A MINIMAL SET OF CONSTRAINTS AND A MINIMAL PARAMETERIZATION FOR THE TRIFOCAL TENSOR

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ABSTRACT:

The topic of this paper is the so-called trifocal tensor (TFT), which describes the relative orientation of three uncalibrated images. The TFT is made up of 27 homogenous elements but only has 18 DOF. Therefore, its elements have to fulfil 8 constraints - a new form for these constraints is presented in this paper. Furthermore, a new minimal parameterization for the TFT is presented having exactly 18 DOF and which is generally applicable for any arrangement of the three images - provided not all three projection centers coincide. Constraints and parameterization are found using the so-called *correlation slices*.

1 INTRODUCTION

The trifocal tensor (TFT) allows a linear formulation for the relative orientation of three uncalibrated images. So it basically plays the same role for three images as the fundamental matrix [Loung, Faugeras 1996] plays for two. The TFT has been subject of much research in the past ten years. [Spetsakis, Aloimonos 1990] were the first to discover redundancies within the contents of three calibrated images. For uncalibrated images [Shashua 1995] showed that 27 coefficients and one homologous triple of points in three views form together nine homogenous linear equations (four of them being independent), which he called *trilinearities*, since they consist of products of three image coordinates and one of the 27 coefficients. Furthermore [Hartley 1994] showed that Shashua's 27 coefficients and a homologous triple of lines create two homogenous linear equations. Therefore, the TFT can be linearly computed using at least 7 points or 13 lines or a proper combination. He also proposed for this set of $3 \times 3 \times 3$ coefficients the term *trifocal tensor*.

Although the TFT is made up of 27 elements, it only has 18 degrees of freedom (DOF): $3{\times}11$ DOF/image - 15 (absolute projective orientation). Therefore the TFT's elements must fulfil 9 constraints (one of them is the fixation of the TFT's scale due to the scale ambiguity in the homogenous trilinear relations). If these constraints are neglected, errors in the image data used to compute the TFT might be absorbed by the redundant parameters, yielding a perhaps severely disturbed TFT - and thus a wrong image orientation. So, in the past few years attempts were made to derive algorithms that return a valid trifocal tensor. This can be done in two ways: a) by introducing the necessary number of constraints into the computation, or b) by using a minimal parameterization for the TFT having 18 DOF. For each of these methods some solutions were presented in the past: [Torr, Zisserman 1997], [Papadopoulo, Faugeras 1998], [Canterakis 2000].

In this paper a new minimal set of constraints will be presented. In that course we will also arrive at a new minimal parameterization for the TFT. The paper is organized in the following way: In section 2 the basic formulation of the TFT is presented. The properties of the tensor resp. of its slices will be summarized in section 3. After a summary of the existing solutions in section 4, the new set of constraints is presented in section 5.2, followed by the minimal parameterization in section 6.

1.1 Notation

Geometric objects:

object points:	upper case Roman font, e.g. P
image points:	lower case Roman font, e.g. p
special image lines:	small Greek font, e.g. λ
general image line:	ℓ

Mathematical quantities:

matrices:	upper case `Typewriter` font, e.g. $\mathtt{R}^\top$
vectors:	bold Roman font
	- euclidian/affine vectors, e.g. $\mathbf{O}_\psi$, $\mathbf{e}_1$
	- projective vectors, e.g. $\tilde{\mathbf{v}}_{31}$, $\hat{\mathbf{v}}_{31}$, $\check{\mathbf{v}}_{31}$
	- $\tilde{\mathbf{x}}$ emphasizes, that $\tilde{\mathbf{x}}$ can be replaced by $\lambda \cdot \tilde{\mathbf{x}}$ anytime ($\lambda \neq 0$);
	- $\hat{\mathbf{x}}$ emphasizes, that the scale of $\hat{\mathbf{x}}$ is determined by a specific relation;
	- $\check{\mathbf{x}}$ emphasizes, that the scale of $\check{\mathbf{x}}$ is fixed either by setting $\|\hat{\mathbf{x}}\| = 1$ or $max(\hat{\mathbf{x}}) = 1$.
scalars:	lower case Roman or Greek font; e.g. u, μ

Special objects:

$\tilde{\mathbf{v}}_{xy}$	epipole of image ψ_y in image ψ_x; i.e. the mapping of $\mathbf{O}_y$ into image ψ_x
r_{x1}, r_{x2}, r_{x3}	the principal rays of image ψ_x; c.f. sec 2
π_{x1}, π_{x2}, π_{x3}	the principal planes of image ψ_x

The symbol $\sim$ denotes equality up to scale.

2 BASICS

2.1 The central projection using homogenous coordinates

The formulation of the image geometry and the underlying relations are based on the one used in [Ressl 2000]. The projective relation between a 3D object point P and its image point p can be represented in a very compact way using projective geometry. If the exterior orientation of an image ψ is given by the image's projection center $\mathbf{O}_\psi$ and the rotation matrix $\mathtt{R}_\psi$ (from the image system to the object system), and if the interior orientation of the image is given by the principal point $(x_0\, y_0)$, the principal distance f and two parameters $(\alpha\,\beta)$ modelling affine image deformations, then the central perspective image point p - as a homogenous vector $\tilde{\mathbf{p}}$ - of an object point P can be computed using the projection matrix $\mathtt{P}_\psi$ (equation (1)).

$$\tilde{\mathbf{p}} \sim \mathtt{C}_\psi^{-1} \cdot \mathtt{R}_\psi^\top \cdot [\mathtt{E}_{3\times3}, -\mathbf{O}_\psi] \cdot \tilde{\mathbf{P}} = \mathtt{P}_\psi \cdot \tilde{\mathbf{P}} \tag{1}$$

$$\mathtt{C}_\psi = \begin{pmatrix} 1 & \alpha & -x_0 \\ 0 & \beta & -y_0 \\ 0 & 0 & -f \end{pmatrix}$$

$$\mathtt{E}_{3\times3} = diag(1,1,1)$$

The three columns of $\mathtt{C}_\psi$ together with $\mathtt{R}_\psi$ represent the affine direction vectors of the so-called *principal rays* $r_{\psi1}$, $r_{\psi2}$, $r_{\psi3}$ (lines through the projection center parallel to the image's coordinate axes) as $\mathtt{R}_\psi \cdot \mathtt{C}_\psi$. These three lines span three planes - the so-called *principal planes* $\pi_{\psi1}$, $\pi_{\psi2}$, $\pi_{\psi3}$ [Papadopoulo, Faugeras 1998].

