Fundamental Equations of Acoustics

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Abstract The equations of acoustics are based on the general equations of fluid dynamics: conservation of mass, momentum, energy and closed by the appropriate constitutive equation defining the thermodynamic state. The use of a perturbation ansatz, which decomposes the physical quantities density, pressure and velocity into mean, incompressible fluctuating and compressible fluctuating ones, allows to derive linearized acoustic conservation equations and its state equation. Thereby, we derive acoustic wave equations both for homogeneous and inhomogeneous media, and the equations model both vibrational- and flow-induced sound generation and its propagation.

1 Overview

Acoustics has developed into an interdisciplinary field encompassing the disciplines of physics, engineering, speech, audiology, music, architecture, psychology, neuroscience, and others (see, e.g., Rossing 2007). Therewith, the arising multi-field problems range from classical airborne sound over underwater acoustics (e.g., ocean acoustics) to ultrasound used in medical application. Here, we concentrate on the basic equations of acoustics describing acoustic phenomena. Thereby, we start with the mass, momentum and energy conservation equations of fluid dynamics as well as the constitutive equations. Furthermore, we introduce the Helmholtz decomposition to split the overall fluid velocity in a pure solenoidal (incompressible part) and irrotational (compressible) part. Since, wave propagation needs a compressible medium, we associate this part to acoustics. Furthermore, we apply a perturbation method to derive the acoustic wave equation, and discuss the main physical quantities of acoustics, plane and spherical wave solutions. Finally, we focus towards the two main mechanism of sound generation: aeroacoustics (flow induced sound) and vibroacoustics (sound generation due to mechanical vibrations).

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2 Basic Equations of Fluid Dynamics

We consider the motion of fluids in the continuum approximation, so that a body $\mathcal{B}$ is composed of particles $\mathcal{R}$ as displayed in Fig. 1. Thereby, a particle $\mathcal{R}$ already represents a macroscopic element. On the one hand a particle has to be small enough to describe the deformation accurately and on the other hand large enough to satisfy the assumptions of continuum theory. This means that the physical quantities density ρ, pressure p, velocity $\boldsymbol{v}$, and so on are functions of space and time, and are written as density $\rho(x_i, t)$, pressure $p(x_i, t)$, velocity $\boldsymbol{v}(x_i, t)$, etc. So, the total change of a scalar quantity like the density ρ is

$$\mathrm{d}\rho = \left(\frac{\partial \rho}{\partial t}\right) \mathrm{d}t + \left(\frac{\partial \rho}{\partial x_1}\right) \mathrm{d}x_1 + \left(\frac{\partial \rho}{\partial x_2}\right) \mathrm{d}x_2 + \left(\frac{\partial \rho}{\partial x_3}\right) \mathrm{d}x_3 \,. \tag{1}$$

Therefore, the total derivative (also called substantial derivative) computes by

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1}\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right) + \frac{\partial \rho}{\partial x_2}\left(\frac{\mathrm{d}x_2}{\mathrm{d}t}\right) + \frac{\partial \rho}{\partial x_3}\left(\frac{\mathrm{d}x_3}{\mathrm{d}t}\right) \\ &= \frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial \rho}{\partial x_i}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i}\underbrace{\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)}_{v_i} \,. \end{aligned} \tag{2}$$

Note that in the last line of (2) we have used the summation rule of Einstein.[1] Furthermore, in literature the substantial derivative of a physical quantity is mainly denoted by the capital letter D and for an Eulerian frame of reference writes as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \,. \tag{3}$$

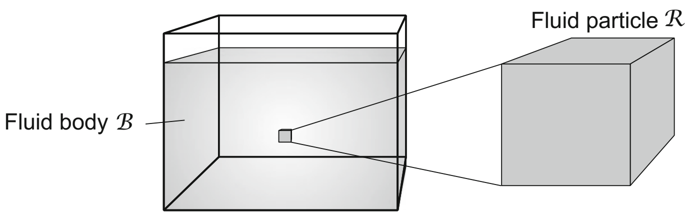


Fig. 1 A body $\mathcal{B}$ composed of particles $\mathcal{R}$

[1] In the following, we will use both vector and index notation; for the main operations see Appendix.

2.1 Spatial Reference Systems

A spatial reference system defines how the motion of a continuum is described i.e., from which perspective an observer views the matter. In a Lagrangian frame of reference, the observer monitors the trajectory in space of each material point and measures its physical quantities. This can be understood by considering a measuring probe which moves together with the material, like a boat on a river. The advantage is that free or moving boundaries can be captured easily as they require no special effort. Therefore, the approach is suitable in the case of structural mechanics. However, its limitation is obtained dealing with large deformation, as in the case of fluid dynamics. In this case, a better choice is the Eulerian frame of reference, in which the observer monitors a single point in space when measuring physical quantities – the measuring probe stays at a fixed position in space. However, contrary to the Lagrangian approach, difficulties arise with deformations on the domain boundary, e.g., free boundaries and moving interfaces.

To derive integral formulations of balance equations, the rate of change of integrals of scalar and vector functions has to be described, which is known as the Reynolds' transport theorem. The volume integral can change for two reasons: (1) scalar or vector functions change (2) the volume changes. The following discussion is directed to scalar valued functions. In an Eulerian context, time derivation must also take the time dependent domain $\Omega(t)$ into account by adding a surface flux term, which can be formulated as a volume term using the integral theorem of Gauß. This results in

$$
\begin{aligned}
\frac{D}{Dt} \int\limits_{\Omega(t)} f \, \mathrm{d}\boldsymbol{x} &= \int\limits_{\Omega(t)} \frac{\partial}{\partial t} f \, \mathrm{d}\boldsymbol{x} + \int\limits_{\Gamma(t)} f \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}s \\
&= \int\limits_{\Omega(t)} \left(\frac{\partial}{\partial t} f + \nabla \cdot (f \boldsymbol{v}) \right) \mathrm{d}\boldsymbol{x} \, .
\end{aligned}
\tag{4}
$$

2.2 Conservation Equations

The basic equations for the flow field are the conservation of mass, momentum and energy. Together with the constitutive equations and equations of state, a full set of partial differential equations (PDEs) is derived.

Conservation of mass The mass m of a body is the volume integral of its density ρ,

$$
m = \int\limits_{\Omega(t)} \rho(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} \, . \tag{5}
$$

Mass conservation states that the mass of a body is conserved over time, assuming there is no source or drain. Therefore, applying Reynolds' transport theorem (4), results in

$$\frac{Dm}{Dt} = \int_{\Omega} \frac{\partial \rho}{\partial t}\, \mathrm{d}\boldsymbol{x} + \int_{\Gamma} \rho \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}s$$
$$= \int_{\Omega} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) \right) \mathrm{d}\boldsymbol{x} = 0. \tag{6}$$

The integral in (6) can be dismissed, as it holds for arbitrary Ω and in the special case of an incompressible fluid ($\rho = const. \quad \forall (\boldsymbol{x}, t) \in \Omega \times \mathbb{R}$), which may be assumed for low Mach numbers (see Sect. 2.4), the time and space derivative of the density vanishes. This leads to the following form of mass conservation equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \text{(compressible fluid)},$$
$$\nabla \cdot \boldsymbol{v} = 0 \qquad \text{(incompressible fluid)}. \tag{7}$$

Conservation of momentum The equation of momentum is implied by Newtons second law and states that momentum $\boldsymbol{I}_{\mathrm{m}}$ is the product of mass m and velocity $\boldsymbol{v}$

$$\boldsymbol{I}_{\mathrm{m}} = m\boldsymbol{v}\,. \tag{8}$$

Derivation in time gives the rate of change of momentum, which is equal to the force $\boldsymbol{F}$ and reveals the relation to Newtons second law in an Eulerian reference system

$$\boldsymbol{F} = \frac{D\boldsymbol{I}_{\mathrm{m}}}{Dt} = \frac{D}{Dt}(m\boldsymbol{v}) = \frac{\partial}{\partial t}(m\boldsymbol{v}) + \nabla \cdot (m\boldsymbol{v} \otimes \boldsymbol{v}), \tag{9}$$

where $\boldsymbol{v} \otimes \boldsymbol{v}$ is a tensor defined by the dyadic product $\otimes$ (see Appendix). The last equality in (9) is derived from Reynolds transport theorem (4) and mass conservation (7).

The forces $\boldsymbol{F}$ acting on fluids can be split up into forces acting on the surface of the body $\boldsymbol{F}_{\Gamma}$, forces due to momentum of the molecules $D\boldsymbol{I}_{\mathrm{m}}/Dt$ and external forces $\boldsymbol{F}_{\mathrm{e}x}$ (e.g. gravity, electromagnetic forces)

$$\boldsymbol{F} = \boldsymbol{F}_{\Gamma} + \frac{D}{Dt}\boldsymbol{I}_{\mathrm{m}} + \boldsymbol{F}_{\mathrm{e}x}. \tag{10}$$

Thereby, the surface force computes by

$$\sum_{i=1}^{3} \boldsymbol{F}_{\Gamma_j} = -\sum_{i=1}^{3} \frac{\partial p}{\partial x_j} \Omega \boldsymbol{n}_j = -\Omega \nabla p. \tag{11}$$

and the total change of momentum $\boldsymbol{I}_{\mathrm{m}}$ by

$$\frac{D}{Dt}\boldsymbol{I}_{\mathrm{m}} = \Omega\, \nabla \cdot [\boldsymbol{\tau}] \tag{12}$$

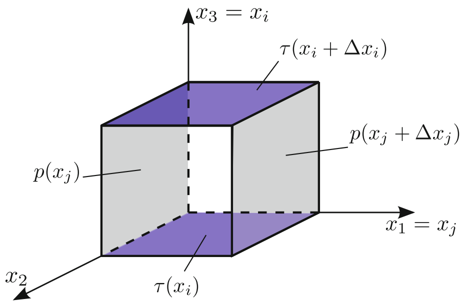


Fig. 2 Forces acting on a fluid element

with the viscous stress tensor $[\boldsymbol{\tau}]$ (see Fig. 2).

