Design of FIR LS Hilbert Transformers Through Fullband Differentiators

Guergana Mollova*

Institute of Communications and Radio-Frequency Engineering
Vienna University of Technology
Gusshausstrasse 25/389, A-1040 Vienna, Austria
E-mail: g.mollova@gmx.net

***Abstract*-This paper presents some new explicit expressions for the impulse responses of the Case 3, Case 4, and differentiating Hilbert transformers. The proposed closed-form design is based on the fullband least-squares differentiator and relations between differentiator and Hilbert transformer. The obtained simple formulas give an efficient way to determine tap-coefficients of designed Hilbert transformers even with a hand calculator. Several numerical examples and comparison with McClellan-Parks algorithm prove the efficiency of this approach.**

I. INTRODUCTION

The Hilbert transformers (HTs) have various applications, namely in speech and image processing, signal modulation, radar techniques, seismic signal processing, etc. Among the most popular approaches for design of HTs are the McClellan-Parks algorithm [1], eigenfilter method [2], and least-squares approach [3]. Several interrelations between digital one/half band filters, low/high order digital differentiators, and discrete/differentiating HTs are discussed in [4]. Other useful relationships are also given in [5-7]. New designs of discrete and differentiating HTs are presented in [8] using their relation with the Taylor series based differentiators. Some other explicit expressions for the impulse response of maximally flat FIR HTs are derived in [9-11]. Le Bihan [9], for example, proposes an efficient algorithm for calculation the coefficients of maximally flat (for midband frequencies) HTs and differentiators. As a result, new closed-form explicit and recursive formulas are derived. A different approach for maximally flat HTs using Taylor configuration is shown in [10]. Further, Khan and Okuda [11] proposed new designs of even and odd length HTs by transforming a design of differentiators with flat magnitude response. Thus, the obtained HTs have relatively narrow transition bands compared to the existing maximally flat designs. The fractional maximally flat FIR HTs are further designed in [12], together with some efficient hardware realization structures. An additional contribution to the problem is given in [13], including a closed-form design of maximally flat FIR HTs, differentiators and fractional delayers based on power series expansion. Efficient implementation structures based on the simple forms of weighting coefficients are also presented.

An application of the Hopfield-type neural network for design of HTs and digital differentiators is shown in [14]. Using the frequency-response masking technique and the relationship between a halfband filter and a transformer, Lim and Yu [15] considered a synthesis of new very sharp HTs.

The aim of this work is to derive several explicit formulas for the tap-coefficients of FIR Hilbert transformers, which will be obtained through fullband least-squares (LS) differentiator. As a starting point we use the exact expressions for the impulse response of a fullband differentiator [16] and interrelations [4]. The problem is stated in Section II. The derivation of the new relations for Case 3/Case 4 HTs and differentiating HTs is shown in Section III and Section IV of this work, respectively. Our examination will finish with some simulation results and conclusions (Section V).

II. PROBLEM FORMULATION

The ideal frequency response $H_{\mathrm{I}}^{\mathrm{HT}}$ of a Hilbert transformer is [2,3]:

$$H_{\mathrm{I}}^{\mathrm{HT}}\left(e^{j\omega}\right)=D^{\mathrm{HT}}(\omega).e^{j\pi/2}, \tag{1}$$

where $D^{\mathrm{HT}}(\omega)=\begin{cases}-1, & \omega_{\mathrm{C}1}\le\omega\le\omega_{\mathrm{C}2}\\ 1, & -\omega_{\mathrm{C}2}\le\omega\le-\omega_{\mathrm{C}1}\end{cases}$,

and $\omega_{\mathrm{C}1}$ and $\omega_{\mathrm{C}2}$ are the lower and upper edge frequencies, respectively. Either Case 3 or Case 4 antisymmetric FIR impulse response sequences can be used to approximate the ideal frequency response (1).

The frequency response of this kind of FIR filter is:

$$H\left(e^{j\omega}\right)=M(\omega).e^{j(\pi/2-\omega(N-1)/2)}, \tag{2}$$

where $M(\omega)$ is real valued, given by:

$$M(\omega)=\begin{cases}\sum\limits_{n=1}^{(N-1)/2}b(n)\sin n\omega, & N\text{ odd}\quad(\text{Case }3)\\ \sum\limits_{n=1}^{N/2}b(n)\sin(n-1/2)\omega, & N\text{ even}\quad(\text{Case }4)\end{cases}, \tag{3}$$

and $b(n)$ can be expressed as a function of the tap-coefficients [3].