An image line ℓ can also be represented by a homogenous vector $\tilde{\ell}$. If the line ℓ contains the point p it holds: $\tilde{\ell}^\top \cdot \tilde{\mathbf{p}} = 0$. The line ℓ defined by two points p and q is given by: $\tilde{\ell} \sim [\,\tilde{\mathbf{p}}\,]_\times \cdot \tilde{\mathbf{q}}$. With $[\,\tilde{\mathbf{p}}\,]_\times$ being the so-called *axiator*:

$$\mathbf{a} \times \mathbf{b} = [\,\mathbf{a}\,]_\times \cdot \mathbf{b} \qquad [\,\mathbf{a}\,]_\times = \begin{pmatrix} 0 & a_z & -a_y \\ -a_z & 0 & a_x \\ a_y & -a_x & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{d} = \mathtt{A} \cdot \mathbf{c} \quad \rightarrow \quad [\,\mathbf{d}\,]_\times = Det(\mathtt{A}) \cdot \mathtt{A}^{-\top} \cdot [\,\mathbf{c}\,]_\times \cdot \mathtt{A}^{-1} \tag{3}$$

2.2 A few basics on tensor calculus

A tensor is an indexed system of numbers. There are two kinds of indices: sub-indices are called *co-variant* and super-indices *contra-variant*. A tensor with contra-variant valence p and co-variant valence q has n^{p+q} components with n being the dimension of the underlying vector-space; i.e. each index runs from 1 to n. Using these indices and Einstein's convention of summation, certain mathematical relations can be expressed in a very efficient way. This convention says that a sum is made up of all the same indices appearing as co- and contra-variant. So, for example, the scalar product $s(\mathbf{x},\mathbf{y}) = \mathbf{x}^\top \cdot \mathbf{y}$ of two vectors $\mathbf{x} = (x_1\,x_2\,x_3)^\top$ and $\mathbf{y} = (y_1\,y_2\,y_3)^\top$ can be written in a shorter way as: $s(\mathbf{x},\mathbf{y}) = x_i\,y^i$. The product $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ can be written as $A^i_j B^j_k = C^i_k$. The contra-variant indices relate to the rows and the co-variant ones to the columns.

2.3 The trifocal tensor

If the orientation of three images ψ_1, ψ_2, ψ_3 is formulated according to equation (1) (with $\mathbf{O}_1 = \mathbf{0}$ and $\mathtt{R}_1 = \mathtt{E}_{3\times3}$) then the TFT can be represented in the following way (equation (4)); c.f. [Ressl 2000].

$$T_i^{j,k} = (\hat{\mathbf{v}}_{21})^j \cdot B_i^k - (\hat{\mathbf{v}}_{31})^k \cdot A_i^j \tag{4}$$

using:

$$\mathtt{A} = \mathtt{C}_2^{-1} \cdot \mathtt{R}_2^\top \cdot \mathtt{C}_1 \qquad \hat{\mathbf{v}}_{21} = -\mathtt{C}_2^{-1} \cdot \mathtt{R}_2^\top \cdot \mathbf{O}_2 \tag{5}$$

$$\mathtt{B} = \mathtt{C}_3^{-1} \cdot \mathtt{R}_3^\top \cdot \mathtt{C}_1 \qquad \hat{\mathbf{v}}_{31} = -\mathtt{C}_3^{-1} \cdot \mathtt{R}_3^\top \cdot \mathbf{O}_3 \tag{6}$$

$\hat{\mathbf{v}}_{xy}$ is the epipole of image ψ_y in image ψ_x, i.e. the image of $\mathbf{O}_y$ in image ψ_x. With ˆ it is emphasized that this epipole is represented as a homogenous vector but in a specific scale (observe the equality-symbol). The other epipoles are:

$$\hat{\mathbf{v}}_{12} = \mathtt{C}_1^{-1} \cdot \mathbf{O}_2 \qquad \hat{\mathbf{v}}_{13} = \mathtt{C}_1^{-1} \cdot \mathbf{O}_3 \tag{7}$$

$$\hat{\mathbf{v}}_{23} = \mathtt{C}_2^{-1} \cdot \mathtt{R}_2^\top \cdot \mathbf{O}_3 + \hat{\mathbf{v}}_{21} \tag{8}$$

$$\hat{\mathbf{v}}_{32} = \mathtt{C}_3^{-1} \cdot \mathtt{R}_3^\top \cdot \mathbf{O}_2 + \hat{\mathbf{v}}_{31} \tag{9}$$

3 THE TENSORIAL SLICES

One can imagine the trifocal tensor T_i^{jk} formed as a 3×3×3 cube of numbers and the cube's edges related to the indices i, j, k. If we keep one index fixed, we slice a 3×3 matrix out of the tensor. Since we have three indices, we get three different kinds of matrices - different also in their geometrical meaning. For didactical reasons we will start with the j and k index.

3.1 The homographic slices $\mathtt{J}_x$ and $\mathtt{K}_x$

If we keep the j-index in equation (4) fixed as $j = x \in \{1,2,3\}$, we get the following matrix $\mathtt{J}_x$ ($\mathbf{e}_x$ being the x^{th} column of $\mathtt{E}_{3\times3}$):

$$\mathtt{J}_x = \mathbf{e}_x^\top \cdot \hat{\mathbf{v}}_{21} \cdot \mathtt{B} - \hat{\mathbf{v}}_{31} \cdot \mathbf{e}_x^\top \cdot \mathtt{A} \tag{10}$$

$\mathtt{J}_x$ describes a mapping (a collineation) of points $\tilde{\mathbf{p}}_1$ in image ψ_1 to points $\tilde{\mathbf{p}}_3$ in image ψ_3 via the principal plane π_{2x}. In [Shashua, Werman 1995] this mapping is termed *homography*. The homography matrices $\mathtt{J}_x$ are distinguished by the properties shown in Table 1.

