partial \boldsymbol{\mathcal{E}}}. \quad (10)$$

Hence, the polarization stress tensor can as well be decomposed into two parts, $\mathbf{S}^{pol} = \mathbf{S}^{pol,el} + \mathbf{S}^{pol,coup}$, with

$$\begin{aligned} \mathbf{S}^{pol,el} &= \boldsymbol{\mathcal{P}}_{el}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} = -\rho_0 \left(\frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}} \right) \boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1}, \\ \mathbf{S}^{pol,coup} &= \boldsymbol{\mathcal{P}}_{coup}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} = \rho_0 \left(2\mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{C}_{me} \cdot \cdot \frac{\partial \mathbf{F}_{el}}{\partial \boldsymbol{\mathcal{E}}} \right) \boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1}. \end{aligned} \quad (11)$$

In particular, we consider $\Psi_{el}(\mathbf{C}, \boldsymbol{\mathcal{E}})$ as

$$\rho_0 \Psi_{el} = -\frac{1}{2} \chi \varepsilon_0 \boldsymbol{\mathcal{E}} \cdot \left(\mathbf{C}^{-1} \cdot \boldsymbol{\mathcal{E}} \right), \quad (12)$$

with the permittivity in vacuum ε_0 and the constant susceptibility χ. This specific form of Ψ_{el} differs from the one used in [3] as it does not involve J. The derivatives with respect to $\boldsymbol{\mathcal{E}}$ and $\mathbf{C}$ are easily computed and result into

$$\begin{aligned} -\rho_0 \frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}} \boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} = \chi \varepsilon_0 \left(\mathbf{C}^{-1} \cdot \boldsymbol{\mathcal{E}} \right) \left(\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} \right) &= \chi \varepsilon_0 \mathbf{C}^{-1} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}) \cdot \mathbf{C}^{-1}, \\ 2\rho_0 \frac{\partial \Psi_{el}}{\partial \mathbf{C}} &= \chi \varepsilon_0 \mathbf{C}^{-1} \cdot (\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}) \cdot \mathbf{C}^{-1}, \end{aligned} \quad (13)$$

such that

$$\mathbf{S}^{pol,el} = -\rho_0 \frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}} \boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} = 2\rho_0 \frac{\partial \Psi_{el}}{\partial \mathbf{C}} \quad (14)$$

holds. Therefore, the constitutive relation for the sum of the stress tensor and the polarization stress tensor due to the coupling polarization $\boldsymbol{\mathcal{P}}_{coup}$, the so-called symmetric electro-mechanical stress tensor $\mathbf{S}^{em} = \mathbf{S} + \boldsymbol{\mathcal{P}}_{coup}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1}$ can be written as

$$\mathbf{S}^{em} = 2\rho_0 \mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{F}_{el}^{-T}. \tag{15}$$

Moreover, with $\mathbf{C}_{me} = \mathbf{F}_{el}^{-T} \cdot \mathbf{C} \cdot \mathbf{F}_{el}^{-1}$, the coupling polarization vector can be written as

$$\boldsymbol{\mathcal{P}}_{coup} = \mathbf{S}^{em} \cdot \mathbf{C} \cdot \mathbf{F}_{el}^{-1} \cdot\cdot \frac{\partial \mathbf{F}_{el}}{\partial \boldsymbol{\mathcal{E}}}. \tag{16}$$

This completes the constitutive modelling, which we summarize as follows:

$$\mathbf{S}^{S} = \mathbf{S} + \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} = \mathbf{S}^{em} + \mathbf{S}^{pol,el} \quad \text{and} \quad \boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_{el} + \boldsymbol{\mathcal{P}}_{coup}, \tag{17}$$

with

$$\begin{aligned} \mathbf{S}^{em} &= 2\rho_0 \mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{F}_{el}^{-T} \quad \text{and} \quad \mathbf{S}^{pol,el} = 2\rho_0 \frac{\partial \Psi_{el}}{\partial \mathbf{C}}, \\ \boldsymbol{\mathcal{P}}_{el} &= -\rho_0 \frac{\partial \Psi_{el}}{\partial \boldsymbol{\mathcal{E}}} \quad \text{and} \quad \boldsymbol{\mathcal{P}}_{coup} = \mathbf{S}^{em} \cdot \mathbf{C} \cdot \mathbf{F}_{el}^{-1} \cdot\cdot \frac{\partial \mathbf{F}_{el}}{\partial \boldsymbol{\mathcal{E}}}. \end{aligned} \tag{18}$$

