KEY WORDS: Mathematics, Algorithms, Modelling, Orientation, Calibration, Theory

ABSTRACT:
The topic of this paper is the so-called trifocal tensor (TFT), which describes the relative orientation of three uncalibrated images. The TFT is made up of 27 homogenous elements but only has 18 DOF. Therefore, its elements have to fulfill 8 constraints - a new form for these constraints is presented in this paper. Furthermore, a new minimal parameterization for the TFT is presented having exactly 18 DOF and which is generally applicable for any arrangement of the three images proved not all three projection centers coincide. Constraints and parameterization are found using the so-called correlation slices.

1 INTRODUCTION

The trifocal tensor (TFT) allows a linear formulation for the relative orientation of three uncalibrated images. So it basically plays the same role for three images as the fundamental matrix [Loung, Faugeras 1996] plays for two. The TFT has been subject of much research in the past ten years. [Spetsakis, Aloimonos 1990] were the first to discover redundancies within the contents of three calibrated images. For uncalibrated images [Shashua 1995] showed that 27 coefficients and one homologous triple of points in three views form together nine homogenous linear equations (four of them being independent), which he called trilinearities, since they consist of products of three image coordinates and one of the 27 coefficients. Furthermore [Hartley 1994] showed that Shashua’s 27 coefficients and a homologous triple of lines create two homogenous linear equations. Therefore, the TFT can be linearly computed using at least 7 points or 13 lines or a proper combination. He also proposed for this set of 3 × 3 × 3 coefficients the term trifocal tensor.

Although the TFT is made up of 27 elements, it only has 18 degrees of freedom (DOF): 3 × 11 DOF/image - 15 (absolute projective orientation). Therefore the TFT’s elements must fulfill 9 constraints (one of them is the fixation of the TFT’s scale due to the scale ambiguity in the homogenous trilinear relations). If these constraints are neglected, errors in the image data used to compute the TFT might be absorbed by the redundant parameters, yielding a perhaps severely disturbed TFT and thus a wrong image orientation. So, in the past few years attempts were made to derive algorithms that return a valid trifocal tensor. This can be done in two ways: a) by introducing the necessary number of constraints into the computation, or b) by using a minimal parameterization for the TFT having 18 DOF. For each of these methods some solutions were presented in the past: [Torr, Zisserman 1997], [Papadopoulo, Faugeras 1998], [Canterakis 2000].

In this paper a new minimal set of constraints will be presented. In that course we will also arrive at a new minimal parameterization for the TFT. The paper is organized in the following way: In section 2 the basic formulation of the TFT is presented. The properties of the tensor resp. of its slices will be summarized in section 3. After a summary of the existing solutions in section 4, the new set of constraints is presented in section 5.2, followed by the minimal parameterization in section 6.

1.1 Notation

Geometric objects:
- object points: upper case Roman font, e.g. P
- image points: lower case Roman font, e.g. p
- special image lines: small Greek font, e.g. λ
- general image line: ℓ

Mathematical quantities:
- matrices: upper case Typewriter font, e.g. R
- vectors: bold Roman font
  - euclidian/affine vectors, e.g. O, e
  - projective vectors, e.g. v₁₁, v₁₃, v₃₁
- scalars: lower case Roman or Greek font; e.g. u, µ

Special objects:
- \( \tilde{v}_{xy} \) epipole of image \( \psi_y \) in image \( \psi_x \);
  - i.e. the mapping of \( O_y \) into image \( \psi_x \);
- \( r_{x₁}, r_{x₂}, r_{x₃} \) the principal rays of image \( \psi_x \); c.f. sec 2
- \( \pi_{x₁}, \pi_{x₂}, \pi_{x₃} \) the principal planes of image \( \psi_x \)

The symbol \( \sim \) denotes equality up to scale.

2 BASICS

2.1 The central projection using homogenous coordinates

The formulation of the image geometry and the underlying relations are based on the one used in [Ressl 2000]. The projective relation between a 3D object point \( P \) and its image point \( p \) can be represented in a very compact way using projective geometry. If the exterior orientation of an image \( \psi \) is given by the image’s projection center \( O_\psi \) and the rotation matrix \( R_\psi \) (from the image system to the object system), and if the interior orientation of the image is given by the principal point \( (x_0 y_0) \), the principal dis-
tance $f$ and two parameters $(\alpha, \beta)$ modelling affine image deformations, then the central perspective image point $p$ as a homogenous vector $\tilde{p}$ - of an object point $P$ can be computed using the projection matrix $P_\psi$ (equation (1)).