Now, we exploit the fact that $m = \rho\Omega$ and insert the pressure force (11), the viscous force (12) and any external forces per unit volume $\boldsymbol{f}$ acting on the fluid into (9). Thereby, we arrive at the momentum equation

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \nabla \cdot [\boldsymbol{\tau}] + \boldsymbol{f} \tag{13}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\rho\, \boldsymbol{v} \otimes \boldsymbol{v} + p\, [\boldsymbol{I}] - [\boldsymbol{\tau}]) = \boldsymbol{f} \tag{14}$$

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j} \left(\rho v_j v_i + p\delta_{ij} - \tau_{ij}\right) = f_i\,, \tag{15}$$

with $[\boldsymbol{I}]$ the identity tensor. Furthermore, we introduce the momentum flux tensor $[\boldsymbol{\pi}]$ defined by

$$\pi_{ij} = \rho v_i v_j + p\delta_{ij} - \tau_{ij}\,, \tag{16}$$

and the fluid stress tensor $[\boldsymbol{\sigma}_\mathrm{f}]$ by

$$[\boldsymbol{\sigma}_\mathrm{f}] = -p\, [\boldsymbol{I}] + [\boldsymbol{\tau}]\,. \tag{17}$$

To arrive at an alternative formulation for the momentum equation, also called the non-conservative form, we exploit the following identities

$$\nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = \rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \boldsymbol{v}\, \nabla \cdot (\rho \boldsymbol{v}) \tag{18}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} = \rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \frac{\partial \rho}{\partial t} \tag{19}$$

and rewrite (13) by

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\, \frac{\partial \rho}{\partial t} + \boldsymbol{v} \nabla \cdot (\rho \boldsymbol{v}) + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot [\boldsymbol{\tau}] + \boldsymbol{f}\,. \tag{20}$$

Now, we use the mass conservation and arrive at

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot [\boldsymbol{\tau}] + \boldsymbol{f} \tag{21}$$
$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i\,.$$

Conservation of energy The total balance of energy considers the inner, the kinetic and potential energies of a fluid. Since we do not consider gravity, the total change of energy over time for a fluid element with mass m is given by

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(m\left(\frac{1}{2}v^2 + e\right)\right) = m\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2 + e\right) + \left(\frac{1}{2}v^2 + e\right)\frac{\mathrm{D}m}{\mathrm{D}t} \tag{22}$$

with e the inner energy and $v^2 = \boldsymbol{v} \cdot \boldsymbol{v}$. Due to mass conservation, the second term is zero and we obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(m\left(\frac{1}{2}v^2 + e\right)\right) = \rho\Omega\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2 + e\right). \tag{23}$$

This change of energy can be caused by Durst (2006)

- heat production per unit of volume: $q_\mathrm{h}\,\Omega$
- heat conduction energy due to heat flux $\boldsymbol{q}_\mathrm{T}$: $(-\partial q_{\mathrm{T}i}/\partial x_i)\,\Omega$
- energy due to surface pressure force: $(-\partial/\partial x_i(p v_i))\,\Omega$
- energy due to surface shear force: $\left(-\partial/\partial x_i(\tau_{ij} v_j)\right)\Omega$
- mechanical energy due to the force density $\boldsymbol{f}_i$ given by: $(f_i v_i)\,\Omega$

Thereby, we arrive at the conservation of energy given by

$$\rho\,\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2 + e\right) = -\frac{\partial q_{\mathrm{T}i}}{\partial x_i} - \frac{\partial p v_i}{\partial x_i} - \frac{\partial \tau_{ij} v_j}{\partial x_i} + f_i v_i + q_\mathrm{h} \tag{24}$$

or in vector notation by

$$\rho\,\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{1}{2}v^2 + e\right) = -\nabla \cdot \boldsymbol{q}_\mathrm{T} - \nabla \cdot (p\boldsymbol{v}) - \nabla \cdot ([\boldsymbol{\tau}] \cdot \boldsymbol{v}) + \boldsymbol{f} \cdot \boldsymbol{v} + q_\mathrm{h}\,. \tag{25}$$

By further exploring thermodynamic relations (see Sect. 2.3) and the mechanical energy (obtained by inner product of momentum conservation with $\boldsymbol{v}$), we may write (25) by the specific entropy s as follows Howe (1998)

$$\rho T\,\frac{\mathrm{D}s}{\mathrm{D}t} = \tau_{ij}\,\frac{\partial v_i}{\partial x_j} - \frac{\partial q_{\mathrm{T}i}}{\partial x_i} + q_\mathrm{h}\,. \tag{26}$$

When heat transfer is neglected, the flow ist *adiabatic*. It is *isentropic*, when it is adiabatic and reversible, which means that the viscous dissipation can be neglected, which leads to (no heat production)

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = 0\,. \tag{27}$$

Finally, when the fluid is homogeneous and the entropy uniform ($\mathrm{d}s = 0$), we call the flow *homentropic*.

2.3 Constitutive Equations

The conservation of mass, momentum and energy involve much more unknowns than equations. To close the system, additional information is provided by empirical information in form of constitutive equations. A good approximation is obtained by assuming the fluid to be in thermodynamic equilibrium. This implies for a homogeneous fluid that two *intrinsic* state variables fully determine the state of the fluid.

When we apply specific heat production q_h to a fluid element, then the specific inner energy e increases and at the same time the volume changes by $p\,\mathrm{d}\rho^{-1}$. This thermodynamic relation is expressed by

$$\mathrm{d}e = \mathrm{d}q_\mathrm{h} - p\,\mathrm{d}\rho^{-1}\,, \tag{28}$$

where the second term describes the work done on the fluid element by the pressure. If the change occurs sufficiently slowly, the fluid element is always in thermodynamic equilibrium, and we can express the heat input by the specific entropy s

$$\mathrm{d}q_\mathrm{h} = T\,\mathrm{d}s\,. \tag{29}$$

Therefore, we may rewrite (28) and arrive at the fundamental law of thermodynamics

$$\begin{aligned} \mathrm{d}e &= T\,\mathrm{d}s - p\,\mathrm{d}\rho^{-1} \\ &= T\,\mathrm{d}s + \frac{p}{\rho^2}\,\mathrm{d}\rho\,. \end{aligned} \tag{30}$$

Towards acoustics, it is convenient to choose the mass density ρ and the specific entropy s as intrinsic state variables. Hence, the specific inner energy e is completely defined by a relation denoted as the thermal equation of state

$$e = e(\rho,\ s)\,. \tag{31}$$

Therefore, variations of e are given by

$$de = \left(\frac{\partial e}{\partial \rho}\right)_s \mathrm{d}\rho + \left(\frac{\partial e}{\partial s}\right)_\rho \mathrm{d}s\,. \tag{32}$$

A comparison with the fundamental law of thermodynamics (30) provides the thermodynamic equations for the temperature T and pressure p

$$T = \left(\frac{\partial e}{\partial s}\right)_\rho \; ; \quad p = \rho^2 \left(\frac{\partial e}{\partial \rho}\right)_s \, . \tag{33}$$

Since p is a function of ρ and s, we may write

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_s \mathrm{d}\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \mathrm{d}s \, . \tag{34}$$

As sound is defined as isentropic ($ds = 0$) pressure-density perturbations, the isentropic speed of sound is defined by

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \, . \tag{35}$$

Since in many applications the fluid considered is air at ambient pressure and temperature, we may use the ideal gas law

$$p = \rho R T \tag{36}$$

with the specific gas constant R, which computes for an ideal gas as

$$R = c_p - c_\Omega \, . \tag{37}$$

In (37) c_p, c_Ω denote the specific heat at constant pressure and constant volume, respectively. Furthermore, the inner energy e depends for an ideal gas just on the temperature T via

$$\mathrm{d}e = c_\Omega \, \mathrm{d}T \, . \tag{38}$$

Substituting this relations in (30), assuming an isentropic state ($\mathrm{d}s = 0$) and using (36) results in

$$c_\Omega \, \mathrm{d}T = \frac{p}{\rho^2} \, \mathrm{d}\rho \; \rightarrow \; \frac{\mathrm{d}T}{T} = \frac{R}{c_\Omega} \frac{\mathrm{d}\rho}{\rho} \, . \tag{39}$$

Using (36), the total change $\mathrm{d}p$ normalized to p computes as

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}T}{T} \, . \tag{40}$$

This relation and applying (39), (37) leads to

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}\rho}{\rho} + \frac{R}{c_\Omega} \frac{\mathrm{d}\rho}{\rho} = \frac{c_p}{c_\Omega} \frac{\mathrm{d}\rho}{\rho} = \kappa \frac{\mathrm{d}\rho}{\rho} \tag{41}$$

with κ the specific heat ratio (also known as adiabatic exponent). A comparison of (41) with (35) yields

$$c = \sqrt{\kappa p/\rho} = \sqrt{\kappa R T}\,. \tag{42}$$

We see that the speed of sound c of an ideal gas depends only on the temperature. For air κ has a value of 1.402 so that we obtain a speed of sound c at $T = 15\,^\circ\mathrm{C}$ of 341 m/s. For most practical applications, we can set the speed of sound to 340 m/s within a temperature range of 5–25 °C. Combining (41) and (42), we obtain the general pressure-density relation for an isotropic state

$$\frac{\mathrm{d}p}{\mathrm{d}t} = c^2 \frac{\mathrm{d}\rho}{\mathrm{d}t}\,. \tag{43}$$

Furthermore, since we use an Eulerian frame of reference, we may rewrite (43) by

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt}\,. \tag{44}$$

For liquids, such as water, the pressure-density relation is written by the adiabatic bulk modulus K_s (or its reciprocal $1/K_\mathrm{s}$, known as the adiabatic compressibility) and (43) reads as

$$\frac{Dp}{Dt} = \frac{K_\mathrm{s}}{\rho} \frac{D\rho}{Dt}\,. \tag{45}$$

2.4 *Characterization of Flows by Dimensionless Numbers*

Two flows around geometric similar models are physically similar if all characteristic numbers coincide (Schlichting and Gersten 2006). Especially for measurement setups, these similarity considerations are important as it allows measuring of down sized geometries. Furthermore, the characteristic numbers are used to classify a flow situation. The Reynolds number is defined by

$$\mathrm{Re} = \frac{vl}{\nu} \tag{46}$$

with the characteristic flow velocity v, flow length l and kinematic viscosity ν. It provides the ratio between stationary inertia forces and viscous forces. Thereby, it allows to subdivide flows into laminar and turbulent ones. The Mach number allows for an approximative subdivision of a flow in compressible (Ma > 0.3) and incompressible (Ma ≤ 0.3), and is defined by

$$\mathrm{Ma} = \frac{v}{c} \tag{47}$$

with c the speed of sound. In unsteady problems, periodic oscillating flow structures may occur, e.g. the Kármán vortex street in the wake of a cylinder. The dimensionless frequency of such an oscillation is denoted as the Strouhal number, and is defined by

$$\mathrm{St} = f\,\frac{l}{v} \tag{48}$$

with f the shedding frequency.