With the aid of these constitutive relation the thermodynamic relation can also be written as

$$\rho_0 \dot{\Psi} = \underbrace{\mathbf{F}_{el} \cdot \mathbf{S}^{em} \cdot \mathbf{F}_{el}^{T} \cdot\cdot \frac{1}{2}\dot{\mathbf{C}}_{me}}_{=\rho_0 \dot{\Psi}_{me}(\mathbf{C}_{me})} + \underbrace{\mathbf{S}^{pol,el} \cdot\cdot \frac{1}{2}\dot{\mathbf{C}} - \boldsymbol{\mathcal{P}}_{el} \cdot \dot{\boldsymbol{\mathcal{E}}}}_{=\rho_0 \dot{\Psi}_{el}}. \tag{19}$$

In this sub-section the multiplicative decomposition was introduced as proposed by [3]; in addition the constitutive relations resulting from this decomposition have been discussed in detail and an alternative form for the electrical part of the free energy was proposed.

1.2 Total stress formulation

In this sub-section we introduce the notion of the the Maxwell stress tensor and of the total stress tensor, which are common in the field of nonlinear electro-elasticity to account for pondomotive forces. We begin by augmenting the free energy by a term accounting for the polarization in vacuum,

$$\rho_0 \Omega = \rho_0 \Psi - \frac{1}{2}\varepsilon_0 J \boldsymbol{\mathcal{E}} \cdot \left(\mathbf{C}^{-1} \cdot \boldsymbol{\mathcal{E}}\right) = \rho_0 \Psi + \rho_0 \Psi_{aug}. \tag{20}$$

Now we compute $\rho_0 \dot{\Omega}$; with $\Psi_{aug} = \Psi_{aug}(\mathbf{C}, \boldsymbol{\mathcal{E}})$ this results into

$$\rho_0 \dot{\Omega} = \rho_0 \dot{\Psi} + \rho_0 \frac{\partial \Psi_{aug}}{\partial \mathbf{C}} \cdot\cdot \dot{\mathbf{C}} + \rho_0 \frac{\partial \Psi_{aug}}{\partial \boldsymbol{\mathcal{E}}} \cdot \dot{\boldsymbol{\mathcal{E}}}, \tag{21}$$

which immediately finds

$$\rho_0\dot{\Omega} = \left(\mathbf{S} + \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{E}}\cdot\mathbf{C}^{-1} + 2\rho_0\frac{\partial\Psi_{aug}}{\partial\mathbf{C}}\right)\cdot\cdot\frac{1}{2}\dot{\mathbf{C}} - \left(\boldsymbol{\mathcal{P}} - \rho_0\frac{\partial\Psi_{aug}}{\partial\boldsymbol{\mathcal{E}}}\right)\cdot\dot{\boldsymbol{\mathcal{E}}}. \tag{22}$$

With the derivative of the augmentation term in the free energy with respect to $\boldsymbol{\mathcal{E}}$, which is

$$\rho_0\frac{\partial\Psi_{aug}}{\partial\boldsymbol{\mathcal{E}}} = -\varepsilon_0 J\mathbf{C}^{-1}\cdot\boldsymbol{\mathcal{E}} \tag{23}$$

and the derivative with respect to $\mathbf{C}$,

$$\rho_0\frac{\partial\Psi_{aug}}{\partial\mathbf{C}} = \frac{1}{2}\varepsilon_0 J\left[\mathbf{C}^{-1}\cdot\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}} - \frac{1}{2}\mathbf{I}\left(\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}\cdot\cdot\mathbf{C}^{-1}\right)\right]\cdot\mathbf{C}^{-1} = \frac{1}{2}\mathbf{S}^{Max} \tag{24}$$

we have

$$\rho_0\dot{\Omega} = \left(\mathbf{S} + \boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{E}}\cdot\mathbf{C}^{-1} - \frac{1}{2}\varepsilon_0 J\mathbf{C}^{-1}\left(\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}\cdot\cdot\mathbf{C}^{-1}\right)\right)\cdot\cdot\frac{1}{2}\dot{\mathbf{C}} - \boldsymbol{\mathcal{D}}\cdot\dot{\boldsymbol{\mathcal{E}}}, \tag{25}$$

in which $\boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{P}} + \varepsilon_0 J\mathbf{C}^{-1}\cdot\boldsymbol{\mathcal{E}}$ is the material electric displacement vector. The bracketed term is the total second Piola-Kirchhoff stress tensor $\mathbf{S}^{tot}$, which is the sum of the mechanical stress $\mathbf{S}$, the polarisation stress $\mathbf{S}^{pol}$ and the Maxwell stress $\mathbf{S}^{Max}$. Therefore,

$$\rho_0\dot{\Omega} = \mathbf{S}^{tot}\cdot\cdot\frac{1}{2}\dot{\mathbf{C}} - \boldsymbol{\mathcal{D}}\cdot\dot{\boldsymbol{\mathcal{E}}} \tag{26}$$