For the design of linear-phase first-order digital differentiators (DDs), the impulse response $d(n)$ is also antisymmetric [2,3]. Therefore, we have $d(n)=-d(N-1-n)$ and $d((N-1)/2)=0$ (for N odd) and the transfer function is:

* The author is on leave of absence from the Dept. of Computer-Aided Eng., University of Architecture, Civil Eng. and Geodesy, 1 Hr. Smirnenski Blvd., 1046 Sofia, Bulgaria

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$$H(z) = \sum_{n=0}^{N-1} d(n).z^{-n} . \tag{4}$$

The ideal DD has the following frequency response:

$$H_{\mathrm{I}}^{\mathrm{DD}}\left(e^{j\omega}\right) = D^{\mathrm{DD}}(\omega).e^{j\pi/2} , \tag{5}$$

where $D^{\mathrm{DD}}(\omega) = \omega$ for $0 \le \omega \le \omega_p \le \pi$ and ω_p denotes the passband edge frequency of a differentiator.

It is shown [16], that for first-order fullband DD (N even, $\omega_p = \pi$) designed by LS method the following compact relation for the coefficients $b(n)$ from (3) could be derived:

$$b(n) = \frac{8.(-1)^{n+1}}{\pi.(2n-1)^2}, \quad 1 \le n \le \frac{N}{2} .$$

For clarity, we introduce below an integer t instead of n. Using the fact that:

$$b(t) = 2d\left(\frac{N}{2} - t\right), \quad 1 \le t \le \frac{N}{2}$$

we can write the expressions for the impulse response of the fullband LS DD (obtained as a result of the method [16]) as:

$$d\left(\frac{N}{2} - t\right) = -d\left(\frac{N}{2} - 1 + t\right) = \frac{4.(-1)^{t+1}}{\pi.(2t-1)^2}, \quad 1 \le t \le \frac{N}{2} . \tag{6}$$

By that means, the need to solve the system of linear equations for the case of fullband DD [16] is avoided. Below, we present new designs of even and odd length HTs by transforming an existing design of fullband DDs (6), which is obtained using the LS method.

III. Design of FIR Case 3 and Case 4 Hilbert Transformers

The following relation between the coefficients of Case 3 HT and $d(n)$ is given in [4]:

$$h_3(n) = \begin{cases} (-1)^{(n-N)/2}.[(N-1)/2 - n/2]d(n/2), & n \text{ even} \\ 0, & n \text{ odd}, \end{cases} \tag{7}$$

where the transfer function is: $H_3(z) = \sum_{n=0}^{2(N-1)} h_3(n).z^{-n}$ and N is the length of differentiator (see Fig. 1). The length of obtained Case 3 Hilbert transformer is therefore $2N-1$.

We would like to express the tap-coefficients $h_3(n)$ as a function of $d\left(\frac{N}{2} - t\right)$ in order to use the known relation (6). Therefore, we set $\frac{n}{2} = \frac{N}{2} - t$ for $1 \le t \le \frac{N}{2}$ and obtain:

$$h_3(N-2t) = (-1)^{-t}\,\frac{2t-1}{2}\,d\left(\frac{N}{2} - t\right), \quad N-2t \text{ even} . \tag{8}$$

Taking into account that $N-2t$ is always an even number for t integer and fullband DD (N even), and using equations (6) and (7), we get:

$$\begin{aligned} &h_3(N-2t) = -h_3(N-2+2t) = -\frac{2}{\pi(2t-1)}, \quad 1 \le t \le \frac{N}{2} \\ &h_3(N-2r) = h_3(N-2+2r) = 0, \quad r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{N-1}{2} \end{aligned} \tag{9}$$