2.5 *Towards Acoustics*

According to the Helmholtz decomposition, the velocity vector $\boldsymbol{v}$ (as any vector field) can be split into an irrotational part and a solenoidal part

$$\boldsymbol{v} = \nabla\phi + \nabla \times \boldsymbol{\Psi}\,, \tag{49}$$

where ϕ is a scalar potential and $\boldsymbol{\Psi}$ a vector potential. Thereby, we call a flow being purely described by a scalar potential via

$$\boldsymbol{v} = \nabla\phi$$

a *potential flow*. Using (49), mass conservation (see (7)) may be written as

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) &= \frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{v} \\ &= \frac{D\rho}{Dt} + \rho\nabla\cdot\nabla\phi + \rho\underbrace{\nabla\cdot\nabla\times\boldsymbol{\Psi}}_{=0} \\ \frac{1}{\rho}\,\frac{D\rho}{Dt} &= -\nabla\cdot\nabla\phi\,. \end{aligned} \tag{50}$$

This result obviously leads us to the interpretation that the flow related to the acoustic field is an irrotational flow and that the acoustic field is the unsteady component of the gradient of the velocity potential ϕ. On the other hand, taking the curl of (49) results in the vorticity of the flow

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{v} = \nabla\times\nabla\times\boldsymbol{\Psi} + \underbrace{\nabla\times\nabla\phi}_{=0} = \nabla\times\nabla\times\boldsymbol{\Psi}\,. \tag{51}$$

We see that this quantity is fully defined by the vector potential and characterizes the solenoidal part of the flow field.

3 Basic Equations of Acoustics

3.1 Acoustic Wave Equation

We assume an isentropic case, where the total variation of the entropy is zero and the pressure is only a function of the density (see (44)). Furthermore, we restrict ourself to a perfect (non-viscous) fluid (setting the viscous fluid tensor $[\boldsymbol{\tau}]$ to zero) and neglect external force density $\boldsymbol{f}$. Thereby, we arrive, according to Sect. 2, to the following set of equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{52}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p = 0 \tag{53}$$

$$\frac{Dp}{Dt} = c^2 \frac{D\rho}{Dt} \,. \tag{54}$$

In a first step, we consider the static case with mean pressure p_0, mean density ρ_0 and velocity $\boldsymbol{v}_0$ being zero. Therefore, (52) is fulfilled identically, while (53) results in

$$\nabla p_0 = 0 \,. \tag{55}$$

Furthermore, (54) is automatically satisfied by some function c_0 (independent of t) defined by means of some virtual non-static variations of the solution. In a next step, we consider a non-static solution of very small order according to a perturbation of the mean quantities

$$p = p_0 + p_{\rm a}\,; \quad \rho = \rho_0 + \rho_{\rm a}\,; \quad \boldsymbol{v} = \boldsymbol{v}_{\rm a} \tag{56}$$

with the following relations

$$p_{\rm a} \ll p_0\,; \quad \rho_{\rm a} \ll \rho_0 \,. \tag{57}$$

We name $p_{\rm a}$ the acoustic pressure, $\rho_{\rm a}$ the acoustic density and $\boldsymbol{v}_{\rm a}$ the acoustic particle velocity. Using the perturbation ansatz (56) and substituting it into (52)–(54), results in

$$\frac{\partial(\rho_0 + \rho_{\rm a})}{\partial t} + \nabla \cdot \Big((\rho_0 + \rho_a) \boldsymbol{v}_{\rm a} \Big) = 0 \tag{58}$$

$$(\rho_0 + \rho_{\rm a}) \frac{\partial \boldsymbol{v}_{\rm a}}{\partial t} + \Big((\rho_0 + \rho_{\rm a}) \boldsymbol{v}_{\rm a} \Big) \cdot \nabla \boldsymbol{v}_{\rm a} + \nabla \big(p_0 + p_{\rm a} \big) = 0 \tag{59}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\rm a} \cdot \nabla \right) (p_0 + p_{\rm a}) - c_0^2 \left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\rm a} \cdot \nabla \right) (\rho_0 + \rho_{\rm a}) = 0 \,. \tag{60}$$

In a next step, we are allowed to cancel second order terms (e.g., such as $\rho_a \boldsymbol{v}_a$), consider that p_0 does not vary over space (see (55)) and arrive at

$$\frac{\partial \rho_a}{\partial t} + \nabla(\rho_0 \boldsymbol{v}_a) = q_{ma} \tag{61}$$

$$\rho_0 \frac{\partial \boldsymbol{v}_a}{\partial t} + \nabla p_a = \boldsymbol{q}_{mo} \tag{62}$$

$$\frac{\partial p_a}{\partial t} = c_0^2 \left(\frac{\partial \rho_a}{\partial t} + \boldsymbol{v}_a \cdot \nabla \rho_0 \right) . \tag{63}$$

Here, we have included possible modeled source terms in (61) (linearized conservation of mass) and (62) (linearized conservation of momentum). Please note that just in the case of constant mean density, i.e. $\nabla \rho_0 = 0$, we are allowed to express the acoustic pressure-density relation by

$$p_a = c_0^2 \rho_a \, , \tag{64}$$

Now, we use (61), substitute it into (63) and obtain the final two equations for linear acoustics

$$\frac{1}{\rho_0 c_0^2} \frac{\partial p_a}{\partial t} + \nabla \cdot \boldsymbol{v}_a = \frac{1}{\rho_0} q_{ma} \tag{65}$$

$$\frac{\partial \boldsymbol{v}_a}{\partial t} + \frac{1}{\rho_0} \nabla p_a = \frac{1}{\rho_0} \boldsymbol{q}_{mo} \, . \tag{66}$$

Applying $\partial/\partial t$ to (65), $\nabla\cdot$ to (66) and subtracting the resulting equations provides the linear wave equation for an inhomogeneous medium (density depending on space)

$$\frac{1}{\rho_0 c_0^2} \frac{\partial^2 p_a}{\partial t^2} - \nabla \cdot \frac{1}{\rho_0} \nabla p_a = \frac{1}{\rho_0} \frac{\partial q_{ma}}{\partial t} - \nabla \cdot \frac{\boldsymbol{q}_{mo}}{\rho_0} \, . \tag{67}$$

Furthermore, since the term $\rho_0 c_0^2$ is constant in space and time, we may rewrite (67) by

$$\frac{\partial^2 p_a}{\partial t^2} - \nabla \cdot c_0^2 \nabla p_a = c_0^2 \frac{\partial q_{ma}}{\partial t} - \nabla \cdot \left(c_0^2 \boldsymbol{q}_{mo} \right) \, . \tag{68}$$

This form of wave equation is mainly used when considering the influence of temperature gradient (speed of sound c_0 depends on temperature, see (42)) on wave propagation. For liquids, (67) may be written as

$$\frac{1}{K_s} \frac{\partial^2 p_a}{\partial t^2} - \nabla \cdot \frac{1}{\rho_0} \nabla p_a = \frac{1}{\rho_0} \frac{\partial q_{ma}}{\partial t} - \nabla \cdot \frac{\boldsymbol{q}_{mo}}{\rho_0} \, . \tag{69}$$

By applying the chain rule

$$\nabla \cdot \frac{1}{\rho_0} \nabla p_\mathrm{a} = \frac{1}{\rho_0} \nabla \cdot \nabla p_\mathrm{a} - \frac{1}{\rho_0^2} \nabla \rho_0 \cdot \nabla p_\mathrm{a} \,,$$

we arrive at

$$\frac{1}{K_\mathrm{s}} \frac{\partial^2 p_\mathrm{a}}{\partial t^2} - \frac{1}{\rho_0} \nabla \cdot \nabla p_\mathrm{a} + \frac{1}{\rho_0^2} \nabla \rho_0 \cdot \nabla p_\mathrm{a} = \frac{1}{\rho_0} \frac{\partial q_\mathrm{ma}}{\partial t} - \nabla \cdot \frac{\boldsymbol{q}_{\mathrm{m}o}}{\rho_0} \,. \tag{70}$$

This form of the wave equation explicitly shows the influence of a space dependent density ρ_0.