holds. Finally, without presenting a derivation, we note that the material electric displacement vector can be written in the well known form

$$\boldsymbol{\mathcal{D}} = -\rho_0\frac{\partial\Psi_{me}}{\partial\boldsymbol{\mathcal{E}}} - \rho_0\frac{\partial\Psi_{el}}{\partial\boldsymbol{\mathcal{E}}} - \rho_0\frac{\partial\Psi_{aug}}{\partial\boldsymbol{\mathcal{E}}} = -\rho_0\frac{\partial\Omega}{\partial\boldsymbol{\mathcal{E}}} \tag{27}$$

as the derivative of the augmented free energy with respect to the material electric field, and the total second Piola-Kirchhoff stress tensor follows from

$$\mathbf{S}^{tot} = 2\rho_0\frac{\partial\Psi_{me}}{\partial\mathbf{C}} + 2\rho_0\frac{\partial\Psi_{el}}{\partial\mathbf{C}} + 2\rho_0\frac{\partial\Psi_{aug}}{\partial\mathbf{C}} = 2\frac{\partial\Omega}{\partial\mathbf{C}} \tag{28}$$

as twice the derivative of the augmented free energy with respect to the right Cauchy-Green tensor. Both, the total stress tensor as well as the material displacement vector are involved in the balance / equilibrium conditions as well as the continuity conditions. We introduce the total first Piola-Kirchhoff stress tensor as $\mathbf{P}^{tot} = \mathbf{F}\cdot\mathbf{S}^{tot}$ and note the relations

$$\nabla_0\cdot\mathbf{P}^{tot} = \mathbf{b}_{me}\,, \quad \mathbf{n}\cdot[\![\mathbf{P}^{tot}]\!] = \mathbf{0} \quad \text{and} \quad \nabla_0\cdot\boldsymbol{\mathcal{D}} = 0\,, \quad \mathbf{n}\cdot[\![\boldsymbol{\mathcal{D}}]\!] = 0, \tag{29}$$

which can be found for instance in [4],[5]. $\mathbf{b}_{me}$ are mechanical body forces and ∇_0 is the invariant differential operator of the reference configuration.

1.3 The concept of a fictitous intermediate "*stress-free*" configuration

The symmetric electro-mechanical stress tensor

$$\mathbf{S}^{em} = \mathbf{S} + \boldsymbol{\mathcal{P}}_{coup}\boldsymbol{\mathcal{E}} \cdot \mathbf{C}^{-1} = 2\rho_0 \mathbf{F}_{el}^{-1} \cdot \frac{\partial \Psi_{me}}{\partial \mathbf{C}_{me}} \cdot \mathbf{F}_{el}^{-T} \tag{30}$$

vanishes, if $\mathbf{C}_{me} = \mathbf{I}$ as long as the specific form of the mechanical part of the free energy $\Psi_{me}(\mathbf{C}_{me})$ ensures this condition. In the absence of a rigid-body rotation this also results into $\mathbf{F}_{me} = \mathbf{I}$, and hence, $\mathbf{F} = \mathbf{F}_{el}$, which constitutes an intermediate configuration. The deformation from this configuration to the actual configuration by means of $\mathbf{F}_{me}$ results into a stress $\mathbf{S}^{em}$, which can be computed from a purely elastic constitutive relation. A

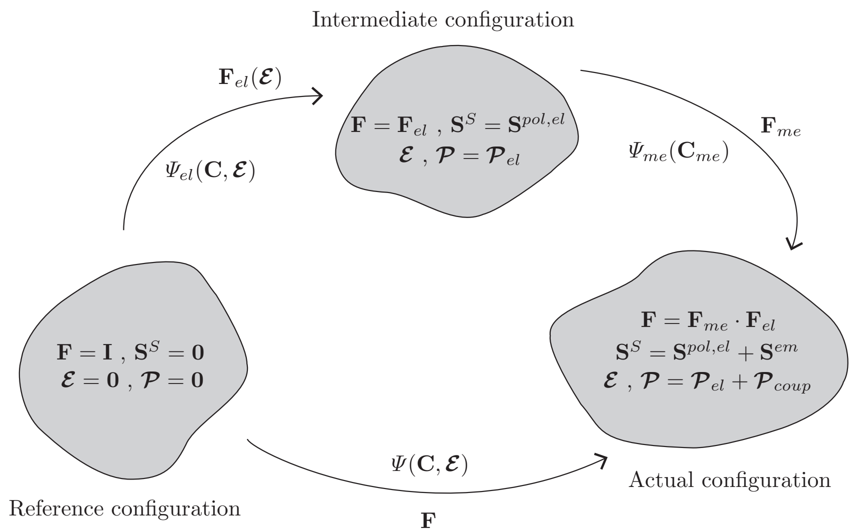

Figure 1: Schematic overview of involved configurations for constitutive modelling within the multiplicative decomposition

schematic overview of the three configurations is shown in Figure 1.