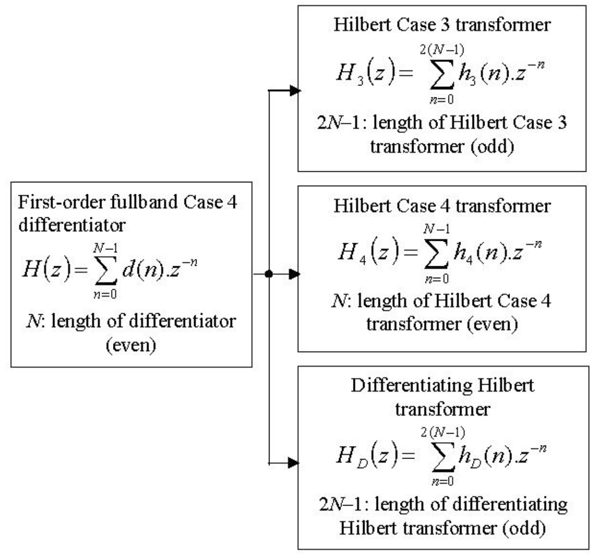

Fig. 1. Design transformations.

The last formula gives a very simple expression for the coefficients of Case 3 HT designed through fullband Case 4 DD with length N. As we have started our design from fullband LS DD, we can conclude now, that the Hilbert transformer (9) also possesses least-squares features. Indeed, the examples given in Section V clearly show typical LS magnitude responses of designed Hilbert transformers.

The second equation from (9) corresponds to inserting a zero between every two successive coefficients, i.e. $h_3(n)=0$ when n is odd number ($n=1,3,5,\ldots,2N-3$).

If we remove the zero-valued samples of the Case 3 LS HT expressed by (9), we will further obtain Case 4 LS HT. Reference [4] proposed the following relation for even-length HT:

$$h_4(n) = h_3(2n) = (-1)^{(2n-N)/2}\left(\frac{N-1}{2} - n\right)d(n), \quad 0 \le n \le N-1 \tag{10}$$

which transfer function is: $H_4(z) = \sum_{n=0}^{N-1} h_4(n).z^{-n}$.

From (6) and (10), it is easy to derive the following new relation for Case 4 HT designed by LS technique:

$$h_4\left(\frac{N}{2} - t\right) = -h_4\left(\frac{N}{2} - 1 + t\right) = -\frac{2}{\pi(2t-1)}, \quad 1 \le t \le \frac{N}{2} . \tag{11}$$

It is clear, that he total number of non-zero impulse response coefficients (and therefore the computational complexity) of designed Case 4 HTs is the same as this one of the fullband DDs (for N even number).

As an example, we have designed the resulting HTs for length of differentiator $N=6$ using the proposed new expressions (9) and (11). The numerical values of the coefficients $d(n)$, $h_3(n)$, and $h_4(n)$ are shown in Table I.

TABLE I
COEFFICIENTS' VALUES OF DESIGNED HTS AND FULLBAND DD

n	0	1	2	3	4	5	6	7	8	9	10
$d(n)$	$\frac{4}{25\pi}$	$-\frac{4}{9\pi}$	$\frac{4}{\pi}$	$-\frac{4}{\pi}$	$\frac{4}{9\pi}$	$-\frac{4}{25\pi}$					
$h_3(n)$	$-\frac{2}{5\pi}$	0	$-\frac{2}{3\pi}$	0	$-\frac{2}{\pi}$	0	$\frac{2}{\pi}$	0	$\frac{2}{3\pi}$	0	$\frac{2}{5\pi}$
$h_4(n)$	$-\frac{2}{5\pi}$	$-\frac{2}{3\pi}$	$-\frac{2}{\pi}$	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$					
$h_D(n)$	$-\frac{2}{25\pi}$	0	$-\frac{2}{9\pi}$	0	$-\frac{2}{\pi}$	$\frac{\pi}{2}$	$-\frac{2}{\pi}$	0	$-\frac{2}{9\pi}$	0	$-\frac{2}{25\pi}$

Additionally, taking into account (3), we can determine the magnitude response of a designed Case 4 HT as follow:

$$M(\omega) = -\frac{4}{\pi}\sum_{n=1}^{N/2}\frac{1}{2n-1}\sin(n-1/2)\omega, \quad N \text{ even} .$$

The above result is a very simple and allows a fast calculation of the $M(\omega)$.