$$\tilde{p} \sim C_\psi^{-1} \cdot R_\psi^\top \cdot [E_{3\times3}, -O_\psi] \cdot \tilde{P} = P_\psi \cdot \tilde{P}$$  \hspace{1cm} (1)

$$C_\psi = \begin{pmatrix} 1 & \alpha & -x_0 \\ 0 & \beta & -y_0 \\ 0 & 0 & -f \end{pmatrix}$$

$$E_{3\times3} = \text{diag}(1,1,1)$$

The three columns of $C_\psi$ together with $R_\psi$ represent the affine direction vectors of the so-called principal rays $r_{\psi_1}, r_{\psi_2}, r_{\psi_3}$ (lines through the projection center parallel to the image’s coordinate axes) as $R_\psi \cdot C_\psi$. These three lines span three planes - the so-called principal planes $\pi_{\psi_1}, \pi_{\psi_2}, \pi_{\psi_3}$ [Papadopoulos, Fauergus 1998].

An image line $\ell$ can also be represented by a homogenous vector $\tilde{\ell}$. If the line $\ell$ contains the point $p$ it holds: $\tilde{\ell}^\top \tilde{p} = 0$. The line $\ell$ defined by two points $p$ and $q$ is given by: $\tilde{\ell} \sim [\tilde{p}]_x \cdot \tilde{q}$. With $[\tilde{p}]_x$ being the so-called axiator:

$$a \times b = [a]_x \cdot b \quad [a]_x = \begin{pmatrix} 0 & a_z & -a_y \\ -a_z & 0 & a_x \\ a_y & -a_x & 0 \end{pmatrix}$$  \hspace{1cm} (2)

$$d = A \cdot c \quad \rightarrow \quad [d]_x = \text{Det}(A) \cdot A^{-\top} \cdot [c]_x \cdot A^{-1}$$  \hspace{1cm} (3)

### 2.2 A few basics on tensor calculus

A tensor is an indexed system of numbers. There are two kinds of indices: sub-indices are called co-variant and super-indices contra-variant. A tensor with contra-variant valence $p$ and co-variant valence $q$ has $n^{p+q}$ components with $n$ being the dimension of the underlying vector-space; i.e. each index runs from 1 to $n$. Using these indices and Einstein’s convention of summation, certain mathematical relations can be expressed in a very efficient way. This convention says that a sum is made up of all the same indices appearing as co- and contra-variant. So, for example, the scalar product $s(x, y) = x^\top \cdot y$ of two vectors $x = (x_1, x_2, x_3)^\top$ and $y = (y_1, y_2, y_3)^\top$ can be written in a shorter way as: $s(x, y) = x^\top \cdot y$. The product $A \cdot B = C$ of two matrices $A$ and $B$ can be written as $A^\top B = C^\top$. The contra-variant indices relate to the rows and the co-variant ones to the columns.

### 2.3 The trifocal tensor

If the orientation of three images $\psi_1, \psi_2, \psi_3$ is formulated according to equation (1) (with $O_1 = 0$ and $R_1 = E_{3\times3}$) then the TFT can be represented in the following way (equation (4)); c.f. [Ressl 2000].

$$T^{i,k} = (\bar{v}_{21})^\top \cdot B^i_k - (\bar{v}_{31})^k \cdot A^i_k$$  \hspace{1cm} (4)

$$\bar{A} = C_2^{-1} \cdot R_2^\top \cdot C_1$$

$$\bar{B} = C_3^{-1} \cdot R_3^\top \cdot C_1$$

$$\bar{v}_{21} = -C_2^{-1} \cdot R_2^\top \cdot O_2$$

$$\bar{v}_{31} = -C_3^{-1} \cdot R_3^\top \cdot O_3$$

$\bar{v}_{xy}$ is the epipole of image $\psi_x$ in image $\psi_y$, i.e. the image of $O_y$ in image $\psi_x$. With “it is emphasized that this epipole is represented as a homogenous vector but in a specific scale (observe the equality-symbol). The other epipoles are:

$$\bar{v}_{12} = C_1^{-1} \cdot O_2$$

$$\bar{v}_{13} = C_1^{-1} \cdot O_3$$

$$\bar{v}_{23} = C_2^{-1} \cdot R_2^\top \cdot O_3 + \bar{v}_{21}$$

$$\bar{v}_{32} = C_3^{-1} \cdot R_3^\top \cdot O_2 + \bar{v}_{31}$$

### 3 THE TENSORIAL SLICES

One can imagine the trifocal tensor $T^{i,k}$ formed as a $3 \times 3 \times 3$ cube of numbers and the cube’s edges related to the indices $i, j, k$. If we keep one index fixed, we slice a $3 \times 3$ matrix out of the tensor. Since we have three indices, we get three different kinds of matrices - different also in their geometrical meaning. For didactical reasons we will start with the $j$ and $k$ index.