Analogously, if we keep the k-index fixed, we get a matrix $\mathtt{K}_x$, which describes a homography from image ψ_1 to image ψ_2 via the principal plane π_{3x}.

$$\mathtt{K}_x = \hat{\mathbf{v}}_{21} \cdot \mathbf{e}_x^\top \cdot \mathtt{B} - \mathbf{e}_x^\top \cdot \hat{\mathbf{v}}_{31} \cdot \mathtt{A} \tag{11}$$

3.2 The correlation slices $\mathtt{I}_x$

If we keep the i-index fixed, we get analogously a 3x3 matrix $\mathtt{I}_x$. For the homographic matrices $\mathtt{J}_x$ resp. $\mathtt{K}_x$ their form resulted directly from the co-variant (i) and contra-variant (k resp. j) indices in equation (4). When the i-index is fixed, only the contra-variant indices (j, k) remain, and therefore one of them has to be chosen for the columns. We choose the j-index.

$$\mathtt{I}_x = \mathtt{B} \cdot \mathbf{e}_x \cdot \hat{\mathbf{v}}_{21}^\top - \hat{\mathbf{v}}_{31} \cdot \mathbf{e}_x^\top \cdot \mathtt{A}^\top \tag{12}$$

$\mathtt{I}_x$ describes a mapping (a dual correlation) of lines ℓ_2 in image ψ_2 to points $\tilde{\mathbf{p}}_3$ in image ψ_3 via the principal ray r_{1x}. $\mathtt{I}_x^\top$ would map the lines ℓ_3 in image ψ_3 to points $\tilde{\mathbf{p}}_2$ in image ψ_2 via the principal ray r_{1x}.

Due to [Papadopoulo, Faugeras 1998] the correlation matrices $\mathtt{I}_x$ are distinguished by the properties shown in Table 2. Since the correlation slices are the basic input for the constraints and parameterization to be presented, they are investigated in more detail in section 5.1.

Note: The relations between the homographic and correlation slices are the following: The y^{th} column of $\mathtt{J}_x$ resp. $\mathtt{K}_x$ is the x^{th} column of $\mathtt{I}_y$ resp. $\mathtt{I}_y^\top$.

4 PREVIOUS CONSTRAINTS AND MINIMAL PARAMETERIZATIONS

Two sets of constraints and two minimal parameterizations (i.e. having 18 DOF) were discussed in the literature so far.

[Torr, Zisserman 1997] present a minimal parameterization for the TFT. By assigning projective canonical coordinates to the image and the space points, they show, that it is possible to compute the tensor from six homologous point triples across three images. Of the 36 observed

	Property	Remark
(a)	$\hat{\mathbf{v}}_{31} \sim \mathtt{J}_x \cdot \hat{\mathbf{v}}_{13}$	provided $rank(\mathtt{J}_x) = 3$
general eigenvalue problem $(\mathtt{J}_y - \mu \cdot \mathtt{J}_x) \cdot \tilde{\mathbf{x}} = \mathbf{0}$, $\quad x, y, z \in \{1, 2, 3\}$, pairwise diff.		
(b)	gen. eigenvalue $\mu_1 = \frac{\mathbf{e}_y^\top \cdot \tilde{\mathbf{v}}_{21}}{\mathbf{e}_x^\top \cdot \tilde{\mathbf{v}}_{21}}$	gen. eigenvector $\tilde{\mathbf{x}}_1 \sim \tilde{\mathbf{v}}_{13}$
(c)	gen. eigenvalue $\mu_2 = \mu_3 = \frac{\mathbf{e}_y^\top \cdot \tilde{\mathbf{v}}_{23}}{\mathbf{e}_x^\top \cdot \tilde{\mathbf{v}}_{23}}$	gen. eigenvector $\tilde{\mathbf{x}}_{2,3} \sim \alpha \cdot \tilde{\mathbf{v}}_{12} + \beta \cdot \mathtt{A}^{-1} \cdot \mathbf{e}_z$

Table 1: Properties of the homographic slices $\mathtt{J}_x$

	Property	Remark
(a)	$rank(\mathtt{I}_x) \leq 2$	
(b)	$rank(\sum_{x=1}^{3} a_x \cdot \mathtt{I}_x) \leq 2$	
(c)	$\mathtt{R}^\top \cdot \tilde{\mathbf{v}}_{21} = \mathbf{0}$	using $\mathtt{R} = [\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3]$ with $\mathtt{I}_x \cdot \tilde{\rho}_x = \mathbf{0}$
(d)	$\mathtt{L}^\top \cdot \tilde{\mathbf{v}}_{31} = \mathbf{0}$	using $\mathtt{L} = [\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3]$ with $\mathtt{I}_x^\top \cdot \tilde{\lambda}_x = \mathbf{0}$

Table 2: Properties of the correlation slices $\mathtt{I}_x$; c.f. [Papadopoulo, Faugeras 1998]

image-coordinates in six homologous triples, convenient 18 coordinates are kept fixed. In this way a minimal parameterization of the tensor is achieved. The unknowns themselves are obtained as (up to 3) solutions of a cubic equation. Due to this fixing of erroneous observations in the images one might be suspicious that errors in the calculated tensor may be induced, furthermore no correct minimization of the measurement-errors in all observations is possible. And as it is shown by the results in [Torr, Zisserman 1997] the standard deviation depends on the choice of the 6 points resp. the fixed 18 coordinates, which is not obvious in the beginning. However, this method of keeping the proper number of image-coordinates fixed, could be helpful also for other tasks, where a minimal parameterization is needed, but cannot be formulated easily.

[Papadopoulo, Faugeras 1998] introduce a minimal parameterization together with a set of 12 sufficient constraints - not minimal, since any number of constraints greater than eight must contain dependencies. Their set of constraints are entirely based on the correlation slices $\mathtt{I}_x$ and are made of the properties (b), (c), (d) shown in Table 2. Their minimal parameterization looks like the following: The left kernels of the correlation slices are parameterized using 2 parameters for their common epipole $\tilde{\mathbf{v}}_{31}$ and 1 parameter (a direction angle) for each kernel - thus 5 parameters in total. With other 5 parameters the right kernels and epipole $\tilde{\mathbf{v}}_{21}$ are parameterized. With the left and right kernels the correlation slices $\mathtt{I}_x$ can be parameterized by 8 coefficients. This way of parameterization results in a very large number of maps $(9 \cdot 3^2 \cdot 3^6)$ and it is not clear how this parameterization is applicable in case of $rank(\mathtt{I}_x) < 2$ - because the kernels need to be lines.

In [Canterakis 2000] the first set of minimal constraints is presented, which are entirely based on the homographic slices $\mathtt{J}_x$ and are derived from the properties shown in Table 1, i.e. each general eigenvalue problem set up with two homographic slices has one general eigenvalue with multiplicity 2 ($\mu_2 = \mu_3$) ($\rightarrow$ 1 constr.), the corresponding general eigenvector is 2-dimensional ($\rightarrow$ 2 constr.), the general eigenvector $\tilde{\mathbf{x}}_1$ corresponding to the single general eigenvalue μ_1 is the same (up to scale) for all three pairs of $\mathtt{J}_x$ ($\rightarrow$ 2 constr.). This general eigenvalue problem can be independently set up twice yielding the required number of 8 constraints. Open questions with this set of constraints are, how are they applicable in case of $rank(\mathtt{J}_x) < 3$ and how to implement them efficiently in a computer program (e.g. constraint ($\tilde{\mathbf{x}}_1$(of pair (x,y)) $\sim$ $\tilde{\mathbf{x}}_1$(of pair (x,z)) requires this general eigenvector to be expressed in terms of the 27 tensor elements).