IV. Design of differentiating Hilbert transformers

Cizek [17] proposed a differentiating HT, the output of which is the derivative of the Hilbert transform of the input signal. It was later proved by authors [4], that the coefficients of differentiating HT can be expressed as:

$$h_D(n) = \begin{cases} (-1)^{(n-N)/2}.\dfrac{d(n/2)}{2}, & n \text{ even} \\ 0, & n \text{ odd} \neq N-1 \\ \dfrac{\pi}{2}, & n = N-1 \end{cases} , \tag{12}$$

where $H_D(z) = \sum_{n=0}^{2(N-1)} h_D(n).z^{-n}$. Hence, the length of designed differentiating HT will be $2N-1$, where N is the length of a fullband differentiator.

Using (6) and (12), we obtain the following new relations for the coefficients of differentiating HT:

$$\begin{aligned} &h_D(N-2t) = h_D(N-2+2t) = -\frac{2}{\pi(2t-1)^2}, \qquad 1 \le t \le \frac{N}{2} \\ &h_D(N-1) = \frac{\pi}{2} \\ &h_D(n) = 0, \quad \text{where } n = 1,3,5,\ldots,2N-3 \text{ and } n \neq N-1 . \end{aligned} \tag{13}$$

It is obvious, that all non-zero coefficients of $h_D(n)$ are negative and symmetric (except for n=N–1). This is in contrast to $h_3(n)$, where non-zero coefficients are antisymmetric. The values of $h_D(n)$ calculated for N=6 are also given in Table I.

V. Examples and conclusions

In order to test our relations, we have created a few simple Matlab programs. As an example, we design below a Case 3 HT with length $2N$–1=59 (N=30: length of fullband DD). In view of comparison, two amplitude responses are obtained (Fig. 2(a)): using our method (solid line) and McClellan-Parks algorithm (dashed line). The edge frequencies could be easily determined from the amplitude response as ω_{C1}=0.0154π and ω_{C2}=π–ω_{C1}=0.9846π (see for details the smaller figure in Fig. 2(a)). A typical LS behaviour of a designed Hilbert transformer in contrast to the equiripple HT can be observed. The error functions and the impulse responses for the same example are given in Fig. 2(b), Fig. 2(c), and Fig. 2(d). It is known that the equiripple HTs are optimal in minimax sense. Our results show that a better error function for the proposed HTs is obtained (see Fig. 2(b)) in most of the frequency band, except in very narrow regions at the band edges.

Concerning the impulse responses, we can conclude that the values obtained with the above considered methods are very close, except the values of the first and last samples (i.e. $h_3(0)$

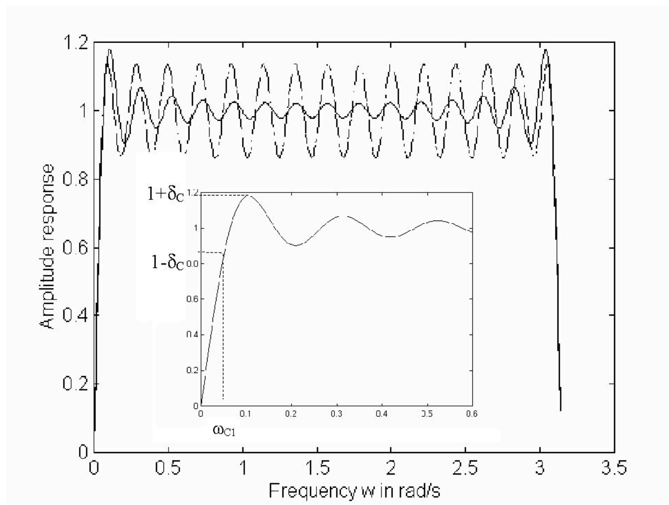

Fig. 2(a). Amplitude responses of Hilbert Case 3 transformer with ω_{C1}=0.0154π, ω_{C2}=0.9846π, and length $2N$–1=59 designed by our method (solid line) and McClellan-Parks algorithm (dashed line).