#### 3.1 The homograhpic slices $J_x$ and $K_x$

If we keep the $j$-index in equation (4) fixed as $j = x \in \{1, 2, 3\}$, we get the following matrix $J_x$ (being the $x^{th}$ column of $E_{3\times3}$):

$$J_x = e_x^\top \cdot \bar{v}_{21} \cdot B - \bar{v}_{31}^\top \cdot e_x^\top \cdot A$$  \hspace{1cm} (10)

$J_x$ describes a mapping (a collineation) of points $\tilde{p}_1$ in image $\psi_1$ to points $\tilde{p}_3$ in image $\psi_3$ via the principal plane $\pi_{\psi_2}$. In [Shashua, Werman 1995] this mapping is termed homography. The homography matrices $J_x$ are distinguished by the properties shown in Table 1.

Analogously, if we keep the $k$-index fixed, we get a matrix $K_x$, which describes a homography from image $\psi_1$ to image $\psi_2$ via the principal plane $\pi_{\psi_3}$.

$$K_x = v_{21} \cdot e_x \cdot B - e_x^\top \cdot v_{31} \cdot A$$  \hspace{1cm} (11)

#### 3.2 The correlation slices $I_x$

If we keep the $i$-index fixed, we get analogously a $3 \times 3$ matrix $I_x$. For the homography matrices $J_x, K_x$ their form resulted directly from the co-variant ($i$) and contra-variant ($k$ resp. $j$) indices in equation (4). When the $i$-index is fixed, only the contra-variant indices ($j, k$) remain, and therefore one of them has to be chosen for the columns. We choose the $j$-index.

$$I_x = B \cdot e_x \cdot \bar{v}_{21} - \bar{v}_{31} \cdot e_x^\top \cdot A^\top$$  \hspace{1cm} (12)

$I_x$ describes a mapping (a dual correlation) of lines $\ell_2$ in image $\psi_2$ to points $\tilde{p}_3$ in image $\psi_3$ via the principal ray $r_{\psi_2}$. $I_x^\top$ would map the lines $\ell_3$ in image $\psi_3$ to points $\tilde{p}_2$ in image $\psi_2$ via the principal ray $r_{\psi_3}$.

Due to [Papadopoulos, Fauergus 1998] the correlation matrices $I_x$ are distinguished by the properties shown in Table 2. Since the correlation slices are the basic input for the constraints and parameterization to be presented, they are investigated in more detail in section 5.1.

Note: The relations between the homographic and correlation slices are the following: The $y^{th}$ column of $J_x$ resp. $K_x$ is the $y^{th}$ column of $I_x$, resp. $I_x^\top$.

### 4 PREVIOUS CONSTRAINTS AND MINIMAL PARAMETERIZATIONS

Two sets of constraints and two minimal parameterizations (i.e. having 18 DOF) were discussed in the literature so far. [Torr, Zisserman 1997] present a minimal parameterization for the TFT. By assigning projective canonical coordinates to the image and the space points, they show, that it is possible to compute the tensor from six homologous point triples across three images. Of the 36 observed
image-coordinates in six homologous triples, convenient 18 coordinates are kept fixed. In this way a minimal parameterization of the tensor is achieved. The unknowns themselves are obtained as (up to 3) solutions of a cubic equation. Due to this fixing of erroneous observations in the images one might be suspicious that errors in the calculated tensor may be induced, furthermore no correct minimization of the measurement-errors in all observations is possible. And as it is shown by the results in [Torr, Zisserman 1997] the standard deviation depends on the choice of the 6 points resp. the fixed 18 coordinates, which is not obvious in the beginning. However, this method of keeping the proper number of image-coordinates fixed, could be helpful also for other tasks, where a minimal parameterization is needed, but cannot be formulated easily.

[Papadopoulo, Faugeras 1998] introduce a minimal parameterization together with a set of 12 sufficient constraints - not minimal, since any number of constraints greater than eight must contain dependencies. Their set of constraints are entirely based on the correlation slices $I_x$ and are made of the properties (b), (c), (d) shown in Table 2. Their minimal parameterization looks like the following: The left kernels of the correlation slices are parameterized using 2 parameters for their common epipole $\tilde{v}_{31}$ and 1 parameter (a direction angle) for each kernel - thus 5 parameters in total. With other 5 parameters the right kernels and epipole $\tilde{v}_{21}$ are parameterized. With the left and right kernels the correlation slices $I_x$ can be parameterized by 8 coefficients. This way of parameterization results in a very large number of maps (9-3²·3³) and it is not clear how this parameterization is applicable in case of $\text{rank}(I_x) < 2$. In this case the kernels need to be lines.