- In the reference configuration the electric field vector and the electric polarization vector vanish, $\boldsymbol{\mathcal{E}} = \mathbf{0}$ and $\boldsymbol{\mathcal{P}} = \mathbf{0}$, and the deformation gradient tensor equals the identity tensor, $\mathbf{F} = \mathbf{I}$. Hence, the symmetric stress tensor is trivial as well, $\mathbf{S}^S = \mathbf{0}$.

- In the intermediate configuration, which emerges due to to the electric field $\boldsymbol{\mathcal{E}}$ by means of the electrical deformation gradient tensor $\mathbf{F}_{el} = \mathbf{F}_{el}(\boldsymbol{\mathcal{E}})$, the polarization vector is equal to its electrical part, $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_{el}$, and the symmetric stress tensor equals the electrical part of the polarization stress tensor, $\mathbf{S}^S = \mathbf{S}^{pol,el}$. To a certain degree the constitutive process, which corresponds to the deformation from the reference configuration to the intermediate configuration, is characterized by the electrical part of the free energy $\Psi_{el}(\mathbf{C}, \boldsymbol{\mathcal{E}})$.

- The actual configuration, which is a result of any type of sources, emerges by means of the mechanical deformation gradient tensor $\mathbf{F}_{me}$; hence, it is charcterized by $\mathbf{F} = \mathbf{F}_{me} \cdot \mathbf{F}_{el}$. The symmtric electro-mechanical stress tensor $\mathbf{S}^{em}$ is assigned to this deformation, and the symmetric stress tensor is $\mathbf{S}^{S} = \mathbf{S}^{pol,el} + \mathbf{S}^{em}$. The electric field vector $\boldsymbol{\mathcal{E}}$ is kept constant, but the polarization vector contains the coupling polarization as well, $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_{el} + \boldsymbol{\mathcal{P}}_{coup}$. This constitutive process is characterized by the mechanical part of the free energy $\Psi_{me}(\mathbf{C}_{me})$.

Allthough this concept of an intermediate "*stress-free*" configuration in terms of the symmetric electro-mechanical stress tensor is quite appealing for the constitutive modelling, this intermediate configuration does not exist in the real problem, even if $\mathbf{F}_{el}$ is compatible and no mechanical forces act on the body. This can be explained with the presence of the Maxwell stress tensor as soon as an electrical deformation gradient $\mathbf{F}_{el} = \mathbf{F}_{el}(\boldsymbol{\mathcal{E}})$, and therefore an electric field is applied. The Maxwell stress tensor represents additional source terms - in terms of a body force as well as a surface traction - preventing this "*stress-free*" intermediate configuration to exist. It is however possible that a "*stress-free*" configuration with $\mathbf{F} \neq \mathbf{F}_{el}$ exists, in which the total stress tensor vanishes, rather than the electro-mechanical stress tensor.

1.4 Specific constitutive relations

We introduce the electrical part of the deformation gradient tensor $\mathbf{F}_{el}$ as

$$\mathbf{F}_{el} = \exp\mathbf{D}, \tag{31}$$

in which $\mathbf{D}$ is a second rank symmetric tensor; this specific law was proposed in [3]. Then, we may as well write $\mathbf{D} = \ln\mathbf{F}_{el}$ and by a proper choice of $\mathbf{D}$ we ensure that the electrical part of the deformation gradient tensor is actually an electrical right stretch tensor, $\mathbf{F}_{el} = \mathbf{R}_{el} \cdot \mathbf{U}_{el} \equiv \mathbf{U}_{el}$. Therefore, $\mathbf{D}$ is identical to an electrical logarthmic strain tensor $\mathbf{D} = \ln\mathbf{U}_{el}$. We introduce a unit vector in the direction of the material electric field vector $\boldsymbol{\mathcal{E}}$, which we denote as $\mathbf{m}$. Then, we choose

$$\mathbf{D} = c_1 \left(\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}\right) \mathbf{m}\mathbf{m} + c_2 \left(\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}\right) \left(\mathbf{I} - \mathbf{m}\mathbf{m}\right), \tag{32}$$

in order to account for electrostriction, which we consider to be quadratic in $\boldsymbol{\mathcal{E}}$. c_1 and c_2 are electrostrictive material parameters. Moreover, we can write $\mathbf{F}_{el} = \lambda_{el,3}\mathbf{m}\mathbf{m} + \lambda_{el} \left(\mathbf{I} - \mathbf{m}\mathbf{m}\right)$, in which

$$\lambda_{el,3} = \exp\left(c_1 \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}\right) \quad \text{and} \quad \lambda_{el} = \exp\left(c_2 \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\mathcal{E}}\right). \tag{33}$$