A wave equation for the particle velocity $\boldsymbol{v}_\mathrm{a}$ may be derived by rewriting (65), (66) as

$$\frac{\partial p_\mathrm{a}}{\partial t} + \rho_0 c_0^2 \nabla \cdot \boldsymbol{v}_\mathrm{a} = c_0^2\, q_\mathrm{ma} \tag{71}$$

$$\rho_0 \frac{\partial \boldsymbol{v}_\mathrm{a}}{\partial t} + \nabla p_\mathrm{a} = \boldsymbol{q}_\mathrm{mo} \,. \tag{72}$$

Now, we apply ∇ to (71), $\partial/\partial t$ to (72) and by subtract the resulting equations we arrive at

$$\rho_0 \frac{\partial^2 \boldsymbol{v}_\mathrm{a}}{\partial t^2} - \nabla \rho_0 c_0^2 \nabla \cdot \boldsymbol{v}_\mathrm{a} = \frac{\partial \boldsymbol{q}_\mathrm{mo}}{\partial t} - \nabla c_0^2 q_\mathrm{ma} \,. \tag{73}$$

It is a vectorial wave equation coupling the three components of the particle velocity. Since the particle velocity $\boldsymbol{v}_\mathrm{a}$ is irrotational, we may express it by the scalar acoustic potential ψ_a via

$$\boldsymbol{v}_\mathrm{a} = -\nabla \psi_\mathrm{a} \,. \tag{74}$$

Substituting this relation into (73), assuming zero source terms and constant density condition ($\nabla \rho_0 = 0$) results in

$$\nabla \left(\frac{\partial^2 \psi_\mathrm{a}}{\partial t^2} - c_0^2 \nabla \cdot \nabla \psi_\mathrm{a} \right) = 0 \,. \tag{75}$$

This equation is clearly satisfied, when ψ_a fulfills

$$\frac{1}{c_0^2} \frac{\partial^2 \psi_\mathrm{a}}{\partial t^2} - \nabla \cdot \nabla \psi_\mathrm{a} = 0 \,. \tag{76}$$

Finally, we provide the most used wave equation in terms of the acoustic pressure p_a, which however does not model an inhomogeneous fluid. It is obtained from (67) assuming a space constant speed of sound c_0 and mean density ρ_0

$$\frac{1}{c_0^2} \frac{\partial^2 p_\mathrm{a}}{\partial t^2} - \nabla \cdot \nabla p_\mathrm{a} = \frac{\partial q_\mathrm{ma}}{\partial t} - \nabla \cdot \boldsymbol{q}_\mathrm{mo} \,. \tag{77}$$

By performing a Fourier transform, we arrive at Helmholtz equation

$$\nabla \cdot \nabla \hat{p}_\mathrm{a} + k^2 \hat{p}_\mathrm{a} = -j\omega \hat{q}_\mathrm{ma} + \nabla \cdot \hat{\boldsymbol{q}}_\mathrm{mo} \tag{78}$$

with the Fourier-transformed acoustic pressure $\hat{p}_\mathrm{a}$ and source terms $\hat{q}_\mathrm{ma}$, $\hat{\boldsymbol{q}}_\mathrm{mo}$ as well as angular frequency ω, wave number k and imaginary unit j (see (86)).

3.2 *Simple Solutions*

In order to get some physical insight in the propagation of acoustic sound, we will consider two special cases: plane and spherical waves. Let's start with the simpler case, the propagation of a plane wave as displayed in Fig. 3. Thus, we can express the acoustic pressure by $p_\mathrm{a} = p_\mathrm{a}(x,t)$ and the particle velocity by $\boldsymbol{v}_\mathrm{a} = v_\mathrm{a}(x,t)\boldsymbol{e}_x$. Using these relations together with the linear pressure-density law (assuming constant mean density, see (64)), we arrive at the following 1D linear wave equation

$$\frac{\partial^2 p_\mathrm{a}}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p_\mathrm{a}}{\partial t^2} = 0\,, \tag{79}$$

which can be rewritten in factorized version as

$$\left(\frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c_0}\frac{\partial}{\partial t}\right) p_\mathrm{a} = 0\,. \tag{80}$$

This version of the linearized, 1D wave equation motivates us to introduce the following two functions (solution according to d'Alembert)

$$\xi = t - x/c_0\,; \quad \eta = t + x/c_0$$

with properties

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\,; \qquad \frac{\partial}{\partial x} = \frac{1}{c_0}\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right).$$

Therewith, we obtain for the factorized operator

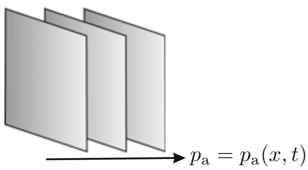


Fig. 3 Propagation of a plane wave

$$\frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial t} = -\frac{2}{c_0}\frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial x} + \frac{1}{c_0}\frac{\partial}{\partial t} = \frac{2}{c_0}\frac{\partial}{\partial \eta}$$

and the linear, 1D wave equation transfers to

$$-\frac{4}{c_0^2}\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} p_\mathrm{a} = 0\,.$$

The general solution computes as a superposition of arbitrary functions of ξ and η

$$p_\mathrm{a} = f(\xi) + f(\eta) = f(t - x/c_0) + g(t + x/c_0)\,. \tag{81}$$

This solution describes waves moving with the speed of sound c_0 in $+x$ and $-x$ direction, respectively. In a next step, we use the linearized conservation of momentum according to (62), and rewrite it for the 1D case (assuming zero source term)

$$\rho_0 \frac{\partial v_\mathrm{a}}{\partial t} + \frac{\partial p_\mathrm{a}}{\partial x} = 0\,. \tag{82}$$

Now, we just consider a forward propagating wave, i.e. $g(t) = 0$, substitute (81) into (82) and obtain

$$\begin{aligned} v_\mathrm{a} &= -\frac{1}{\rho_0}\int \frac{\partial p_\mathrm{a}}{\partial x}\,\mathrm{d}t = \frac{1}{\rho_0 c_0}\int \frac{\partial f(t - x/c_0)}{\partial t}\,\mathrm{d}t \\ &= \frac{1}{\rho_0 c_0} f(t - x/c_0) = \frac{p_\mathrm{a}}{\rho_0 c_0}\,. \end{aligned} \tag{83}$$

Therewith, the value of the acoustic pressure over acoustic particle velocity for a plane wave is constant. To allow for a general orientation of the coordinate system, a free field plane wave may be expressed by

$$p_\mathrm{a} = f(\boldsymbol{n}\cdot\boldsymbol{x} - c_0 t)\,; \quad \boldsymbol{v}_\mathrm{a} = \frac{\boldsymbol{n}}{\rho_0 c_0} f(\boldsymbol{n}\cdot\boldsymbol{x} - c_0 t)\,, \tag{84}$$

where the direction of propagation is given by the unit vector $\boldsymbol{n}$. A time harmonic plane wave of angular frequency $\omega = 2\pi f$ is usually written as

$$p_\mathrm{a}\,,\ \boldsymbol{v}_\mathrm{a} \sim e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} \tag{85}$$

with the wave number (also called wave vector) $\boldsymbol{k}$, which computes by

$$\boldsymbol{k} = k\boldsymbol{n} = \frac{\omega}{c_0}\boldsymbol{n}\,. \tag{86}$$

The second case of investigation will be a spherical wave, where we assume a point source located at the origin. In the first step, we rewrite the linearized wave equation in spherical coordinates and consider that the pressure p_a will just depend on the radius r. Therewith, the Laplace-operator reads as

$$\nabla \cdot \nabla p_\mathrm{a}(r,t) = \frac{\partial^2 p_\mathrm{a}}{\partial r^2} + \frac{2}{r}\frac{\partial p_\mathrm{a}}{\partial r} = \frac{1}{r}\frac{\partial^2 r p_\mathrm{a}}{\partial r^2}$$

and we obtain

$$\frac{1}{r}\frac{\partial^2 r p_\mathrm{a}}{\partial r^2} - \frac{1}{c_0^2}\underbrace{\frac{\partial^2 p_\mathrm{a}}{\partial t^2}}_{\frac{1}{r}\frac{\partial^2 r p_\mathrm{a}}{\partial t^2}} = 0\,. \tag{87}$$

A multiplication of (87) with r results in the same wave equation as obtained for the plane case (see (79)), just instead of p_a we have $r p_\mathrm{a}$. Therefore, the solution of (87) reads as

$$p_\mathrm{a}(r,t) = \frac{1}{r}\,\left(f(t - r/c_0) + g(t + r/c_0)\right)\,, \tag{88}$$

which means that the pressure amplitude will decrease according to the distance r from the source.

3.3 Acoustic Quantities and Order of Magnitudes

Let us consider a loudspeaker generating sound at a fixed frequency f and a number of microphones recording the sound as displayed in Fig. 4. In a first step, we measure the sound with one microphone fixed at $\boldsymbol{x}_0$, and we will obtain a periodic signal in time with the same frequency f and period time $T = 1/f$. In a second step, we use all microphones and record the pressure at a fixed time t_0. Drawing the obtained values along the individual positions of the microphone, e.g. along the coordinate x, we again obtain a periodic signal, which is now periodic in space. This periodicity is characterized by the wavelength λ and is uniquely defined by the frequency f and the speed of sound c_0 via the relation

$$\lambda = \frac{c_0}{f}\,. \tag{89}$$

Assuming a frequency of 1 kHz, the wavelength in air takes on the value of 0.343 m ($c_0 = 343$ m/s).

Strictly speaking, each acoustic wave has to be considered as transient, having a beginning and an end. However, for some long duration sound, we speak of continuous wave (cw) propagation and we define for the acoustic pressure p_a a mean square pressure $(p_\mathrm{a})^2_\mathrm{av}$ as well as a root mean squared (rms) pressure $p_\mathrm{a,rms}$

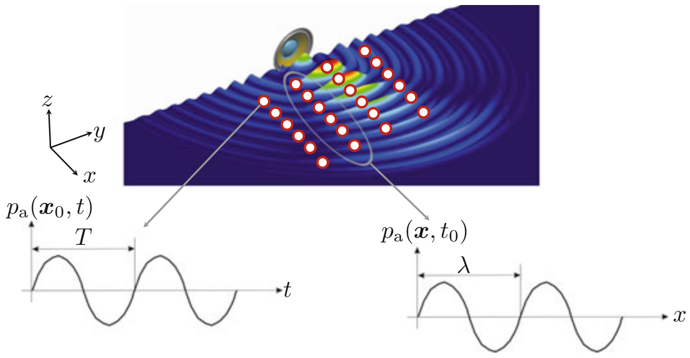


Fig. 4 Sound generated by a loudspeaker and measured by microphones

$$p_{a,rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} (p - p_0)^2 \, dt} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} p_a^2 \, dt} \,. \tag{90}$$

In (90) T denotes the period time of the signal or if we cannot strictly speak of a periodic signal, an interminable long time interval. Now, it has to be mentioned that the threshold of hearing of an average human is at about 20 μPa and the threshold of pain at about 20 Pa, which differs 10^6 orders of magnitude. Thus, logarithmic scales are mainly used for acoustic quantities. The most common one is the *decibel* (dB), which expresses the quantity as a ratio relative to a reference value. Thereby, the sound pressure level L_{p_a} (SPL) is defined by

$$L_{p_a} = 20 \log_{10} \frac{p_{a,\,rms}}{p_{a,ref}} \qquad p_{a,\,ref} = 20 \,\mu Pa \,. \tag{91}$$

The reference pressure $p_{a,ref}$ corresponds to the sound at 1 kHz that an average person can just hear.