In [Canterakis 2000] the first set of minimal constraints is presented, which are entirely based on the homographic slices $J_x$ and are derived from the properties shown in Table 1, i.e. each general eigenvalue problem set up with two homographic slices has one general eigenvalue with multiplicity 2 ($\mu_2 = \mu_3$ ($\rightarrow$ 1 constr.), the corresponding general eigenvector is 2-dimensional ($\rightarrow$ 2 constr.), the general eigenvector $\tilde{x}_1$ corresponding to the single general eigenvalue $\mu_1$ is the same (up to scale) for all three pairs of $J_x$ ($\rightarrow$ 2 constr.). This general eigenvalue problem can be independently set up twice yielding the required number of 8 constraints. Open questions with this set of constraints are, how are they applicable in case of $\text{rank}(I_x) < 3$ and how to implement them efficiently in a computer program (e.g. constraint $\tilde{x}_1(\text{of pair (x,y)}) \sim \tilde{x}_1(\text{of pair (x,z)})$ requires this general eigenvector to be expressed in terms of the 27 tensor elements).

In the following sections a new set of minimal constraints together with a minimal parameterization will be presented. Both are derived very easily, having very simple geometric properties. Their implementation is rather simple (actually the minimal parameterization is easier to realize than the constrained version).

### 5 A NEW MINIMAL SET OF CONSTRAINTS

The basic input for this set of constraints are the correlation slices $I_x$, therefore we will take a closer look at these matrices.

#### 5.1 The correlation slices $I_x$ - Revisited

The correlation slices in equation (12) describe a mapping of lines $\tilde{\ell}_2$ in image $\psi_2$ to points $\tilde{p}_3$ in image $\psi_3$ via the principal ray $r_{12}$, meaning that $\tilde{p}_3$ is the projection of the intersection point of $r_{12}$ with the projection plane of $\tilde{\ell}_2$.

In general, $\text{rank}(I_x) = 2$, since the columns of $I_x$ are linear combinations of two vectors ($B \cdot e_x$ and $\tilde{v}_{31}$) - or the rows are linear combination of two vectors ($A \cdot e_x$ and $\tilde{v}_{21}$). For the same reason, any linear combination of the correlation slices $\sum_{x=1}^{3} a_x \cdot I_x$ will also have $\text{rank} = 2$ in general; c.f. [Papadopoulo, Faugeras 1998].

Using equ. ((5) - (9), (12)), we can find the cases where $\text{rank}(I_x) < 2$:

- $\text{rank}(I_x) = 1$ will result if $B \cdot e_x \sim \tilde{v}_{31}$ ($\rightarrow O_1 \in r_{1x}$ and $I_x \sim \tilde{v}_{31} \cdot \tilde{v}_{21}$) or if $(A \cdot e_x \sim \tilde{v}_{21})$ ($\rightarrow O_2 \in r_{1x}$ and $I_x \sim \tilde{v}_{31} \cdot \tilde{v}_{21}$).

#### Table 1: Properties of the homographic slices $J_x$

<table>
<thead>
<tr>
<th>Property</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$\tilde{v}<em>{31} \sim J_x \cdot \tilde{v}</em>{13}$ provided $\text{rank}(J_x) = 3$</td>
</tr>
<tr>
<td>(b)</td>
<td>general eigenvalue problem $(J_x - \mu \cdot J_x) \cdot \tilde{x} = 0$, $x, y, z \in {1, 2, 3}$, pairwise diff.</td>
</tr>
<tr>
<td>(c)</td>
<td>gen. eigenvalue $\mu_1 = \tilde{v}<em>{31} \cdot \tilde{v}</em>{21}$ gen. eigenvector $\tilde{x}<em>1 \sim \tilde{v}</em>{13}$</td>
</tr>
<tr>
<td>(d)</td>
<td>gen. eigenvalue $\mu_2 = \mu_3 = \tilde{v}<em>{31} \cdot \tilde{v}</em>{21}$ gen. eigenvector $\tilde{x}<em>{2,3} \sim \alpha \cdot \tilde{v}</em>{12} + \beta \cdot A^{-1} \cdot e_x$</td>
</tr>
</tbody>
</table>

#### Table 2: Properties of the homographic slices $I_x$; c.f. [Papadopoulo, Faugeras 1998]

<table>
<thead>
<tr>
<th>Property</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$\text{rank}(I_x) \leq 2$</td>
</tr>
<tr>
<td>(b)</td>
<td>$\text{rank}(\sum_{x=1}^{3} a_x \cdot I_x) \leq 2$</td>
</tr>
<tr>
<td>(c)</td>
<td>$R \cdot \tilde{v}_{21} = 0$ using $R = [\tilde{p}_1, \tilde{p}_2, \tilde{p}_3]$ with $I_x \cdot \tilde{p}_x = 0$</td>
</tr>
<tr>
<td>(d)</td>
<td>$L \cdot \tilde{v}_{31} = 0$ using $L = [\lambda_1, \lambda_2, \lambda_3]$ with $I_x \cdot \lambda_x = 0$</td>
</tr>
</tbody>
</table>
The underlying geometric properties become clearer, when one considers a regular 1-dimensional correlation between the lines of this pencil and the points on \( \lambda_x \).