In the following sections a new set of minimal constraints together with a minimal parameterization will be presented. Both are derived very easily, having very simple geometric properties. Their implementation is rather simple (actually the minimal parameterization is easier to realize than the constrained version).

5 A NEW MINIMAL SET OF CONSTRAINTS

The basic input for this set of constraints are the correlation slices $\mathtt{I}_x$, therefore we will take a closer look at these matrices.

5.1 The correlation slices $\mathtt{I}_x$ - Revisited

The correlation slices in equation (12) describe a mapping of lines $\tilde{\ell}_2$ in image ψ_2 to points $\tilde{\mathbf{p}}_3$ in image ψ_3 via the principal ray r_{1x}, meaning that $\tilde{\mathbf{p}}_3$ is the projection of the intersection point of r_{1x} with the projection plane of $\tilde{\ell}_2$.

In general, $rank(\mathtt{I}_x) = 2$, since the columns of $\mathtt{I}_x$ are linear combinations of two vectors ($\mathtt{B} \cdot \mathbf{e}_x$ and $\hat{\mathbf{v}}_{31}$) - or the rows are linear combination of two vectors ($\mathtt{A} \cdot \mathbf{e}_x$ and $\hat{\mathbf{v}}_{21}$). For the same reason, any linear combination of the correlation slices $\sum_{x=1}^{3} a_x \cdot \mathtt{I}_x$ will also have $rank = 2$ in general; c.f. [Papadopoulo, Faugeras 1998].

Using equ. ((5) - (9), (12)), we can find the cases where $rank(\mathtt{I}_x) < 2$:
$rank(\mathtt{I}_x) = 1$ will result if $\mathtt{B} \cdot \mathbf{e}_x \sim \hat{\mathbf{v}}_{31}$ ($\rightarrow$ $\mathbf{O}_3 \in r_{1x}$ and $\mathtt{I}_x \sim \tilde{\mathbf{v}}_{31} \cdot \tilde{\mathbf{v}}_{23}^\top$) or if ($\mathtt{A} \cdot \mathbf{e}_x \sim \hat{\mathbf{v}}_{21}$) ($\rightarrow$ $\mathbf{O}_2 \in r_{1x}$ and $\mathtt{I}_x \sim \tilde{\mathbf{v}}_{32} \cdot \tilde{\mathbf{v}}_{21}^\top$).

$rank(\mathtt{I}_x) = 0$ will result if $\mathtt{B}\cdot\mathbf{e}_x\cdot\hat{\mathbf{v}}_{21}^\top = \hat{\mathbf{v}}_{31}\cdot\mathbf{e}_x^\top\cdot\mathtt{A}^\top$ ($\rightarrow \mathbf{O}_2 \equiv \mathbf{O}_3 \in r_{1x}$ and $\mathtt{I}_x = \mathtt{0}$).

Since the correlation slices are *always* singular, there always exists a nontrivial null space. The right null space of $\mathtt{I}_x$ is a line $\tilde{\rho}_x$ in image ψ_2 in general. It represents the line for which the correlation matrix $\mathtt{I}_x$ yields no valid point (i.e. $\mathbf{0}$) in image ψ_3. Geometric reason: The projection plane due to $\tilde{\rho}_x$ contains the principal ray r_{1x}. Analogously, the left null space is a line $\tilde{\lambda}_x$ in image ψ_3 in general. It represents the line on which all mapped points $\mathtt{I}_x\cdot\tilde{\ell}_2$ lie. The lines $\tilde{\rho}_x$ and $\tilde{\lambda}_x$ are the epipolar lines of the principal ray r_{1x} in image ψ_2 and ψ_3, respectively. Epipolar lines always pass through the respective epipole ($\tilde{\mathbf{v}}_{21}$ resp. $\tilde{\mathbf{v}}_{31}$). Thus the matrices $\mathtt{L} = [\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3]$ and $\mathtt{R} = [\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3]$ are also of $rank = 2$ in general; c.f. [Papadopoulo, Faugeras 1998].

If one considers the pencil of lines with a carrier $\tilde{\mathbf{c}}_2 \in \tilde{\rho}_x$, then all of its lines ($\not\sim \tilde{\rho}_x$) map to the same point $\tilde{\mathbf{p}}_3$ on $\tilde{\lambda}_x$; i.e. $\tilde{\mathbf{p}}_3 = \tilde{\mathbf{p}}_3(\tilde{\mathbf{c}}_2)$. Thus, $\mathtt{I}_x$ represents a 1-dimensional collineation of the points $\tilde{\mathbf{c}}_2$ on $\tilde{\rho}_x$ in image ψ_2 to the points $\tilde{\mathbf{p}}_3$ on $\tilde{\lambda}_x$ in image ψ_3.[1] If we consider particularly the pencil of lines with carrier $\tilde{\mathbf{v}}_{21}$, then we can prove that all of its lines $\tilde{\ell}_{\tilde{\mathbf{v}}_{21}}$ ($\not\sim \tilde{\rho}_x$) are mapped to $\tilde{\mathbf{v}}_{31}$.

$$\tilde{\ell}_{\tilde{\mathbf{v}}_{21}} = [\,\tilde{\mathbf{v}}_{21}\,]_\times \cdot \tilde{\mathbf{p}}_2 \text{ with } \tilde{\mathbf{p}}_2 \not\sim \tilde{\mathbf{v}}_{21} \text{ and } \tilde{\mathbf{p}}_2 \notin \tilde{\rho}_x$$
$$\rightarrow \mathtt{I}_x \cdot \tilde{\ell}_{\tilde{\mathbf{v}}_{21}} \overset{\text{equ. (12)}}{=} -\hat{\mathbf{v}}_{31}\cdot \underbrace{\mathbf{e}_x^\top \cdot \mathtt{A}^\top \cdot [\,\tilde{\mathbf{v}}_{21}\,]_\times \cdot \tilde{\mathbf{p}}_2}_{\text{scalar } s} \qquad (13)$$
$$s \overset{\text{equ. (5)}}{=} \mathbf{e}_x^\top\cdot\mathtt{A}^\top\cdot[\,-\mathtt{C}_2^{-1}\cdot\mathtt{R}_2^\top\cdot\mathbf{O}_2\,]_\times\cdot\tilde{\mathbf{p}}_2$$
$$s \overset{\text{equ. (3)}}{\sim} \underbrace{\mathbf{e}_x^\top\cdot\mathtt{C}_1^\top\cdot[\mathbf{O}_2]_\times}_{=\mathbf{0}^\top\rightarrow\mathbf{O}_2\in r_{1x}}\cdot\underbrace{\mathtt{R}_2\cdot\mathtt{C}_2\cdot\tilde{\mathbf{p}}_2}_{\neq\mathbf{0}\leftarrow\tilde{\mathbf{p}}_2\not\sim\tilde{\mathbf{v}}_{21}} \neq 0 \leftarrow \tilde{\mathbf{p}}_2\notin\tilde{\rho}_x$$