It remains to specify the specific form of the mechanical part $\Psi_{me}(\mathbf{C}_{me})$ of the free energy $\Psi = \Psi_{me}(\mathbf{C}_{me}) + \Psi_{el}(\mathbf{C}, \boldsymbol{\mathcal{E}})$. In the present paper we will be using a neo-Hookean hyperelastic strain energy function,

$$\rho_0 \Psi_{me}(\mathbf{C}_{me}) = \Psi_{me}\left(I_{\mathbf{C}_{me}}, II_{\mathbf{C}_{me}}, III_{\mathbf{C}_{me}}\right) = \frac{\mu}{2}\left(I_{\mathbf{C}_{me}} - 3 - 2\ln J\right) + K (\ln J)^2, \tag{34}$$

with the Lame parameter μ and the bulk modulus K. The purely electrical part $\rho_0\Psi_{el}(\mathbf{C},\boldsymbol{\mathcal{E}})$ and the augmentation term $\rho_0\Psi_{aug}(\mathbf{C},\boldsymbol{\mathcal{E}})$ have already been defined.

2 Thin dielectric elastomer single layer plates

We consider thin single layer plates made of a dielectric elastomer with a constitutive relation as introduced above. The dielectric elastomer layer is equipped with electrodes at its horizontal surfaces; hence, it is near at hand to approximate the material electric field vector $\boldsymbol{\mathcal{E}}$ as $\boldsymbol{\mathcal{E}} = \mathcal{E}\mathbf{m}$, in which $\mathbf{m}$ is the unit vector in thickness direction in the reference configuration. Moreover, we assume the right-Cauchy Green tensor as $\mathbf{C} = \mathbf{C}_2 + C_{33}\mathbf{mm}$, in which $\mathbf{C}_2$ refers to the plane part of $\mathbf{C}$. Likewise, $\mathbf{C}_{me} = \mathbf{C}_{2,me} + C_{33,me}\mathbf{mm}$ holds due to the specific form of $\mathbf{F}_{el}$. With these assumptions, we specify the augmented free energy Ω as

$$\rho_0\Omega = \rho_0\Psi_{me}\left(I_{\mathbf{C}_{me}}, II_{\mathbf{C}_{me}}, III_{\mathbf{C}_{me}}\right) - \frac{1}{2}\varepsilon_0\left(\chi + J\right)\frac{\mathcal{E}^2}{C_{33}}, \tag{35}$$

in which $I_{\mathbf{C}_{me}} = \text{tr}\mathbf{C}_{me}$, $II_{\mathbf{C}_{me}} = \mathbf{C}_{me}\cdot\cdot\,\mathbf{C}_{me}$ and $III_{\mathbf{C}_{me}} = \det\mathbf{C}_{me}$ are the three invariants of $\mathbf{C}_{me}$. With $\mathbf{C}_{me} = \mathbf{F}_{el}^{-T}\cdot\mathbf{C}\cdot\mathbf{F}_{el}^{-1}$, we can write these invariants as

$$I_{\mathbf{C}_{me}} = \lambda_{el}^{-2}I_{\mathbf{C}_2} + \lambda_{el,3}^{-2}C_{33}\,,\ II_{\mathbf{C}_{me}} = \lambda_{el,3}^{-4}C_{33}^2 + \lambda_{el}^{-4}II_{\mathbf{C}_2}\,,\ III_{\mathbf{C}_{me}} = \lambda_{el}^{-4}\lambda_{el,3}^{-2}C_{33}III_{\mathbf{C}_2}. \tag{36}$$

We study only incompressible materials and we apply the incompressibility condition to both, $\mathbf{F}_{el}$ and $\mathbf{F}$. Concerning $\mathbf{F}_{el}$ this results into $\det\mathbf{F}_{el} = 1$; hence, we have

$$\det\mathbf{F}_{el} = \lambda_{el,3}\lambda_{el}^2 = \exp\left((c_1 + 2c_2)\mathcal{E}^2\right) = 1 \quad\rightarrow\quad c_1 = -2c_2, \tag{37}$$

such that $\lambda_{el}^{-2} = \lambda_{el,3} = \exp\left(c_1\mathcal{E}^2\right) \equiv \lambda_e^{-2}$. Moreover, $\det\mathbf{C} = 1$ results into $C_{33} = \det\mathbf{C}_2^{-1} = III_{\mathbf{C}_2}^{-1}$. Then, the invariants of $\mathbf{C}_{me}$ are

$$I_{\mathbf{C}_{me}} = \lambda_e^{-2}I_{\mathbf{C}_2} + \lambda_e^4 III_{\mathbf{C}_2}^{-1}\,,\quad II_{\mathbf{C}_{me}} = \lambda_e^8 III_{\mathbf{C}_2}^{-2} + \lambda_e^{-4}II_{\mathbf{C}_2}\,,\quad III_{\mathbf{C}_{me}} = 1. \tag{38}$$