Since the correlation slices are always singular, there always exists a nontrivial null space. The right null space of \( I_x \) is a line \( \rho_x \) in image \( \psi_3 \) in general. It represents the line for which the correlation matrix \( I_x \) yields no valid point (i.e., 0) in image \( \psi_3 \). Geometric reason: The projection plane due to \( \rho_x \) contains the principal ray \( r_1 \). Analogously, the left null space is a line \( \lambda_x \) in image \( \psi_3 \) in general. It represents the line on which all mapped points \( I_x \cdot \hat{e}_x \) lie. The lines \( \rho_x \) and \( \lambda_x \) are the epipolar lines of the principal ray \( r_1 \) in image \( \psi_2 \) and \( \psi_3 \), respectively. Epipolar lines always pass through the respective epipole \( (\hat{v}_2, \hat{v}_3) \).

Thus the matrices \( L = [\lambda_1, \lambda_2, \lambda_3] \) and \( R = [\rho_1, \rho_2, \rho_3] \) are also of rank = 2 in general; cf. [Papadopulu, Faugeras 1998].

The matrix \( L \) made of the three left kernels must be singular; provided all three correlation slices \( I_x \) have rank = 2 - otherwise the kernel of \( I_x \) will not be a line \( \lambda_x \).

\[
\text{Det}(I_x) = 0 \quad (15)
\]

These 4 constraints have already been presented in e.g. [Papadopulo, Faugeras 1998].

The 4 remaining constraints are new and will be explained in the following. In section (5.1) we saw, that any line \( \ell_{v_3} \neq \lambda_x \) through \( v_3 \) is mapped by \( I_x \) to the epipole \( \hat{v}_3 \) in image \( \psi_2 \); provided, \( \text{rank}(I_x) = 2 \). So, we can formulate the following constraints:

\[
\begin{align*}
\forall \sigma, \tau, \rho \in \{1, 2, 3\}, \quad (I_x - \mu_1 \cdot I_y) \cdot \ell_{v_3} &= 0 \\
\forall \sigma, \tau, \rho \in \{1, 2, 3\}, \quad (I_x - \mu_2 \cdot I_y) \cdot \ell_{v_3} &= 0 \\
\end{align*}
\]

(16)

Obviously, the relations (16) produce 4 independent equations (6 equations - 2 additional unknown scales \( \mu_1, \mu_2 \)). However, it needs to be proven that these relations are also independent of the determinant constraints (14) and (15).

If the determinant constraints are satisfied, then the correlation slices can be parameterized in the following way (without loss of generality):

\[
\begin{align*}
I_1 &= [a, b, c] = [s_1, v_1 \cdot s_1 + j \cdot \hat{v}_3, w_1 \cdot s_1 + s \cdot \hat{v}_3] \\
I_2 &= [d, e, f] = [s_2, v_2 \cdot \hat{s}_2 + k \cdot \hat{v}_3, w_2 \cdot \hat{s}_2 + \ell \cdot \hat{v}_3] \\
I_3 &= [g, h, i] = [s_3, v_3 \cdot \hat{s}_3 + l \cdot \hat{v}_3, w_3 \cdot \hat{s}_3 + u \cdot \hat{v}_3] \\
\end{align*}
\]

This parameterization just means, that the columns of the three matrices \( I_x \) are represented as linear combinations of a vector \( v_{31} \) common to all three matrices and individual vectors \( \hat{s}_j \). The vector \( v_{31} \) is the common perpendicular of the three left kernels \( \{\lambda_1, \lambda_2, \lambda_3\} \) and its scale is chosen appropriately (hence \( \text{instead of} \)). If we choose any line \( \ell_{v_3} \) through \( v_{31} \), not to any left kernel, we get:

\[
\begin{align*}
I_1^T \cdot \ell_{v_3} &= \left( \begin{array}{c} s_1^T \cdot \ell_{v_3} \\
 v_1 \cdot s_1 + j \cdot \hat{v}_3 \\
w_1 \cdot s_1 + s \cdot \hat{v}_3 \\
\end{array} \right) \sim \left( \begin{array}{c} 1 \\
v_1 \\
w_1 \\
\end{array} \right) \\
I_2^T \cdot \ell_{v_3} &= \left( \begin{array}{c} s_2^T \cdot \ell_{v_3} \\
v_2 \cdot \hat{s}_2 + k \cdot \hat{v}_3 \\
w_2 \cdot \hat{s}_2 + \ell \cdot \hat{v}_3 \\
\end{array} \right) \sim \left( \begin{array}{c} 1 \\
v_2 \\
w_2 \\
\end{array} \right) \\
I_3^T \cdot \ell_{v_3} &= \left( \begin{array}{c} s_3^T \cdot \ell_{v_3} \\
v_3 \cdot \hat{s}_3 + l \cdot \hat{v}_3 \\
w_3 \cdot \hat{s}_3 + u \cdot \hat{v}_3 \\
\end{array} \right) \sim \left( \begin{array}{c} 1 \\
v_3 \\
w_3 \\
\end{array} \right)
\end{align*}
\]