So we see, that as long as $\mathbf{O}_2 \notin r_{1x}$ ($\rightarrow rank(\mathtt{I}_x) = 2$) any line $\tilde{\ell}_{\tilde{\mathbf{v}}_{21}} \not\sim \tilde{\rho}_x$ through the epipole $\tilde{\mathbf{v}}_{21}$ is mapped by any $\mathtt{I}_x$ to $\tilde{\mathbf{v}}_{31}$. Analogously, any line $\tilde{\ell}_{\tilde{\mathbf{v}}_{31}} \not\sim \tilde{\lambda}_x$ through the epipole $\tilde{\mathbf{v}}_{31}$ is mapped by any $\mathtt{I}_x^\top$ to $\tilde{\mathbf{v}}_{21}$ - provided $\mathbf{O}_3 \notin r_{1x}$. Table 3 summarizes these mapping properties (the others can be derived similarly to (13)) for $\mathtt{I}_x$ depending on its rank.

5.2 The minimal set of constraints

The underlying geometric properties become clearer, when we consider the columns of these matrices: $\mathtt{I}_1 = [\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}]$, $\mathtt{I}_2 = [\hat{\mathbf{d}}, \hat{\mathbf{e}}, \hat{\mathbf{f}}]$, $\mathtt{I}_3 = [\hat{\mathbf{g}}, \hat{\mathbf{h}}, \hat{\mathbf{i}}]$. Since $\mathtt{I}_x$ describes a mapping of lines $\tilde{\ell}_2$ in image ψ_2 to points $\tilde{\mathbf{p}}_3$ in image ψ_3 these columns can be interpreted as being points in image ψ_3 - the mappings of the ψ_2-lines $(1,0,0)^\top$, $(0,1,0)^\top$, $(0,0,1)^\top$ - (again ˆ emphasizes that these columns are projective points in a specific scale). It always holds that $rank(\mathtt{I}_x) \leq 2$. Thus we already have 3 constraints:

$$Det(\mathtt{I}_x) = 0 \qquad x \in \{1,2,3\} \qquad (14)$$

The geometric interpretation of (14) is, that the columns of $\mathtt{I}_x$ interpreted as image points are collinear: $\{\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}\} \in \tilde{\lambda}_1$, $\{\tilde{\mathbf{d}}, \tilde{\mathbf{e}}, \tilde{\mathbf{f}}\} \in \tilde{\lambda}_2$ and $\{\tilde{\mathbf{g}}, \tilde{\mathbf{h}}, \tilde{\mathbf{i}}\} \in \tilde{\lambda}_3$. These three lines $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\}$ are the left kernels of the correlation slices. Following section (5.1) we saw that these left kernels are epipolar lines, which always pass through the respective epipole ($\tilde{\mathbf{v}}_{31}$ in this case). This yields the next constraint.

[1] If the carrier $\tilde{\mathbf{c}}_2 \notin \tilde{\rho}_x$, then $\mathtt{I}_x$ describes a regular 1-dimensional correlation between the lines of this pencil and the points on $\tilde{\lambda}_x$.

The matrix $\mathtt{L}$ made of the three left kernels must be singular; provided all three correlation slices $\mathtt{I}_x$ have $rank = 2$ - otherwise the kernel of $\mathtt{I}_x$ will not be a line $\tilde{\lambda}_x$.

$$Det(\mathtt{L}) = 0 \qquad (15)$$

These 4 constraints have already been presented in e.g. [Papadopoulo, Faugeras 1998].

The 4 remaining constraints are new and will be explained in the following. In section (5.1) we saw, that *any* line $\ell_{\tilde{\mathbf{v}}_{31}} \not\sim \lambda_x$ through $\tilde{\mathbf{v}}_{31}$ is mapped by $\mathtt{I}_x^\top$ to the epipole $\tilde{\mathbf{v}}_{21}$ in image ψ_2; provided, $rank(\mathtt{I}_x) = 2$. So, we can formulate the following constraints:

$$\begin{array}{ll} (\mathtt{I}_q^\top - \mu_1\cdot\mathtt{I}_p^\top)\cdot\tilde{\ell}_{\tilde{\mathbf{v}}_{31}} = \mathbf{0} & {\scriptstyle p,q,r\in\{1,2,3\},} \\ (\mathtt{I}_r^\top - \mu_2\cdot\mathtt{I}_p^\top)\cdot\tilde{\ell}_{\tilde{\mathbf{v}}_{31}} = \mathbf{0} & {\scriptstyle \text{pairwise different}} \end{array} \qquad (16)$$

Obviously, the relations (16) produce 4 independent equations (6 equations - 2 additional unknown scales ($\mu_1\,\mu_2$)). However, it needs to be proven that theses relations are also independent of the determinant constraints (14) and (15).