For the incompressible neo-Hookean material we write the augmented free energy as

$$\rho_0\Omega = \frac{\mu}{2}\left(\lambda_e^{-2}\text{tr}\mathbf{C}_2 + \lambda_e^4 III_{\mathbf{C}_2}^{-1} - 3\right) - \frac{1}{2}\varepsilon III_{\mathbf{C}_2}\mathcal{E}^2 \equiv \rho_0\Omega_2, \tag{39}$$

with the permittivity $\varepsilon = \varepsilon_0(\chi + 1) = \varepsilon_r\varepsilon_0$, with the relative permittivity $\varepsilon_r = \chi + 1$ and the electrical stretch $\lambda_e = \exp\left(c_2\mathcal{E}^2\right) = \exp\left((-c_1/2)\mathcal{E}^2\right)$. The plane part $\mathbf{S}_2^{tot}$ of the total stress tensor and the thickness component $\mathcal{D}_3$ of the material electric displacement vector, which are the only non-zero parts of the total stress tensor and of the electric displacement vector, are

$$\mathbf{S}_2^{tot} = 2\rho_0\frac{\partial\Omega_2}{\partial\mathbf{C}_2} \quad\text{and}\quad \mathcal{D}_3 = -\rho_0\frac{\partial\Omega_2}{\partial\mathcal{E}} = \varepsilon III_{\mathbf{C}_2}\mathcal{E} - \rho_0\frac{\partial\Omega_2}{\partial\lambda_{el}}\frac{\partial\lambda_{el}}{\partial\mathcal{E}}. \tag{40}$$

3 Electromechanical stability

As a preliminary study we consider a plate, for which the deformation is not constrained and no mechanical forces are applied. We denote the thickness with h and we apply a voltage V between the two electrodes; hence, $\mathcal{E}_3 = V/h \equiv \mathcal{E}$ holds. In such a problem the resulting in-plane deformation is homogenous and characterized by a constant spherical plane right Cauchy-Green tensor. Therefore, $\mathbf{C}_2 = C\mathbf{I}_2$ with $C = \lambda^2$ applies; here, λ is the principal stretch in both in-plane directions, $\lambda_1 = \lambda_2 = \lambda$, and $\mathbf{I}_2 = \mathbf{I} - \mathbf{mm}$ is the plane part of $\mathbf{I}$. Moreover, the two invariants of $\mathbf{C}_2$ are

$$I_{\mathbf{C}_2} = 2C = 2\lambda^2 \ , \quad III_{\mathbf{C}_2} = C^2 = \lambda^4. \tag{41}$$

Under these conditions, the augmented free energy for the incompressible neo-Hookean material is written in terms of the principal stretch λ as

$$\rho_0\Omega_2 = \frac{\mu}{2}\left(2\lambda_e^{-2}\lambda^2 + \lambda_e^4\lambda^{-4} - 3\right) - \frac{1}{2}\varepsilon\lambda^4\mathcal{E}^2. \tag{42}$$

The plane part of the total second Piola-Kirchhoff stress tensor follows from

$$\mathbf{S}_2^{tot} = 2\rho_0\frac{\partial\Omega_2}{\partial\mathbf{C}_2} = 2\rho_0\frac{\partial\Omega_2}{\partial\lambda}\frac{\partial\lambda}{\partial\mathbf{C}_2} = \rho_0\frac{1}{\lambda}\frac{\partial\Omega_2}{\partial\lambda}\mathbf{I}_2; \tag{43}$$

here, we have used the relation $\partial\mathbf{C}_2/\partial\lambda = 2\lambda\mathbf{I}_2$. We assume the contribution to the surface tractions at the vertical edges, which results from the Maxwell stress in the surrounding medium to be negligible, such that the plane part of the total second Piola-Kirchhoff stress tensor vanishes in this specific problem. With the augmented free energy $\rho_0\Omega_2$ for the conservative problem at hand, the equilibrium condition can be stated in the form of the Principle of Gibbs,

$$\rho_0\frac{\partial\Omega_2}{\partial\lambda} = \left(\lambda_e^{-2}\lambda - \lambda_e^4\lambda^{-5}\right) - \lambda^3\frac{\varepsilon}{\mu}\mathcal{E}^2 = 0, \tag{44}$$