(18)

Since all the right sides in (18) should be similar to the same vector (i.e. epipole \( v_{31} \)), this only can be achieved, if for the coefficients holds: \( v_1 = v_2 = v = v_3 \) and \( w_1 = w_2 = w_3 = w \) - and thus the constraints (16) are independent of the determinant constraints (14) and (15).

Actually the constraints (16) correspond to the already known property (b) in Table 2 with \( a = (0, 1, \mu_2) \). The constraints (16) hold for any line \( \ell_{v_3} \neq \lambda_x \) through \( v_{31} \), therefore the components parallel to \( v_{31} \) of the column vectors \( (I_x - \mu_1 \cdot I_y) \) resp. \( (I_x - \mu_2 \cdot I_y) \) are of no concern and only the components orthogonal to \( v_{31} \) need to be considered. Consequently we get a more preferable form for the constraints (16) by:

\[
\begin{align*}
\lambda_1^T \cdot [\hat{v}_3 \cdot \hat{v}_3] \cdot I_1^T \sim \lambda_2^T \cdot [\hat{v}_3 \cdot \hat{v}_3] \cdot I_2^T \sim \lambda_3^T \cdot [\hat{v}_3 \cdot \hat{v}_3] \cdot I_3^T
\end{align*}
\]

(19)
Summing it up, the presented minimal set of constraints, ensures, that the three correlation slices of a TFT are singular mappings of the lines from one image \((\psi_2)\) to the points of another image \((\psi_3)\) via three concurrent 3D-lines, which are made up by the principal lines of image \(\psi_1\).

6 The minimal parameterization for the TFT

In the previous section we did not only prove, that (14), (15) and (19) constitute a minimal set of constraints for the TFT, but we also found a minimal parameterization for it. If we adopt the equality of coefficients \((v_1 = v_2 = v_3 = v\) and \(w_1 = w_2 = w_3 = w\)) to the parameterization (17), we get this minimal parameterization (having 18 DOF):

\[
\begin{align*}
I_1 &= [\hat{a}, \hat{b}, \hat{c}] = [s_1, v \cdot s + j \cdot \psi_{31}, w \cdot s + 1 + s \cdot \psi_{31}] \\
I_2 &= [d, e, f] = [s_2, v \cdot s + k \cdot \psi_{31}, w \cdot s + t \cdot \psi_{31}] \\
I_3 &= [g, h, i] = [s_3, v \cdot s + l \cdot \psi_{31}, w \cdot s + u \cdot \psi_{31}]
\end{align*}
\]

A few remarks need to be given:

- Obviously, this parameterization is not linear. Thus approximations are required, which can be obtained by an initial solution using the well-known eigenvalue or linear solution for the TFT; e.g. [Hartley 1994].
- The scale of \(\psi_{31}\) needs to be fixed, e.g. by setting its length to 1.
- Observe, that the vectors \(\{s_1, s_2, s_3\}\) in (20) parameterize the same column (index \(c_i\)) in the matrices \(I_x\). For numerical reasons, this index should be that one, for which the respective columns are farthest away from \(c_i\). This index \(c_i\) may be found by

\[
\prod_{i=1}^3 ||I_x \cdot e_{c_i} \times \psi_{31}|| \rightarrow \text{Max}.
\]

- The overall scale in this parameterization needs also to be fixed, e.g. by setting the length of the concatenated vectors \(\{s_1, s_2, s_3\}\) to 1. This yields in total 3 possible mappings; i.e. the choice of \(c_i\).

With this parameterization (20) (i.e. \(c_i = 1\)) we get the homographic slices \(J_x\) as follows:

\[
\begin{align*}
J_1 &= [\hat{a}, \hat{d}, \hat{g}] = [s_1, s_2, s_3] \\
J_2 &= [\hat{b}, \hat{e}, \hat{h}] = [v \cdot s_1 + j \cdot \psi_{31}, v \cdot s_2 + k \cdot \psi_{31}, v \cdot s_3 + l \cdot \psi_{31}] \\
J_3 &= [\hat{c}, \hat{f}, \hat{i}] = [w \cdot s_1 + s \cdot \psi_{31}, w \cdot s_2 + t \cdot \psi_{31}, w \cdot s_3 + u \cdot \psi_{31}]
\end{align*}
\]