If the determinant constraints are satisfied, then the correlation slices can be parameterized in the following way (without loss of generality):

$$\begin{array}{rcl} \mathtt{I}_1 = [\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}] & = & [\hat{\mathbf{s}}_1\,,\, v_1\cdot\hat{\mathbf{s}}_1 + j\cdot\check{\mathbf{v}}_{31}\,,\, w_1\cdot\hat{\mathbf{s}}_1 + s\cdot\check{\mathbf{v}}_{31}] \\ \mathtt{I}_2 = [\hat{\mathbf{d}}, \hat{\mathbf{e}}, \hat{\mathbf{f}}] & = & [\hat{\mathbf{s}}_2\,,\, v_2\cdot\hat{\mathbf{s}}_2 + k\cdot\check{\mathbf{v}}_{31}\,,\, w_2\cdot\hat{\mathbf{s}}_2 + t\cdot\check{\mathbf{v}}_{31}] \\ \mathtt{I}_3 = [\hat{\mathbf{g}}, \hat{\mathbf{h}}, \hat{\mathbf{i}}] & = & [\hat{\mathbf{s}}_3\,,\, v_3\cdot\hat{\mathbf{s}}_3 + l\cdot\check{\mathbf{v}}_{31}\,,\, w_3\cdot\hat{\mathbf{s}}_3 + u\cdot\check{\mathbf{v}}_{31}] \end{array} \qquad (17)$$

This parameterization just means, that the columns of the three matrices $\mathtt{I}_x$ are represented as linear combinations of a vector $\check{\mathbf{v}}_{31}$ common to all three matrices and individual vectors $\hat{\mathbf{s}}_x$. The vector $\check{\mathbf{v}}_{31}$ ($\sim \hat{\mathbf{v}}_{31}$) is the common perpendicular of the three left kernels $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\}$ - and its scale is chosen appropriately (hence ˇ instead of ˆ). If we choose any line $\ell_{\check{\mathbf{v}}_{31}}$ through $\check{\mathbf{v}}_{31}$, $\not\sim$ to any left kernel, we get:

$$\begin{array}{rcl} \mathtt{I}_1^\top\cdot\tilde{\ell}_{\tilde{\mathbf{v}}_{31}} & = & \begin{pmatrix} \hat{\mathbf{s}}_1^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \\ v_1\cdot\hat{\mathbf{s}}_1^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \\ w_1\cdot\hat{\mathbf{s}}_1^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \end{pmatrix} \sim \begin{pmatrix} 1 \\ v_1 \\ w_1 \end{pmatrix} \\ \mathtt{I}_2^\top\cdot\tilde{\ell}_{\tilde{\mathbf{v}}_{31}} & = & \begin{pmatrix} \hat{\mathbf{s}}_2^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \\ v_2\cdot\hat{\mathbf{s}}_2^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \\ w_2\cdot\hat{\mathbf{s}}_2^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \end{pmatrix} \sim \begin{pmatrix} 1 \\ v_2 \\ w_2 \end{pmatrix} \\ \mathtt{I}_3^\top\cdot\tilde{\ell}_{\tilde{\mathbf{v}}_{31}} & = & \begin{pmatrix} \hat{\mathbf{s}}_3^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \\ v_3\cdot\hat{\mathbf{s}}_3^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \\ w_3\cdot\hat{\mathbf{s}}_3^\top\cdot\tilde{\ell}_{\check{\mathbf{v}}_{31}} \end{pmatrix} \sim \begin{pmatrix} 1 \\ v_3 \\ w_3 \end{pmatrix} \end{array} \qquad (18)$$

Since all the right sides in (18) should be similar to the same vector (i.e. epipole $\tilde{\mathbf{v}}_{21}$), this only can be achieved, if for the coefficients holds: $v_1 = v_2 = v_3 = v$ and $w_1 = w_2 = w_3 = w$ - and thus the constraints (16) are independent of the determinant constraints (14) and (15).

Actually the constraints (16) correspond to the already known property (b) in Table 2 with $\mathbf{a} = (1, 0, \mu_1)$ resp. $\mathbf{a} = (0, 1, \mu_2)$. The constraints (16) hold for any line $\tilde{\ell}_{\tilde{\mathbf{v}}_{31}} \not\sim \tilde{\lambda}_x$ through $\tilde{\mathbf{v}}_{31}$, therefore the components parallel to $\tilde{\mathbf{v}}_{31}$ of the column vectors $(\mathtt{I}_q - \mu_1\cdot\mathtt{I}_p)$ resp. $(\mathtt{I}_r - \mu_2\cdot\mathtt{I}_p)$ are of no concern and only the components orthogonal to $\tilde{\mathbf{v}}_{31}$ need to be considered. Consequently we get a more preferable form for the constraints (16) by:

$$\tilde{\lambda}_1^\top\cdot[\,\check{\mathbf{v}}_{31}\,]_\times\cdot\mathtt{I}_1 \sim \tilde{\lambda}_2^\top\cdot[\,\check{\mathbf{v}}_{31}\,]_\times\cdot\mathtt{I}_2 \sim \tilde{\lambda}_3^\top\cdot[\,\check{\mathbf{v}}_{31}\,]_\times\cdot\mathtt{I}_3 \qquad (19)$$

$rank(\mathtt{I}_x)$	Properties for $\mathtt{I}_x$	Properties for $\mathtt{I}_x^\top$
2	$\mathtt{I}_x \cdot \tilde{\ell}_{v21} \sim \begin{cases} \mathbf{0} & \tilde{\ell}_{v21} \sim \tilde{\rho}_x \\ \tilde{\mathbf{v}}_{31} & \end{cases}$	$\mathtt{I}_x^\top \cdot \tilde{\ell}_{v31} \sim \begin{cases} \mathbf{0} & \tilde{\ell}_{v31} \sim \tilde{\lambda}_x \\ \tilde{\mathbf{v}}_{21} & \end{cases}$
1 : $\mathbf{O}_2 \in r_{1x} \qquad \mathtt{I}_x \sim \tilde{\mathbf{v}}_{32} \cdot \tilde{\mathbf{v}}_{21}^\top$ $\tilde{\rho}_x =$ pencil of lines in $\tilde{\mathbf{v}}_{21}$	$\mathtt{I}_x \cdot \tilde{\ell}_2 \sim \begin{cases} \mathbf{0} & \tilde{\ell}_2 \ni \tilde{\mathbf{v}}_{21} \\ \tilde{\mathbf{v}}_{32} & \end{cases}$	$\mathtt{I}_x^\top \cdot \tilde{\ell}_3 \sim \begin{cases} \mathbf{0} & \tilde{\ell}_3 \ni \tilde{\mathbf{v}}_{32} \\ \tilde{\mathbf{v}}_{21} & \end{cases}$
1 : $\mathbf{O}_3 \in r_{1x} \qquad \mathtt{I}_x \sim \tilde{\mathbf{v}}_{31} \cdot \tilde{\mathbf{v}}_{23}^\top$ $\tilde{\lambda}_x =$ pencil of lines in $\tilde{\mathbf{v}}_{31}$	$\mathtt{I}_x \cdot \tilde{\ell}_2 \sim \begin{cases} \mathbf{0} & \tilde{\ell}_2 \ni \tilde{\mathbf{v}}_{23} \\ \tilde{\mathbf{v}}_{31} & \end{cases}$	$\mathtt{I}_x^\top \cdot \tilde{\ell}_3 \sim \begin{cases} \mathbf{0} & \tilde{\ell}_3 \ni \tilde{\mathbf{v}}_{31} \\ \tilde{\mathbf{v}}_{23} & \end{cases}$

Table 3: Additional mapping properties of the correlation slices $\mathtt{I}_x$; c.f. Table 2

Summing it up, the presented minimal set of constraints, ensures, that the three correlation slices of a TFT are singular mappings of the lines from one image (ψ_2) to the points of another image (ψ_3) via three concurrent 3D-lines, which are made up by the principal lines of image ψ_1.