In addition, the acoustic intensity $\boldsymbol{I}_a$ is defined by the product of the acoustic pressure and particle velocity

$$\boldsymbol{I}_a = p_a \boldsymbol{v}_a \,. \tag{92}$$

The intensity level L_{I_a} is then defined by

$$L_{I_a} = 10 \log_{10} \frac{I_a^{av}}{I_{a,ref}} \qquad I_{a,ref} = 10^{-12} \, W/m^2 \,, \tag{93}$$

with $I_{a,ref}$ the reference sound intensity corresponding to $p_{a,ref}$. Again, we use an averaged value for defining the intensity level, which computes by

$$I_\mathrm{a}^\mathrm{av} = \left|\boldsymbol{I}_\mathrm{a}^\mathrm{av}\right| = \left| \frac{1}{T} \int_{t_0}^{t_0+T} \boldsymbol{v}_\mathrm{a} p_\mathrm{a} \, \mathrm{d}t \, . \right| \tag{94}$$

Finally, we compute the acoustic power by integrating the acoustic intensity (unit $\mathrm{W/m^2}$) over a closed surface

$$P_\mathrm{a} = \oint_\Gamma \boldsymbol{I}_\mathrm{a} \cdot \mathrm{d}\boldsymbol{s} = \oint_\Gamma \boldsymbol{I}_\mathrm{a} \cdot \boldsymbol{n} \, \mathrm{d}s \, . \tag{95}$$

Then, the sound-power level $L_{\mathrm{P_a}}$ computes as

$$L_{\mathrm{P_a}} = 10 \, \log_{10} \frac{P_\mathrm{a}^\mathrm{av}}{P_\mathrm{a,ref}} \qquad P_\mathrm{a,ref} = 10^{-12}\,\mathrm{W}\,, \tag{96}$$

with $P_\mathrm{a,ref}$ the reference sound power corresponding to $p_\mathrm{a,ref}$. In Tables 1 and 2 some typical sound pressure and sound power levels are listed.

A useful quantity in acoustics is impedance, which is a measure of the amount by which the motion induced by a pressure applied to a surface is impeded. However, a quantity that varies with time and depends on initial values is not of interest. Thus the impedance is defined via the Fourier transform by

$$\hat{Z}_\mathrm{a}(\boldsymbol{x}, \omega) = \frac{\hat{p}_\mathrm{a}(\boldsymbol{x}, \omega)}{\hat{\boldsymbol{v}}_\mathrm{a}(\boldsymbol{x}, \omega) \cdot \boldsymbol{n}(\boldsymbol{x})} \tag{97}$$

at a point $\boldsymbol{x}$ on the surface Γ with unit normal vector $\boldsymbol{n}$. It is in general a complex number and its real part is called *resistance*, its imaginary part *reactance* and its inverse the *admittance* denoted by $\hat{Y}_\mathrm{a}(\boldsymbol{x}, \omega)$. For a plane wave (see Sect. 3.2) the acoustic impedance $\hat{Z}_\mathrm{a}$ is constant

$$\hat{Z}_\mathrm{a}(\boldsymbol{x}, \omega) = \rho_0 c_0 \, . \tag{98}$$

Table 1 Typical sound pressure levels SPL

Threshold of hearing	Voice at 5 m	Car at 20 m	Pneumatic hammer at 2 m	Jet at 3 m
0 dB	60 dB	80 dB	100 dB	140 dB

Table 2 Typical sound power levels and in parentheses the absolute acoustic power P_a

Voice	Fan	Loudspeaker	Jet airliner
30 dB (25 μW)	110 dB (0.05 W)	128 dB (60 W)	170 dB (50 kW)

Often the acoustic impedance $\hat{Z}_\mathrm{a}$ is normalized to this value and then named *s*pecific impedance (is a dimensionless value).

For a quiescent fluid the acoustic power across a surface Γ computes for time harmonic fields by

$$\begin{aligned} P_\mathrm{a}^{\mathrm{av}} &= \int_\Gamma \left(\frac{1}{T} \int_0^T \mathrm{Re}\left(\hat{p}_\mathrm{a} e^{j\omega t}\right) \mathrm{Re}\left(\hat{\boldsymbol{v}}_\mathrm{a} \cdot \boldsymbol{n} e^{j\omega t}\right) \, \mathrm{d}t \right) \mathrm{d}s \\ &= \frac{1}{4} \int_\Gamma \left(\hat{p}_\mathrm{a} \hat{\boldsymbol{v}}_\mathrm{a}^* + \hat{p}_\mathrm{a}^* \hat{\boldsymbol{v}}_\mathrm{a}\right) \cdot \boldsymbol{n} \, \mathrm{d}s \\ &= \frac{1}{2} \int_\Gamma \mathrm{Re}\left(\hat{p}_\mathrm{a}^* \hat{\boldsymbol{v}}_\mathrm{a}\right) \cdot \boldsymbol{n} \, \mathrm{d}s \end{aligned} \tag{99}$$

with * denoting the conjugate complex. Now, we use the impedance $\hat{Z}$ of the surface and arrive at

$$P_\mathrm{a}^{\mathrm{av}} = \frac{1}{2} \int_\Gamma \mathrm{Re}\left(\hat{Z}_\mathrm{a}\right) |\hat{\boldsymbol{v}}_\mathrm{a} \cdot \boldsymbol{n}|^2 \, \mathrm{d}s \,. \tag{100}$$

Hence, the real part of the impedance (equal to the resistance) is related to the energy flow. If $\mathrm{Re}\left(\hat{Z}_\mathrm{a}\right) > 0$ the surface is *passive* and absorbs energy, and if $\mathrm{Re}\left(\hat{Z}_\mathrm{a}\right) < 0$ the surface is *active* and produces energy.

In a next step, we analyze what happens, when an acoustic wave propagates from one fluid medium to another one. For simplicity, we restrict to a plane wave, which is described by (see (81))

$$p_\mathrm{a}(t) = f/t - x/c_0) + g(t + x/c_0) \tag{101}$$

In the frequency domain, we may write

$$\hat{p}_\mathrm{a} = \hat{f} e^{-j\omega x/c_0} + \hat{g} e^{j\omega x/c_0} = p^+ e^{j\omega t - jkx} + p^- e^{j\omega t + jkx} \,. \tag{102}$$

Thereby, p^+ is the amplitude of the wave incident at $x = 0$ from $x < 0$ and p^- the amplitude of the reflected wave at $x = 0$ by an impedance $\hat{Z}_\mathrm{a}$. Using the linear conservation of momentum, we obtain the particle velocity

$$\hat{v}_\mathrm{a}(x) = \frac{1}{\rho_0 c_0} \left(p^+ e^{-jkx} - p^- e^{jkx}\right) \,. \tag{103}$$

Defining the reflection coefficient R by

$$R = \frac{p^-}{p^+} \,, \tag{104}$$

we arrive with $\hat{Z}_a = \hat{p}(0)/\hat{v}(0)$ at

$$R = \frac{\hat{Z}_a - \rho_0 c_0}{\hat{Z}_a + \rho_0 c_0}\,. \tag{105}$$

In two dimensions, we consider a plane wave with direction $(\cos\theta, \sin\theta)$, where θ is the angle with the y-axis and the wave approaches from $y < 0$ and hits an impedance $\hat{Z}_a$ at $y = 0$. The overall pressure may be expressed by

$$\hat{p}_a(x, y) = e^{-jkx\sin\theta}\left(p^+ e^{-ky\cos\theta} + p^- e^{jky\cos\theta}\right)\,. \tag{106}$$

Furthermore, the y-component of the particle velocity computes to

$$\hat{v}_a(x, y) = \frac{\cos\theta}{\rho_0 c_0} e^{-jkx\sin\theta}\left(p^+ e^{-ky\cos\theta} - p^- e^{jky\cos\theta}\right)\,. \tag{107}$$

Thereby, the impedance is

$$\hat{Z}_a = \frac{\hat{p}(x,0)}{\hat{v}(x,0)} = \frac{\rho_0 c_0}{\cos\theta}\frac{p^+ + p^-}{p^+ - p^-} = \frac{\rho_0 c_0}{\cos\theta}\frac{1+R}{1-R} \tag{108}$$

so that the reflection coefficient computes as

$$R = \frac{\hat{Z}_a\cos\theta - \rho_0 c_0}{\hat{Z}_a\cos\theta + \rho_0 c_0}\,. \tag{109}$$

4 Boundary Conditions

For realistic simulations, a good approximation of the actual physical boundary conditions is essential. In the two simple cases - acoustically hard and soft boundary - the solution is easy:

- Acoustically hard boundary: Here, the reflection coefficient R gets 1 (total reflection), which means that the surface impedance has to approach infinity. According to (97), the term $\boldsymbol{n}\cdot\boldsymbol{v}_a$ has to be zero. Using the linearized momentum equation (62) with zero source term, we arrive at the Neumann boundary condition

$$\boldsymbol{n}\cdot\nabla p_a = \frac{\partial p_a}{\partial \boldsymbol{n}} = 0\,. \tag{110}$$

- Acoustically soft boundary: In this case, the acoustic impedance gets zero, which simply results in a homogeneous Dirichlet boundary condition

$$p_{\mathrm{a}} = 0\,. \tag{111}$$

Since real surfaces (boundaries) are never totally hard or totally soft, it seems to be a good idea to use a Robin boundary condition as a model

$$\frac{\partial p_{\mathrm{a}}}{\partial \boldsymbol{n}} + \alpha\, p_{\mathrm{a}} = 0\,. \tag{112}$$

In the time harmonic case, we can explore (62) with zero source term and apply a dot product with the normal vector $\boldsymbol{n}$

$$j\rho_0\omega \boldsymbol{n}\cdot\hat{\boldsymbol{v}}_{\mathrm{a}} + \frac{\partial \hat{p}_{\mathrm{a}}}{\partial \boldsymbol{n}} = 0 \tag{113}$$

By using (97) we obtain

$$\frac{\partial \hat{p}_{\mathrm{a}}}{\partial \boldsymbol{n}} + j\rho_0\omega\frac{\hat{p}_{\mathrm{a}}}{\hat{Z}_{\mathrm{a}}} = 0 \tag{114}$$

and identify the parameter α as

$$\alpha = \frac{j\rho_0\omega}{\hat{Z}_{\mathrm{a}}} = j\rho_0\omega\,\hat{Y}_{\mathrm{a}}\,. \tag{115}$$

As known from measurements, $\hat{Z}_{\mathrm{a}}$ is a function of frequency and therefore a inverse Fourier transform to arrive at a time domain formulation results in a convolution integral. Furthermore, $\hat{Z}_{\mathrm{a}}$ depends on the incident angle of the acoustic wave, which makes acoustic computations of rooms quite complicated. Therefore, often the computational domain is not limited by an impedance boundary condition, but the surrounded elastic body is taken into account (see Sect. 6).