And so we can look at the general eigen-value problem \(J_2 - \mu \cdot J_1\). It is easy to see, that \(\mu = v\) yields a 2-dimensional general eigen-space, the line \((j k l)^\top\). Thus \(v\) is an eigen-value with multiplicity 2. This is in accordance with [Canterakis 2000]. And so we can summarize the geometrical interpretation of this minimal parameterization in the following way:

- \(\psi_{31}\) is the epipole of base \(\hat{O}_1\hat{O}_3\) in image \(\psi_3\).
- \([s_1, s_2, s_3]\) is a homography from image \(\psi_1\) to image \(\psi_3\); i.e. the homographic slice \(J_1\).
- \((v \; w)^\top\) is the epipole \(\psi_{21}\) (of base \(\hat{O}_1\hat{O}_2\) in image \(\psi_2\)) - its component at position \(c_i\) is set to 1.
- \((j \; k \; l)^\top\) is the 2-dimensional general eigenspace of \((J_2 - \mu \cdot J_1)\). Since \(J_1\) resp. \(J_2\) is a homography due to \(\pi_{21}\) resp. \(\pi_{23}\), the general eigenvector of \((J_2 - \mu \cdot J_1)\) must be the projection of the intersection of these two principal planes of image \(\psi_2\); i.e. the projection of the principal ray \(r_{22}\) of image \(\psi_2\) into image \(\psi_1\).
- \((s \; t \; u)^\top\) is the 2-dimensional general eigenspace of \((J_3 - \nu \cdot J_1)\), i.e. the projection of the principal ray \(r_{22}\) of image \(\psi_2\) into image \(\psi_1\).

Of interest are the critical configurations for this minimal parameterization. Since it is part of the parameterization, that the length of \(\psi_{31}\) and one component in \(\psi_{21}\) are set to 1, problems surely arise if either of these epipoles is the zero-vector - \(O_1 = O_3\) resp. \(O_1 = O_2\). This problem can be solved - as long as not all three projection centers coincide - by changing the role of the images in the way that the image with the unique projection center plays the role of image \(\psi_1\). Note: The identity of two or all three projection centers might be of practical relevance during the work with a moving camera acquiring images in a constant frequency and which stops at a particular position for a moment. In case of \(O_1 = O_2 = O_3\) the respective TFT becomes the zero-tensor.

Another problem with this parameterization could come from the fact, that the vectors \(\{s_1, s_2, s_3\}\) parameterize the same column (with index \(c_i\)) in all three correlation slices. Still keep in mind that we choose the best column for this parameterization - the one that is farthest away from \(\psi_{31}\). If we take the minimal parameterization exactly as it is given in equation (20), we see, that the columns of \(I_1\) are parameterized by \(\psi_{31}\) and the vector \(\hat{s}_i\). Thus it must be assured, that \(\hat{s}_1\) is different from \(\psi_{31}\) and different from the zero-vector, because otherwise the column vectors \(\hat{a}\) and/or \(\hat{c}\) (being different from \(\psi_{31}\) and \(0\)) cannot be parameterized by \(s_1\) and \(\psi_{31}\). Of course, if \(\hat{b}\) and \(\hat{c}\) are similar to \(\psi_{31}\) or \(0\), than we would have no problem. So,
we will prove, that the first case can not occur. This prove
is outlined in the following.

First we arrange the three matrices \( I_x \) as the rows of a
large matrix \( Z \), which then has 9 rows and 3 columns. We,
however, consider the elements of \( Z \) being the column
vectors of the \( I_x \) matrices. So, \( Z \) has \( 3 \times 3 \) elements and
the element at row \( \xi \) and col \( \eta \) is the \( \eta \)th column vector in
matrix \( I_x \), which is the image of the intersection point of
principal plane \( \pi_{20} \) with the principal ray \( r_{1\xi} \).

Now, we consider that one element (row \( \xi \), col \( \eta \)) of \( Z \) shall
be \( \sim \mathbf{v}_{31} \). This may happen due to two situations: A1)
\( O_3 \in r_{1\xi} \) or A2) \( O_3 \in \pi_{20} \). Then, we consider that one
element (row \( \xi \), col \( \eta \)) of \( Z \) shall be \( 0 \). This may happen
also due to two situations: B1) \( O_3 = \pi_{20} \cap r_{1\xi} \) or B2)
\( r_{1\xi} \in \pi_{20} \). However, B1) implies \( O_3 \in r_{1\xi} \) (\( \rightarrow A1 \)) and
B2) implies \( O_3 \in \pi_{20} \) (\( \rightarrow A2 \)). Thus, the only possible
situations, that may return one element in \( Z \) being \( \sim \mathbf{v}_{31}
\) or \( 0 \) are the ones of A1) and A2).