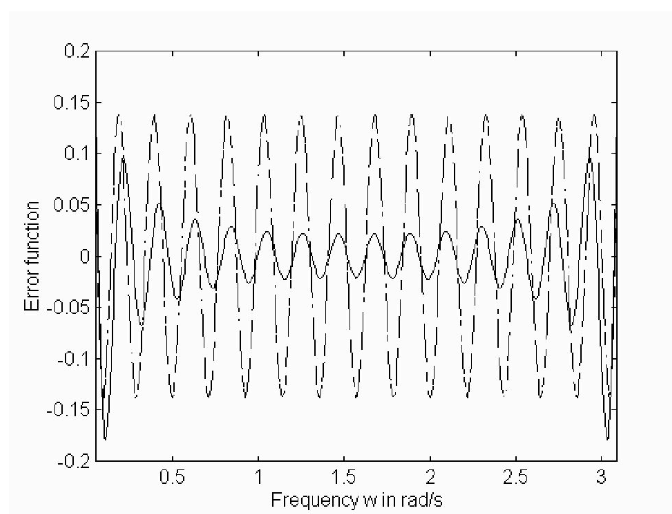

Fig. 2(b). Error functions, for our method (solid line) and McClellan-Parks algorithm (dashed line).

and $h_3(2N-2)$, see Fig. 2(c) , Fig. 2(d)).

The amplitude response of designed differentiating HT with the same length ($2N-1=59$) is given in Fig. 3(a). In our approach, we could obtain even length HT only with Case 4 type characteristic. The amplitude response of such a HT (with length $N=50$) is shown as an example in Fig. 3(b).

We consider below the performance of designed Case 3 HTs using relations (9). Let we denote the passband (or middle band) width of the Case 3 HTs with $B=\omega_{c2}-\omega_{c1}$. The obtained magnitude responses of these HTs have relatively wide passband width B (as can be observed from the examples). The above result is a consequence of the fullband differentiator case (with $\omega_p=\pi$) which is a starting point of the proposed method. We have investigated how the value of B depends on the length of designed HTs. Next figure (Fig. 4) presents edge frequency ω_{c2} as a function of the length $2N-1$ for Case 3 HT. We have determined fourteen different cases of HTs designed with lengths values between 7 and 191. As boundary cases, we have obtained the following parameters of designed response: $\omega_{c2}=0.8895\pi$ ($B=2.4473$ rad/s) and $\omega_{c2}=0.9952\pi$ ($B=3.1114$ rad/s) for HTs with lengths 7 and 191, respectively.

As a conclusion, we can summarize that this paper presents some new simple relations for design of three types of Hilbert transformers based on least-squares approach. These explicit formulas give an efficient and easy way for computation of tap-coefficients of HTs even with a hand calculator. Design examples and comparison with McClellan-Parks algorithm prove the effectiveness of proposed approach. The accuracy is very good taking into account that also we avoid application of complex iterative procedures.

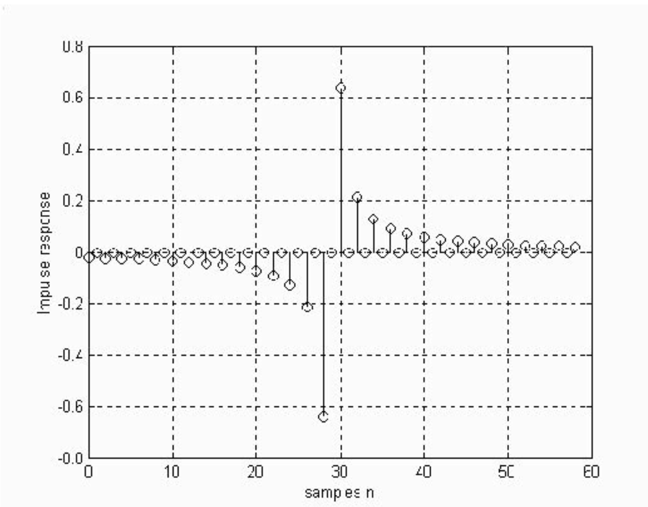

Fig. 2(c). Impulse response $h_3(n)$: our method.