One of the great challenges for wave propagation is the efficient and stable computation of waves in unbounded domains. The crucial point for these computations is that the numerical scheme avoids any reflections at the boundaries, even in case the diameter of the computational domain is just a fraction of a wavelength. Since the eighties of the last century, several numerical techniques have been developed to deal with this topic: infinite elements, Dirichlet-to-Neumann operators based on truncated Fourier expansions, absorbing boundary conditions, etc. The advantages and drawbacks of these different approaches have been widely discussed in literature, see e.g. (Ihlenburg 1998; Givoli 2008). Especially higher order absorbing boundary conditions (ABCs) have gained increasing interest, since these methods do not involve high order derivatives (Hagstrom and Warburton 2009; Bécache et al. 2010).

An alternative approach to approximate free radiation is to surround the computational domain by an additional damping layer and guarantee within the formulation, that no reflections occur at its interface with the computational domain. This so-called perfectly matched layer (PML) technique was first introduced by Berenger (1994) using a splitting of the physical variables and considering a system of first order partial differential equations (PDEs) for electromagnetics. In the framework of time-harmonic wave propagation, the PML can be interpreted as a complex-valued coordinate stretching (Teixeira and Chew 2000).

5 Aeroacoustics

The sound generated by a flow in an unbounded fluid is usually called *aerodynamic sound*. Most unsteady flows in technical applications are of high Reynolds number, and the acoustic radiation is a very small by-product of the motion. Thereby, the turbulence is usually produced by fluid motion over a solid body and/or by flow instabilities.

Since the beginning of aeroacoustics several numerical methodologies have been proposed. Each of these trying to overcome the challenges that the specific problems pose for an effective and accurate computation of the radiated sound. The main difficulties include (Hardin and Hussaini 1992a, b):

- *Energy disparity and acoustic inefficiency*: There is a large disparity between the overall energy of the flow and the part which is converted to acoustic energy (see Fig. 5). In general, the total radiated power of a turbulent jet scales with $O(v^8/c^5)$, and for a dipole source arising from pressure fluctuations on surfaces inside the flow scales with $O(v^6/c^3)$, where v denotes the characteristic flow velocity and c the speed of sound.
- *Length scale disparity*: A large disparity also occurs between the size of an eddy in the turbulent flow and the wavelength of the generated acoustic sound (see Fig. 5). Low Mach number eddies have a characteristic length scale l and velocity v. This eddy will then radiate acoustic waves of the same characteristic frequency, but with a much larger length scale, expressed by the acoustic wavelength λ

$$\lambda \propto c\frac{l}{v} = \frac{l}{M}\,.$$

- *Simulation of unbounded domains*: As a main issue for the simulation of unbounded domains using volume discretization methods remains the boundary treatment which needs to be applied to avoid the reflection of the outgoing waves on the truncating boundary of the computational domain (see Sect. 4).

Currently, available aeroacoustic methodologies overcome only some of these broad range of numerical and physical issues, which restricts their applicability, making them, in many cases, problem dependent methodologies. In a Direct Numerical Simulation (DNS), all relevant scales of turbulence are resolved and no turbulence modeling is employed. The application of DNS is becoming more feasible with the permanent advancement in computational resources. However, due to the large disparities of length and time scales between fluid and acoustic fields, DNS remains restricted to low Reynolds number flows. Therefore, although some promising work

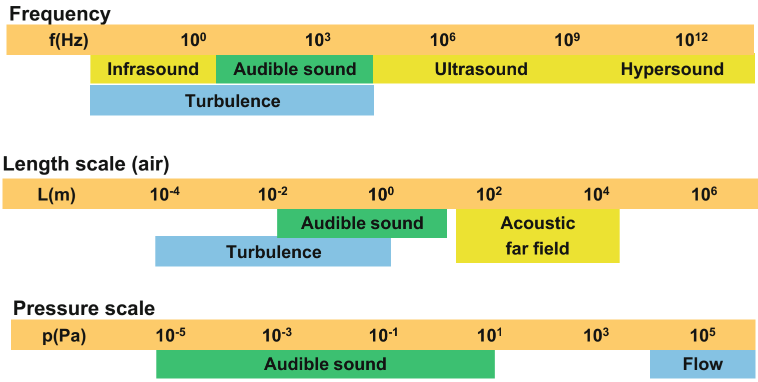


Fig. 5 Flow and acoustic scales

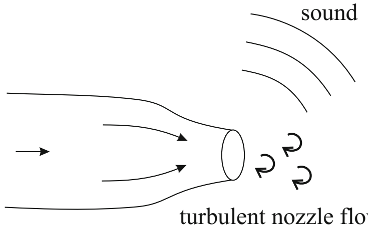


Fig. 6 Turbulent nozzle flow

has been done in this direction (Freund et al. 2000), the simulation of practical problems involving high Reynolds numbers requires very high resolutions and capabilities of supercomputers (Dumbser and Munz 2005; Frank and Munz 2016). Hence, hybrid methodologies have been established as the most practical methods for aeroacoustic computations, due to the separate treatment of the fluid and the acoustic computations. In these schemes, the computational domain is split into a nonlinear source region and a wave propagation region, and different numerical schemes are used for the flow and acoustic computations. Herewith, first a turbulence model is used to compute the unsteady flow in the source region. Secondly, from the fluid field, acoustic sources are evaluated which are then used as input for the computation of the acoustic propagation. In these coupled simulations it is generally assumed that no significant physical effects occur from the acoustic to the fluid field.

5.1 Lighthill's Acoustic Analogy

Lighthill was initially interested in solving the problem, illustrated in Fig. 6, of the sound produced by a turbulent nozzle and arrived at the inhomogeneous wave equation (Lighthill 1952; 1954). For the derivation, we start at Reynolds form of the momentum equation, as given by (15) neglecting any force density $\boldsymbol{f}$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot [\boldsymbol{\pi}] = 0\,, \tag{116}$$

with the momentum flux tensor $\pi_{ij} = \rho v_i v_j + (p - p_0)\delta_{ij} - \tau_{ij}$, where the constant pressure p_0 is inserted for convenience. In an ideal, linear acoustic medium, the momentum flux tensor contains only the pressure

$$\pi_{ij} \rightarrow \pi_{ij}^0 = (p - p_0)\delta_{ij} = c_0^2(\rho - \rho_0)\delta_{ij} \tag{117}$$

and Reynolds momentum equation reduces to

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_i}\left(c_0^2(\rho - \rho_0)\right) = 0\,. \tag{118}$$

Rewriting the conservation of mass in the form

$$\frac{\partial}{\partial t}(\rho - \rho_0) + \frac{\partial \rho v_i}{\partial x_i} = 0 \tag{119}$$

allows us to eliminate the momentum density ρv_i in (118). Therefore, we perform a time derivative on (119), a spatial derivative on (118) and subtract the two resulting equations. These operations leads to the equation of linear acoustics satisfied by the perturbation density

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla \cdot \nabla\right)\left(c_0^2(\rho - \rho_0)\right) = 0\,. \tag{120}$$

Because flow is neglected, the unique solution of this equation satisfying the radiation condition is $\rho - \rho_0 = 0$.

Now, it can be asserted that the sound generated by the turbulence in the *real fluid* is exactly equivalent to that produced in the ideal, stationary acoustic medium forced by the stress distribution

$$L_{ij} = \pi_{ij} - \pi_{ij}^0 = \rho v_i v_j + \left((p - p_0) - c_0^2(\rho - \rho_0)\right)\delta_{ij} - \tau_{ij}\,, \tag{121}$$

where $[\boldsymbol{L}]$ is called the *Lighthill stress tensor*.

Indeed, we can rewrite (116) as the momentum equation for an ideal, stationary acoustic medium of mean density ρ_0 and speed of sound c_0 subjected to the externally applied stress L_{ij}

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \pi_{ij}^0}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(\pi_{ij} - \pi_{ij}^0\right)\,, \tag{122}$$

or equivalent

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(c_0^2(\rho - \rho_0)\right) = -\frac{\partial L_{ij}}{\partial x_j}\,. \tag{123}$$

By eliminating the momentum density ρv_i using (119) we arrive at **Lighthill's equation**

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla\cdot\nabla\right)\left(c_0^2(\rho - \rho_0)\right) = \frac{\partial^2 L_{ij}}{\partial x_i \partial x_j}\,. \tag{124}$$

It has to be noted that $(\rho - \rho_0) = \rho'$ is a fluctuating density not being equal to the acoustic density $\rho_{\rm a}$, but a superposition of flow and acoustic parts within flow regions.

Neglecting viscous dissipation and assuming an isentropic case, we may approximate the Lighthill tensor by

$$L_{ij} \approx \rho_0 v_i v_j \quad \text{for Ma}^2 \ll 1\,. \tag{125}$$

Please note that with this assumptions, the divergence of (15) provides the following equivalence (assuming an incompressible flow $\nabla\cdot\boldsymbol{v} = 0$ and $\boldsymbol{f} = 0$)

$$\nabla\cdot\nabla p_{\rm ic} = -\rho_0 \frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} \tag{126}$$

with the incompressible flow pressure $p_{\rm ic}$. Therefore, we may rewrite Lighthill's inhomogeneous wave equation (124) for the fluctuating pressure p' as

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla\cdot\nabla p' = \nabla\cdot\nabla p_{\rm ic}\,. \tag{127}$$

This equation is a quite good model for the computation of sound generated by low Mach and high Reynolds number flows.