These situations, however, not only return the element at
row \( \xi \) and col \( \eta \) of matrix \( Z \) to be \( \sim \mathbf{v}_{31} \) or \( 0 \), they further
imply: A1) returns that all elements in row \( \xi \) of \( Z \) are
\( \sim \mathbf{v}_{31} \) i.e. the entire matrix \( I_x \). And so all columns \( I_x \) can
be parameterized by \( \mathbf{s}_3 \) (being \( \sim \mathbf{v}_{31} \) or \( 0 \)) and \( \mathbf{v}_{31} \). A2)
returns that all elements in column \( \eta \) of \( Z \) are \( \sim \mathbf{v}_{31} \); i.e. the
\( \eta \)th column in all three correlation matrices \( I_x \). Again,
the parameterization of these columns is not difficult, but
what’s more important: When situation A2) occurs, the
\( \eta \)th column in the three correlation matrices \( I_x \) will never
be used as the vectors \( \{ \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \} \) in the parameterization,
since they are not far away from \( \mathbf{v}_{31} \).

This completes the proof, that it is impossible, that one of
the three vectors \( \{ \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \} \) is \( \sim \mathbf{v}_{31} \) or \( 0 \), but one of the
other columns in \( I_x \) is different from \( \mathbf{v}_{31} \) and \( 0 \). Thus, the
minimal parameterization (20) holds for any image config-
uration - provided not all three projection centers coincide.

7 SUMMARY AND FUTURE WORK

In this paper a new minimal set of constraints as well as
a new minimal parameterization for the trifocal tensor
(TFT) were presented. They were found using the so-called
correlation slices \( I_x \), together with a new discov-
ered property of them (equ. (19)). Especially the minimal
parameterization, which is applicable for any image con-
figuration (provided not all three projection centers coin-
cide), will help to get new insights into the geometric re-
lations and properties of the TFT. With these constraints
resp. minimal parameterization it is possible to compute the
TFT with minimal (i.e. 18) DOF. Since both rely on
non-linear relations an initial solution for the TFT is re-
quired; e.g. using the well-known linear solution.

So far, the presented constraints and the minimal parame-
terization have been implemented and it will be among the
future work to investigate the advantages of each method.
The experiments so far show a benefit for the minimal
parameterized solution (equ. (20)), which can be imple-
mented rather simple and works for all practical image
configurations - as opposed to the constrained solution,
which relies on correlation slices having \( \text{rank} = 2 \) for the
constraints (15) and (19).

Also of interest are the additional constraints resp. the
minimal parameterization that arise when the interior ori-
entation of the images is known, or if it is unknown but
common to all three images. The latter is of special in-
terest for camera calibration, which needs at least three
images taken by the same camera; e.g. [Hartley 1997].

During the future work we will also investigate, what
amount of error is induced in the resulting TFT (and
thus in the image orientation), when the constraints are
neglected and/or algebraic error is minimized instead of
measurement error. This is especially of interest when the
TFT-solution serves only as an initial start for a subse-
quent bundle-adjustment, since there already the linear
solution might be sufficient.

ACKNOWLEDGMENT

This work was supported by the Austrian Science Fund
FWF (P13901-INF).

REFERENCES

Cantarakis, N., 2000. A minimal set of constraints for the
trifocal tensor. Proceedings of the 6th European Confer-
ence on Computer Vision, Dublin, Ireland , Springer.

Hartley, R.I., 1994. Lines and points in three views - a
unified approach. Proceedings of an Image Under-
standing Workshop held in Monterey, California, Nov. 13-16. Vol.
II, pp.1009-1016.

Hartley, R.I., 1997. Kruppa’s equations derived from the
fundamental matrix. IEEE Transactions on Pattern Analysis

Loun, Q.-T., Faugeras, O., 1996. The fundamental ma-
trix: theory, algorithms and stability analysis. The Inter-
national Journal of Computer Vision 1(17).

Papadopoulo, T., Faugeras, O., 1998. A new characteriza-
tion of the trifocal tensor. Proceedings of the 5th European
Conference on Computer Vision, edited by H. Burkhardt
and B. Neumann, Springer.

Ressl, C., 2000. An Introduction to the relative orienta-
tion using the trifocal tensor. International Archives of
Photogrammetry and Remote Sensing, Vol XXXIII, Part B3/2, Amster-
dam, Netherlands, 2000.

IEEE Transactions on Pattern Analysis and Machine In-
telligence Vol. 17. No. 8, pp. 779-788.

Shashua, A., Werman, M., 1995. On the trilinear tensor
of three perspective views and its underlying geometry. Pro-
cedings of the International Conference on Computer
Vision, Boston, MA.

of structure from motion. Proceedings of an Image Un-
derstanding Workshop held at Pittsburgh, Pennsylvania, Sept.
11-13.

Torr, P.H.S., Zisserman, A., 1997. Robust parameteriza-
tion and computation of the trifocal tensor. Image and Vision
Understanding 15, Elsevier.