6 The minimal parameterization for the TFT

In the previous section we did not only prove, that (14), (15) and (19) constitute a minimal set of constraints for the TFT, but we also found a minimal parameterization for it. If we adopt the equality of coefficients ($v_1 = v_2 = v_3 = v$ and $w_1 = w_2 = w_3 = w$) to the parameterization (17), we get this minimal parameterization (having 18 DOF):

$$\begin{aligned} \mathtt{I}_1 &= [\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}] = [\hat{\mathbf{s}}_1\,,\ v \cdot \hat{\mathbf{s}}_1 + j \cdot \check{\mathbf{v}}_{31}\,,\ w \cdot \hat{\mathbf{s}}_1 + s \cdot \check{\mathbf{v}}_{31}] \\ \mathtt{I}_2 &= [\hat{\mathbf{d}}, \hat{\mathbf{e}}, \hat{\mathbf{f}}] = [\hat{\mathbf{s}}_2\,,\ v \cdot \hat{\mathbf{s}}_2 + k \cdot \check{\mathbf{v}}_{31}\,,\ w \cdot \hat{\mathbf{s}}_2 + t \cdot \check{\mathbf{v}}_{31}] \\ \mathtt{I}_3 &= [\hat{\mathbf{g}}, \hat{\mathbf{h}}, \hat{\mathbf{i}}] = [\hat{\mathbf{s}}_3\,,\ v \cdot \hat{\mathbf{s}}_3 + l \cdot \check{\mathbf{v}}_{31}\,,\ w \cdot \hat{\mathbf{s}}_3 + u \cdot \check{\mathbf{v}}_{31}] \end{aligned} \tag{20}$$

A few remarks need to be given:

- Obviously, this parameterization is not linear. Thus approximations are required, which can be obtained by an initial solution using the well-known eigen-value or linear solution for the TFT; e.g. [Hartley 1994].
- The scale of $\check{\mathbf{v}}_{31}$ needs to be fixed, e.g. by setting its length to 1.
- Observe, that the vectors $\{\hat{\mathbf{s}}_1\,, \hat{\mathbf{s}}_2\,, \hat{\mathbf{s}}_3\}$ in (20) parameterize the same column (index c_I) in the matrices $\mathtt{I}_x$. For numerical reasons, this index should be that one, for which the respective columns are *farthest away* from $\check{\mathbf{v}}_{31}$. This index c_I may be found by $\prod_{x=1}^{3} \|\mathtt{I}_x \cdot \mathbf{e}_{c_I} \times \check{\mathbf{v}}_{31}\| \to Max$.
- The overall scale in this parameterization needs also to be fixed, e.g. by setting the length of the concatenated vectors $\{\hat{\mathbf{s}}_1\,, \hat{\mathbf{s}}_2\,, \hat{\mathbf{s}}_3\}$ to 1. This yields in total 3 possible mappings; i.e. the choice of c_I.

With this parameterization (20) (i.e. $c_I = 1$) we get the homographic slices $\mathtt{J}_x$ as follows:

$$\begin{aligned} \mathtt{J}_1 &= [\hat{\mathbf{a}}, \hat{\mathbf{d}}, \hat{\mathbf{g}}] = [\hat{\mathbf{s}}_1\,,\ \hat{\mathbf{s}}_2\,,\ \hat{\mathbf{s}}_3] \\ \mathtt{J}_2 &= [\hat{\mathbf{b}}, \hat{\mathbf{e}}, \hat{\mathbf{h}}] = [v \cdot \hat{\mathbf{s}}_1 + j \cdot \check{\mathbf{v}}_{31}\,,\ v \cdot \hat{\mathbf{s}}_2 + k \cdot \check{\mathbf{v}}_{31}\,,\ v \cdot \hat{\mathbf{s}}_3 + l \cdot \check{\mathbf{v}}_{31}] \\ \mathtt{J}_3 &= [\hat{\mathbf{c}}, \hat{\mathbf{f}}, \hat{\mathbf{i}}] = [w \cdot \hat{\mathbf{s}}_1 + s \cdot \check{\mathbf{v}}_{31}\,,\ w \cdot \hat{\mathbf{s}}_2 + t \cdot \check{\mathbf{v}}_{31}\,,\ w \cdot \hat{\mathbf{s}}_3 + u \cdot \check{\mathbf{v}}_{31}] \end{aligned} \tag{21}$$

And so we can look at the general eigen-value problem $\mathtt{J}_2 - \mu \cdot \mathtt{J}_1$. It is easy to see, that $\mu = v$ yields a 2-dimensional general eigen-space, the line $(j\,k\,l)^\top$. Thus v is an eigen-value with multiplicity 2. This is in accordance with [Canterakis 2000]. And so we can summarize the geometrical interpretation of this minimal parameterization in the following way:

- $\check{\mathbf{v}}_{31}$ is the epipole of base $\overline{\mathbf{O}_1\mathbf{O}_3}$ in image ψ_3.
- $[\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_3]$ is a homography from image ψ_1 to image ψ_3; i.e. the homographic slice $\mathtt{J}_1$.
- $(1\,v\,w)^\top$ is the epipole $\tilde{\mathbf{v}}_{21}$ (of base $\overline{\mathbf{O}_1\mathbf{O}_2}$ in image ψ_2) - its component at position c_I is set to 1.
- $(j\,k\,l)^\top$ is the 2-dimensional general eigenspace of $(\mathtt{J}_2 - \mu \cdot \mathtt{J}_1)$. Since $\mathtt{J}_1$ resp. $\mathtt{J}_2$ is a homography due to π_{21} resp. π_{22}, the general eigenvector of $(\mathtt{J}_2 - \mu \cdot \mathtt{J}_1)$ must be the projection of the intersection of these two principal planes of image ψ_2; i.e. the projection of the principal ray r_{23} of image ψ_2 into image ψ_1.
- $(s\,t\,u)^\top$ is the 2-dimensional general eigenspace of $(\mathtt{J}_3 - \nu \cdot \mathtt{J}_1)$, i.e. the projection of the principal ray r_{22} of image ψ_2 into image ψ_1.