from which equilibrium stretches λ are obtained. It remains to specify the material parameters μ, ε_r and c_1. We use a polyurethane elastomer, for which material parameters were reported in [6] as $\varepsilon_r = 8.8$, $Y = 3\mu$ with $Y = 3.6$MPa. In order to proceed with identifying the electrostrictive parameter c_1, we write the equilibrium conditions using the thickness stretch λ_3, which follows from the incompressibilty condition as $\lambda^2 = \lambda_3^{-1}$. Therefore, we have

$$\left(\lambda_e^{-2}\lambda_3^{-1} - \lambda_e^4\lambda_3^2\right) - \lambda_3^{-2}\frac{\varepsilon}{\mu}\mathcal{E}^2 = F(\lambda_3, \mathcal{E}^2) = 0. \tag{45}$$

The nonlinear function $F(\lambda_3, \mathcal{E}^2)$ is approximated in the vicinity of $\lambda_3 \approx 1$ and $\mathcal{E}^2 \approx 0$, which results into the linear relation

$$\varepsilon_3 = -\left(\frac{\varepsilon}{3\mu} - c_1\right)\mathcal{E}^2 = -M\mathcal{E}^2, \tag{46}$$

with the Biot strain $\varepsilon_3 = \lambda_3 - 1$ and an apperant electrostrictive coefficient M. Two particular effects contribute to the strain: the electrostrictive effect, which in our approach is accounted for by the parameter c_1 and the Maxwell effect which is mainly due to Coulomb interaction and charcterized by the first term in this linear relation. The experimentally identified value taken from [6] is $M = 7.07 \times 10^{-16}\mathrm{m}^2\mathrm{V}^{-2}$, which results into $c_1 = -6.86 \times 10^{-16}\mathrm{m}^2\mathrm{V}^{-2}$. Figure 2 shows the equilibrium Biot strain as a function of the square of the

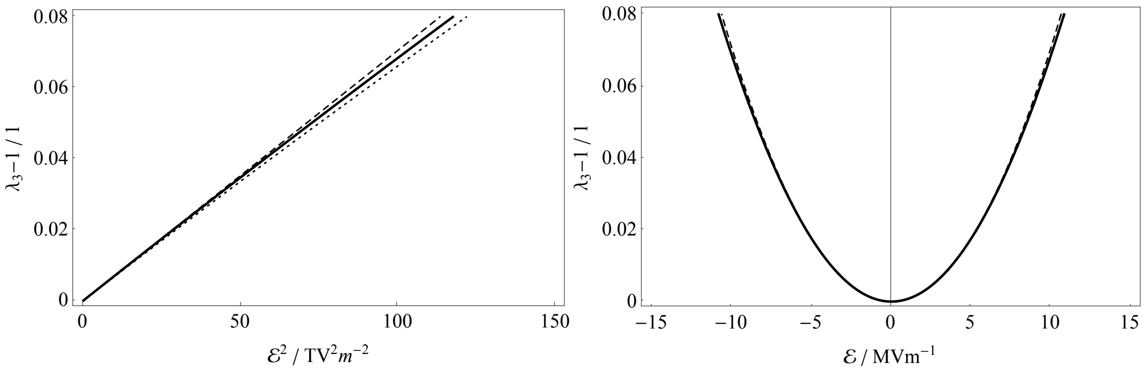

Figure 2: Equilibrium Biot strains in the small signal and deformation regime

electric field (left) and as a function of the electric field (right) for relatively small electric fields and strains. The presented results are very close to the experimental ones provided in [6]. The solid line corresponds to a solution using the multiplicative decomposition as proposed in this paper, the dashed line is the linearized response from eq. (46), and the dotted line in the left plot shows the electrical Biot strain $\varepsilon_{el,3} = \lambda_{el,3} - 1$ with the electrical part of the thickness stretch $\lambda_{el,3} = \lambda_e^{-2} = \exp(c_1\mathcal{E}^2)$. Within the small signal