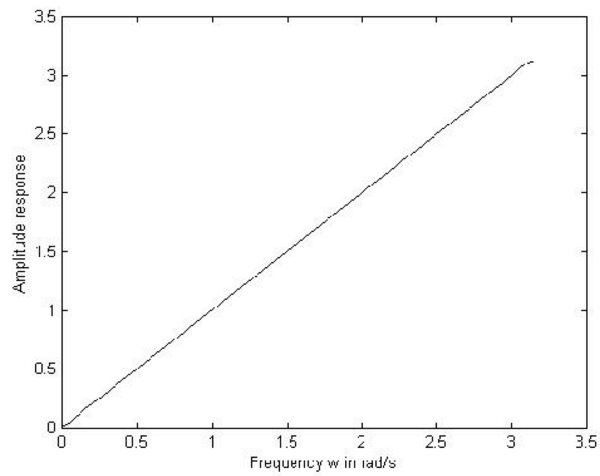

Fig. 3(a). Amplitude response of differentiating HT with length $2N-1=59$.

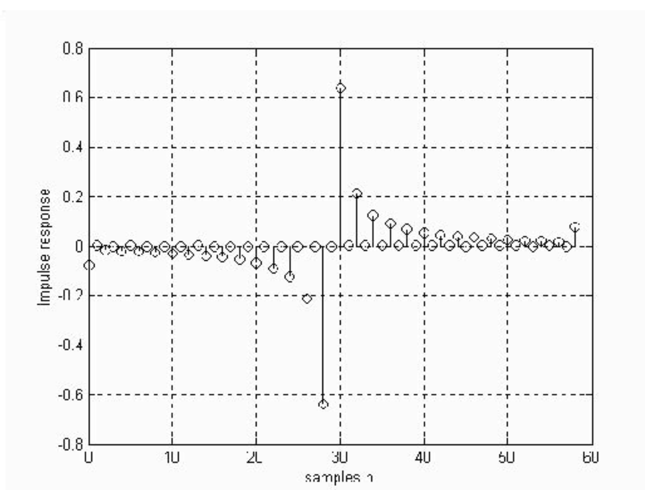

Fig. 2(d). Impulse response $h_3(n)$: McClellan-Parks algorithm.

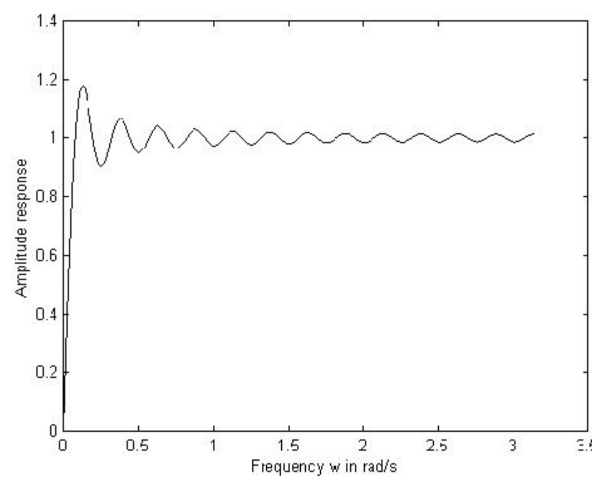

Fig. 3(b). Amplitude response of Case 4 HT with length $N=50$.

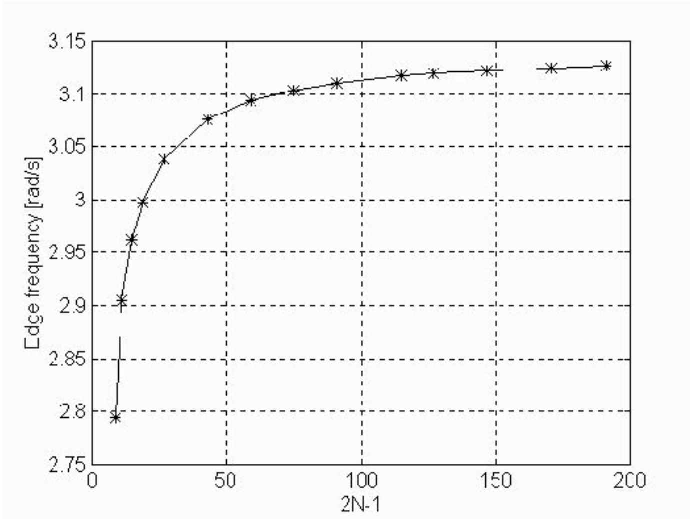

Fig. 4. Edge frequency ω_{C2} as a function of the length of a Hilbert Case 3 transformer for the proposed method.

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