5.2 Perturbation Equations

The acoustic/viscous splitting technique for the prediction of flow induced sound was first introduced in Hardin and Pope (1994), and afterwards many groups presented alternative and improved formulations for linear and non linear wave propagation (Shen and Sørensen 1999; Ewert and Schröder 2003; Seo and Moon 2005; Munz et al. 2007). These formulations are all based on the idea, that the flow field quantities are split into compressible and incompressible parts.

For our derivation, we introduce a generic splitting of physical quantities to the conservation equations. For this purpose, we choose a combination of the two splitting approaches introduced above and define the following

$$p = \bar{p} + p_{\text{ic}} + p_{\text{c}} = \bar{p} + p_{\text{ic}} + p_{\text{a}} \tag{128}$$
$$\boldsymbol{v} = \bar{\boldsymbol{v}} + \boldsymbol{v}_{\text{ic}} + \boldsymbol{v}_{\text{c}} = \bar{\boldsymbol{v}} + \boldsymbol{v}_{\text{ic}} + \boldsymbol{v}_{\text{a}} \tag{129}$$
$$\rho = \rho_0 + \rho_1 + \rho_{\text{a}}\,. \tag{130}$$

Thereby the field variables are split into mean and fluctuating parts just like in the linearized Euler equations (LEE). In addition the fluctuating field variables are split into acoustic and non-acoustic components. Finally, the density correction ρ_1 is build in as introduced above. This choice is motivated by the following assumptions

- The acoustic field is a fluctuating field.
- The acoustic field is irrotational, i.e. $\nabla \times \boldsymbol{v}_{\text{a}} = 0$.
- The acoustic field requires compressible media and an incompressible pressure fluctuation is not equivalent to an acoustic pressure fluctuation.

By doing so, we arrive for an incompressible flow at the following perturbation equations[2]

$$\frac{\partial p_{\text{a}}}{\partial t} + \overline{\boldsymbol{v}} \cdot \nabla p_{\text{a}} + \rho_0 c_0^2 \nabla \cdot \boldsymbol{v}_{\text{a}} = -\frac{\partial p_{\text{ic}}}{\partial t} - \overline{\boldsymbol{v}} \cdot \nabla p_{\text{ic}} \tag{131}$$
$$\rho_0 \frac{\partial \boldsymbol{v}_{\text{a}}}{\partial t} + \rho_0 \nabla \left(\overline{\boldsymbol{v}} \cdot \boldsymbol{v}_{\text{a}}\right) + \nabla p_{\text{a}} = 0 \tag{132}$$

with spatial constant mean density ρ_0 and speed of sound c_0. This system of partial differential equations is equivalent to the previously published ones (Ewert and Schröder 2003). The source term is the substantial derivative of the incompressible flow pressure p_{ic}. Using the acoustic scalar potential ψ_{a} and assuming a spacial constant mean density and speed of sound, we may rewrite (132) by

$$\nabla \left(\rho_0 \frac{\partial \psi_{\text{a}}}{\partial t} + \rho_0\, \overline{\boldsymbol{v}} \cdot \nabla \psi_{\text{a}} - p_{\text{a}} \right) = 0\,, \tag{133}$$

and arrive at

$$p_{\text{a}} = \rho_0 \frac{\partial \psi_{\text{a}}}{\partial t} + \rho_0\, \overline{\boldsymbol{v}} \cdot \nabla \psi_{\text{a}}\,. \tag{134}$$

Now, we substitute (134) into (131) and arrive at

$$\frac{1}{c_0^2} \frac{D^2 \psi_{\text{a}}}{Dt^2} - \Delta \psi_{\text{a}} = -\frac{1}{\rho_0 c_0^2} \frac{D p_{\text{ic}}}{Dt}\,; \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{\boldsymbol{v}} \cdot \nabla\,. \tag{135}$$

This convective wave equation fully describes acoustic sources generated by incompressible flow structures and its wave propagation through flowing media. In addition, instead of the original unknowns p_{a} and $\boldsymbol{v}_{\text{a}}$ we have just the scalar unknown ψ_{a}. In

[2]For a detailed derivation of perturbation equations both for compressible as well as incompressible flows, we refer to Hüppe (2013).

accordance to the acoustic perturbation equations (APE), we name this resulting partial differential equation for the acoustic scalar potential as *Perturbed Convective Wave Equation* (PCWE).

Finally, it is of great interest that by neglecting the mean flow $\bar{\boldsymbol{v}}$ in (131) and (132), we arrive at the linearized conservation equations of acoustics with $\partial p_{\mathrm{ic}}/\partial t$ as a source term

$$\frac{1}{\rho_0 c_0^2}\frac{\partial p_{\mathrm{a}}}{\partial t} + \nabla\cdot\boldsymbol{v}_{\mathrm{a}} = \frac{-1}{\rho_0 c_0^2}\frac{\partial p_{\mathrm{ic}}}{\partial t} \tag{136}$$

$$\frac{\partial \boldsymbol{v}_{\mathrm{a}}}{\partial t} + \frac{1}{\rho_0}\nabla p_{\mathrm{a}} = 0\,. \tag{137}$$

As in the standard acoustic case, we apply $\partial/\partial t$ to (136) and $\nabla\cdot$ to (137) and subtract the two resulting equations to arrive at

$$\frac{1}{c_0^2}\frac{\partial^2 p_{\mathrm{a}}}{\partial t^2} - \nabla\cdot\nabla p_{\mathrm{a}} = \frac{-1}{c_0^2}\frac{\partial^2 p_{\mathrm{ic}}}{\partial t^2}\,. \tag{138}$$

We call this partial differential equation the aeroacoustic wave equation (AWE). Please note, that this equation can also be obtained by starting at Lighthill's inhomogeneous wave equation for incompressible flow, where we can substitute the second spatial derivative of Lighthill's tensor by the Laplacian of the incompressible flow pressure (see (126)) and arrive at (127). Using the decomposition of the fluctuating pressure p'

$$p' = p_{\mathrm{ic}} + p_{\mathrm{a}}\,.$$

results again into (138).

5.3 *Comparison of Different Aeroacoustic Analogies*

As a demonstrative example to compare the different acoustic analogies, we choose a cylinder in a cross flow, as displayed in Fig. 7. Thereby, the computational grid is just up to the height of the cylinder and together with the boundary conditions (bottom and top as well as span-wise direction symmetry boundary condition), we obtain a pseudo two-dimensional flow field. The diameter of the cylinder D is 1 m resulting with the inflow velocity of 1 m/s and chosen viscosity in a Reynolds number of 250 and Mach number of 0.2. From the flow simulations, we obtain a shedding frequency of 0.2 Hz (Strouhal number of 0.2). The acoustic mesh is chosen different from the flow mesh, and resolves the wavelength of two times the shedding frequency with 10 finite elements of second order. At the outer boundary of the acoustic domain we add a perfectly matched layer to efficiently absorb the outgoing waves. For the acoustic field computation we use the following formulations:

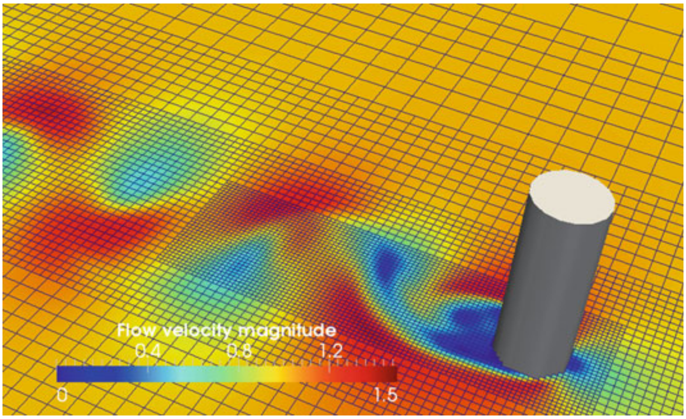


Fig. 7 Computational setup for flow computation

- Lighthill's acoustic analogy with Lighthill's tensor $[\boldsymbol{L}]$ according to (125) as source term
- Lighthill's acoustic analogy with the Laplacian of the incompressible flow pressure p_{ic} as source term (see (126))
- the aeroacoustic wave equation (AWE) according to (138)
- Perturbed Convective Wave Equation (PCWE) according to (135); for comparison, we set the mean flow velocity $\bar{\boldsymbol{v}}$ to zero.

Figure 8 displays the acoustic field for the different formulations. One can clearly see that the acoustic field of PCWE (for comparison with the other formulations we have neglected the convective terms) meets very well the expected dipole structure and is free from dynamic flow disturbances. Furthermore, the acoustic field of AWE is quite similar and exhibits almost no dynamic flow disturbances. Both computations with Lighthill's analogy show flow disturbances, whereby the formulation with the Laplacian of the incompressible flow pressure as source term shows qualitative better result as the classical formulation based on the incompressible flow velocities.

6 Vibroacoustics

In many technical applications, vibrating structures are immersed in an acoustic fluid. Therefore, acoustic waves are generated, which are acting as a surface pressure load on the vibrating structure. In general, we distinguish between the following two situations concerning mechanical-acoustic couplings:

- *Strong Coupling*: In this case, the mechanical and acoustic field equations including their couplings have to be solved simultaneously (two way coupling). A typical example is a piezoelectric ultrasound array immersed in water (see Fig. 9).