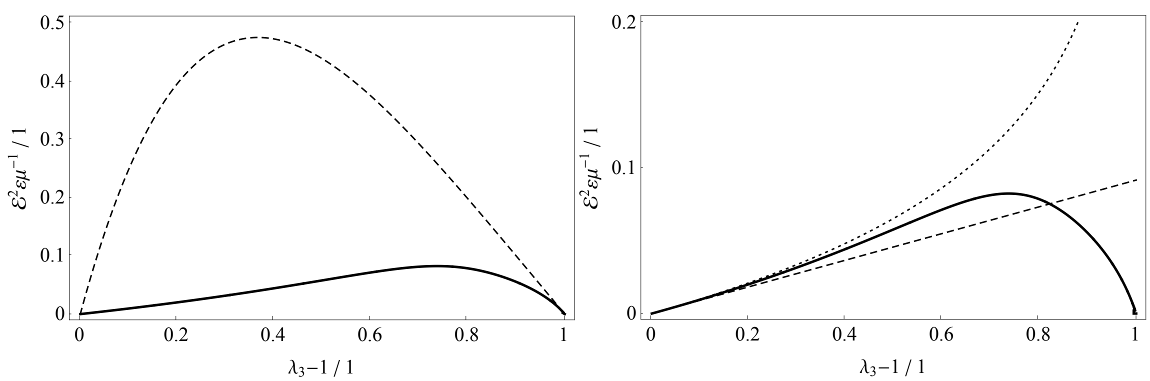

Figure 3: Equilibrium Biot strains in the large deformation regime

and deformation regime the deviation between the different curves and solutions is quite small; this changes, if we increase the electric field, as shown in Figure 3. In the left plot we compare the present solution (solid line) accounting for electrostriction with a solution, for which electrostriction is neglected, $c_1 = 0$, see e.g. [7]. One can see the extremely high importance of electrostriction, which was also pointed out in [6]. Electrostriction also significantly reduces the critical electric field and increases the corresponding critical Biot strain, at which the so-called electromechanical breakdown occurs. Beyond this point (horizontal tangent in the plot) no more stable equilibrium solutions exist. In the right plot of Figure 3 we compare our solution (solid line), the linearized response (dashed line) and the electrical part of the Biot strain (dotted line). As long as the Biot strain and the electric field are relatively small the three curves coincide quite well, a behavior that is fully lost for large strains. The deviation of the actual Biot strain from the electrical part of the Biot strain characterizes the deviation of the actual configuration from the intermediate one, and hence, the evolution of the symmetric electro-mechanical stress tensor. Finally,

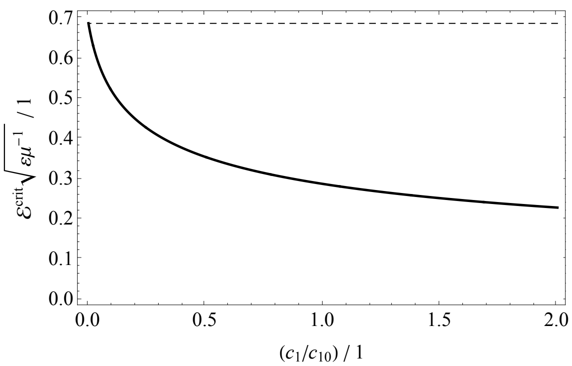

Figure 4: Critical electric field $\mathcal{E}_{crit}$ vs. electrostrictive coefficient c_1

we discuss the effect of electrostriction on the electromechanical breakdown in some more detail. For that sake, we compute the second derivative of the augmented free energy with respect to the stretch λ to

$$\rho_0 \frac{\partial^2 \Omega_2}{\partial \lambda^2} = \left(\lambda_e^{-2} + 5\lambda_e^4 \lambda^{-6}\right) - 3\lambda^2 \frac{\varepsilon}{\mu} \mathcal{E}^2. \tag{47}$$

Stability of an equilibrium point requires this second derivative to be positive. From the equilibrium condition and the stability margin $\rho_0 \frac{\partial^2 \Omega_2}{\partial \lambda^2} = 0$ we find

$$\left(\frac{\lambda_{crit}}{\lambda_{e,crit}}\right)^{-6} = \frac{1}{4} \quad \text{and} \quad \sqrt{\frac{\varepsilon}{\mu}} \left(\exp\left(-c_1 \mathcal{E}_{crit}^2\right) \mathcal{E}_{crit}\right) = \frac{\sqrt{3}}{4^{\frac{2}{3}}} = 0.687, \tag{48}$$

from which we can compute the critical value $\mathcal{E}_{crit}$, $\lambda_{e,crit}$ and furthermore λ_{crit}. For $\lambda_e = 1$, which means $c_1 = 0$, this result is well-known from the literature, see [8]. In Figure 4 we present the dependance of the breakdown electric field $\mathcal{E}_{crit}$ on the electrostrictive coefficient c_1, which is scaled with respect to the value c_{10} we have used in the above results.

4 Concusion & Outlook

The present paper was focussed on the discussion of a multiplicative decomposition of the deformation gradient tensor in dielectric elastomers to account for electrostriction. Only single layered plates were studied. In the future the approach will be extended to geometrically nonlinear shells with layers made of such dielectric elastomers. Moreover, the specific constitutive law for the electrical part of the deformation gradient will be revisited.

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