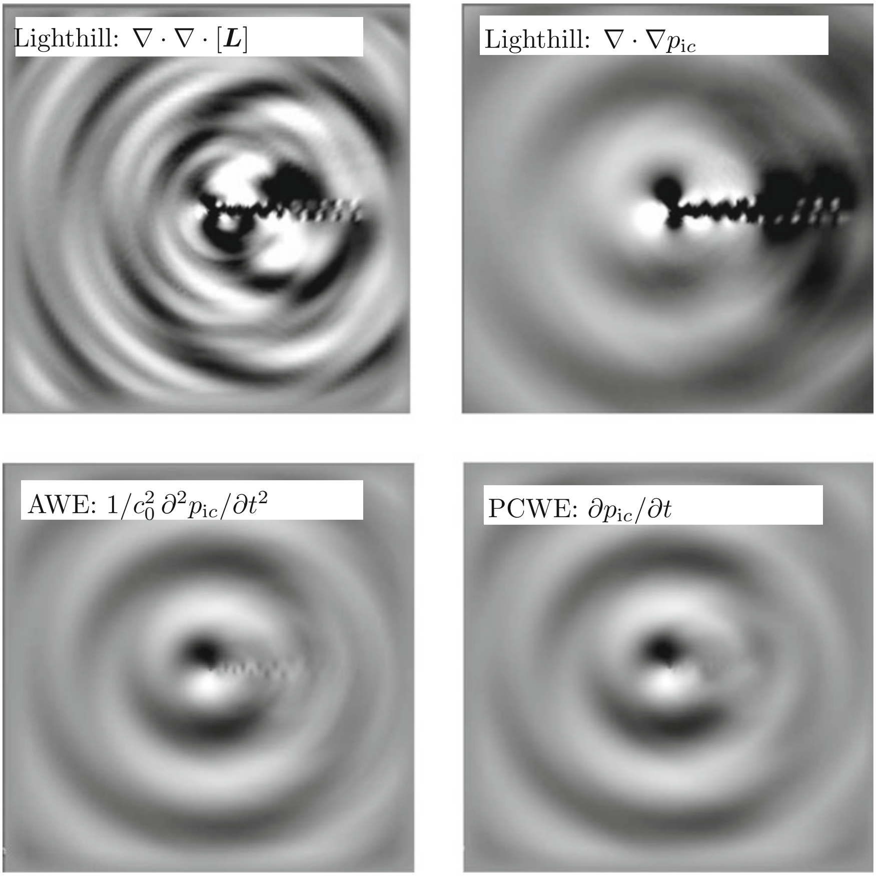


Fig. 8 Computed acoustic field with the different formulations

- *Weak Coupling*: If the pressure forces of the fluid on the solid are negligible, a sequential computation can be performed (one way coupling). For example, the acoustic sound field of an electric transformer as displayed in Fig. 10 can be obtained in this way. Thus, in a first simulation the mechanical surface vibrations are calculated, which are then used as the input for an acoustic field computation.

6.1 Interface Conditions

At a solid–fluid interface, the continuity requires that the normal component of the mechanical surface velocity of the solid must coincide with the normal component of the acoustic velocity of the inviscid fluid (see Fig. 11). Thus, the following relation

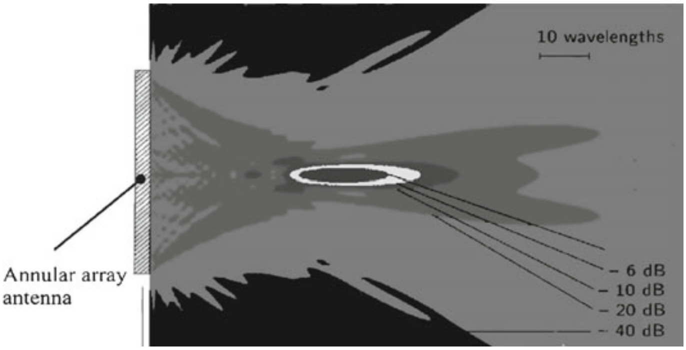


Fig. 9 Acoustic sound field of a piezoelectric ultrasound array antenna

Fig. 10 Noise radiation from the tank of an electric power transformer

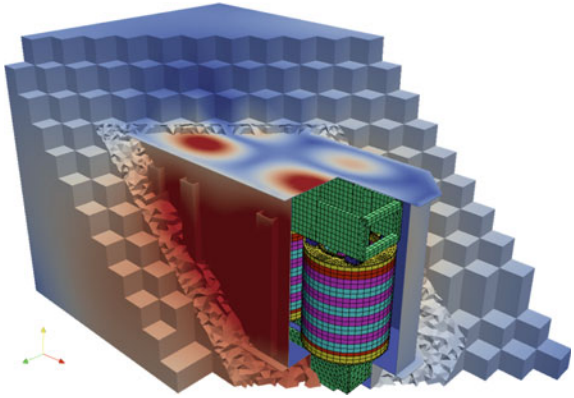
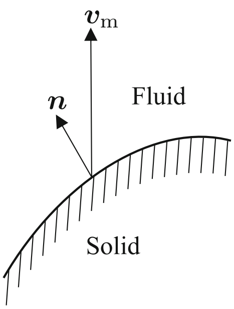


Fig. 11 Solid–fluid interface



between the velocity $\boldsymbol{v}_{\mathrm{m}}$ of the solid expressed by the mechanical displacement $\boldsymbol{u}$ and the acoustic particle velocity $\boldsymbol{v}_{\mathrm{a}}$ expressed by the acoustic scalar potential ψ_{a} arises

$$
\begin{aligned}
\boldsymbol{v}_{\mathrm{m}} = \frac{\partial \boldsymbol{u}}{\partial t} \qquad \boldsymbol{v}_{\mathrm{a}} = -\nabla \psi_{\mathrm{a}} \\
\boldsymbol{n} \cdot (\boldsymbol{v}_{\mathrm{m}} - \boldsymbol{v}_{\mathrm{a}}) = 0 \\
\boldsymbol{n} \cdot \frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{n} \cdot \nabla \psi_{\mathrm{a}} = -\frac{\partial \psi_{\mathrm{a}}}{\partial \boldsymbol{n}} \,.
\end{aligned} \tag{139}
$$

In addition, one has to consider the fact that the ambient fluid causes on the surface a mechanical stress $\boldsymbol{\sigma}_{\mathrm{n}}$

$$
\boldsymbol{\sigma}_{\mathrm{n}} = -\boldsymbol{n} p_{\mathrm{a}} = -\boldsymbol{n} \rho_0 \frac{\partial \psi_{\mathrm{a}}}{\partial t} \,, \tag{140}
$$

which acts like a pressure load on the solid.

When modeling special wave phenomena, we often arrive at a partial differential equation for the acoustic pressure. Therewith, we will also derive the coupling conditions between the mechanical displacement and acoustic pressure at a solid–fluid interface. For the first coupling condition, the continuity of the velocities, we have to establish the relation between the acoustic particle velocity $\boldsymbol{v}_{\mathrm{a}}$ and the acoustic pressure p_{a}. According to the linearized momentum equation (see (62) and assuming zero source term), we can express the normal component of $\boldsymbol{v}_{\mathrm{a}}$ by

$$
\boldsymbol{n} \cdot \frac{\partial \boldsymbol{v}_{\mathrm{a}}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p_{\mathrm{a}}}{\partial \boldsymbol{n}} \,. \tag{141}
$$

Therewith, since $\boldsymbol{n} \cdot \boldsymbol{v}_{\mathrm{m}} = \boldsymbol{n} \cdot \boldsymbol{v}_{\mathrm{a}}$ holds, we get the relation to the mechanical displacement by

$$
\boldsymbol{n} \cdot \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = -\frac{1}{\rho_0} \frac{\partial p_{\mathrm{a}}}{\partial \boldsymbol{n}} \,. \tag{142}
$$

The second coupling condition as defined in (140) is already established for an acoustic pressure formulation.

7 Appendix

Here, we provide often used operations both in vector and index notation.

- Scalar product of two vectors

$$
\boldsymbol{a} \cdot \boldsymbol{b} = c \;\rightarrow\; a_i b_i = c \tag{143}
$$

- Vector product of two vectors

$$
\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \;\rightarrow\; \epsilon_{ijk} a_j b_k = c_i \tag{144}
$$

- Gradient of a scalar

$$
\nabla \phi = \boldsymbol{u} \;\rightarrow\; \frac{\partial \phi}{\partial x_i} = u_i \tag{145}
$$

- Gradient of a vector

$$\nabla \boldsymbol{a} = \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_2}{\partial x_1} & \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_1}{\partial x_2} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_3}{\partial x_2} \\ \frac{\partial a_1}{\partial x_3} & \frac{\partial a_2}{\partial x_3} & \frac{\partial a_3}{\partial x_3} \end{pmatrix} \rightarrow \frac{\partial a_i}{\partial x_j} \tag{146}$$

- Gradient of a second order tensor

$$\nabla [\boldsymbol{A}] = \frac{\partial [\boldsymbol{A}]}{\partial \boldsymbol{x}} = \sum_{i,j,k=1}^{3} \frac{\partial A_{ij}}{\partial x_k} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{147}$$

- Divergence of a vector

$$\nabla \cdot \boldsymbol{a} = b \rightarrow \frac{\partial a_i}{\partial x_i} = b \tag{148}$$

- Divergence of a second order tensor

$$\nabla \cdot [\boldsymbol{A}] = \sum_{i,j=1}^{3} \frac{\partial A_{ij}}{\partial x_j} \boldsymbol{e}_i \tag{149}$$

- Curl of a vector

$$\nabla \times \boldsymbol{a} = \boldsymbol{b} \rightarrow \epsilon_{ijk} \frac{\partial a_k}{\partial x_j} = b_i \tag{150}$$

  with

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123,\ 231 \text{ or } 312 \\ 0 & \text{if any two indices are the same} \\ -1 & \text{if } ijk = 132,\ 213 \text{ or } 321 \end{cases}$$

- Double product or double contraction of two second order tensors

$$[\boldsymbol{A}] : [\boldsymbol{B}] = c \rightarrow A_{ij} B_{ij} = c \tag{151}$$

- Dyadic or tensor product

$$\boldsymbol{a} \otimes \boldsymbol{b} = [\boldsymbol{C}] \rightarrow a_i b_j = C_{ij} \tag{152}$$

$$[\boldsymbol{A}] \otimes \boldsymbol{b} = [\boldsymbol{C}] \rightarrow A_{ij} b_k = C_{ijk} \tag{153}$$

$$[\boldsymbol{A}] \otimes [\boldsymbol{B}] = [\boldsymbol{D}] \rightarrow A_{ij} B_{kl} = D_{ijkl} \tag{154}$$

- Trace of a tensor

$$\mathrm{tr}([\boldsymbol{A}]) = b \rightarrow A_{ii} = b\,. \tag{155}$$

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