Of interest are the critical configurations for this minimal parameterization. Since it is part of the parameterization, that the length of $\check{\mathbf{v}}_{31}$ and one component in $\tilde{\mathbf{v}}_{21}$ are set to 1, problems surely arise if either of these epipoles is the zero-vector $\to$ $\mathbf{O}_1 = \mathbf{O}_3$ resp. $\mathbf{O}_1 = \mathbf{O}_2$. This problem can be solved - as long as not all three projection centers coincide - by changing the role of the images in the way that the image with the unique projection center plays the role of image ψ_1. Note: The identity of two or all three projection centers might be of practical relevance during the work with a moving camera acquiring images in a constant frequency and which stops at a particular position for a moment. In case of $\mathbf{O}_1 = \mathbf{O}_2 = \mathbf{O}_3$ the respective TFT becomes the zero-tensor.

Another problem with this parameterization could come from the fact, that the vectors $\{\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_3\}$ parameterize the same column (with index c_I) in all three correlation slices. Still keep in mind that we choose the best column for this parameterization - the one that is farthest away from $\check{\mathbf{v}}_{31}$. If we take the minimal parameterization exactly as it is given in equation (20), we see, that the columns of $\mathtt{I}_1$ are parameterized by $\check{\mathbf{v}}_{31}$ and the vector $\hat{\mathbf{s}}_1$. Thus it must be assured, that $\hat{\mathbf{s}}_1$ is different from $\check{\mathbf{v}}_{31}$ and different from the zero-vector, because otherwise the column vectors $\hat{\mathbf{b}}$ and/or $\hat{\mathbf{c}}$ (being different from $\check{\mathbf{v}}_{31}$ and $\mathbf{0}$) can not be parameterized by $\hat{\mathbf{s}}_1$ and $\check{\mathbf{v}}_{31}$. Of course, if $\hat{\mathbf{b}}$ *and* $\hat{\mathbf{c}}$ are similar to $\check{\mathbf{v}}_{31}$ or $\mathbf{0}$, than we would have no problem. So, we will prove, that the first case can not occur. This prove is outlined in the following.

First we arrange the three matrices $\mathtt{I}_x$ as the rows of a large matrix Z, which then has 9 rows and 3 columns. We, however, consider the elements of Z being the column vectors of the $\mathtt{I}_x$ matrices. So, Z has 3×3 elements and the element at row ξ and col η is the η^{th} column vector in matrix $\mathtt{I}_\xi$, which is the image of the intersection point of principal plane $\pi_{2\eta}$ with the principal ray $r_{1\xi}$.

Now, we consider that one element (row ξ, col η) of Z shall be $\sim \check{\mathbf{v}}_{31}$. This may happen due to two situations: A1) $\mathbf{O}_3 \in r_{1\xi}$ or A2) $\mathbf{O}_1 \in \pi_{2\eta}$. Then, we consider that one element (row ξ, col η) of Z shall be $= \mathbf{0}$. This may happen also due to two situations: B1) $\mathbf{O}_3 = \{\pi_{2\eta} \cap r_{1\xi}\}$ or B2) $r_{1\xi} \in \pi_{2\eta}$. However, B1) implies $\mathbf{O}_3 \in r_{1\xi}$ ($\rightarrow$ A1)) and B2) implies $\mathbf{O}_1 \in \pi_{2\eta}$ ($\rightarrow$ A2)). Thus, the only possible situations, that may return one element in Z being $\sim \check{\mathbf{v}}_{31}$ or $= \mathbf{0}$ are the ones of A1) and A2).

These situations, however, not only return the element at row ξ and col η of matrix Z to be $\sim \check{\mathbf{v}}_{31}$ or $= \mathbf{0}$, they further imply: A1) returns that all elements in row ξ of Z are $\sim \check{\mathbf{v}}_{31}$; i.e. the entire matrix $\mathtt{I}_\xi$. And so all columns $\mathtt{I}_\xi$ can be parameterized by $\hat{\mathbf{s}}_\xi$ (being $\sim \check{\mathbf{v}}_{31}$ or $= \mathbf{0}$) and $\check{\mathbf{v}}_{31}$. A2) returns that all elements in column η of Z are $\sim \check{\mathbf{v}}_{31}$; i.e. the η^{th} column in all three correlation matrices $\mathtt{I}_x$. Again, the parameterization of these columns is not difficult, but what's more important: When situation A2) occurs, the η^{th} column in the three correlation matrices $\mathtt{I}_x$ will never be used as the vectors $\{\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_3\}$ in the parameterization, since they are not far away from $\check{\mathbf{v}}_{31}$.

This completes the prove, that it is impossible, that one of the three vectors $\{\hat{\mathbf{s}}_x\}$ is $\sim \check{\mathbf{v}}_{31}$ or $= \mathbf{0}$, but one of the other columns in $\mathtt{I}_x$ is different from $\check{\mathbf{v}}_{31}$ and $\mathbf{0}$. Thus, the minimal parameterization (20) holds for any image configuration - provided not all three projection centers coincide.

7 SUMMARY AND FUTURE WORK

In this paper a new minimal set of constraints as well as a new minimal parameterization for the trifocal tensor (TFT) were presented. They were found using the so-called *correlation slices* $\mathtt{I}_x$ together with a new discovered property of them (equ. (19)). Especially the minimal parameterization, which is applicable for any image configuration (provided not all three projection centers coincide), will help to get new insights into the geometric relations and properties of the TFT. With these constraints resp. minimal parameterization it is possible to compute the TFT with minimal (i.e. 18) DOF. Since both rely on non-linear relations an initial solution for the TFT is required; e.g. using the well-known linear solution.

So far, the presented constraints and the minimal parameterization have been implemented and it will be among the future work to investigate the advantages of each method. The experiments so far show a benefit for the minimal parameterized solution (equ. (20)), which can be implemented rather simple and works for all practical image configurations - as opposed to the constrained solution, which relies on correlation slices having $rank = 2$ for the constraints (15) and (19).

Also of interest are the additional constraints resp. the minimal parameterization that arise when the interior orientation of the images is known, or if it is unknown but common to all three images. The latter is of special interest for camera calibration, which needs at least three images taken by the same camera; e.g. [Hartley 1997].

During the future work we will also investigate, what amount of error is induced in the resulting TFT (and thus in the image orientation), when the constraints are neglected and/or algebraic error is minimized instead of measurement error. This is especially of interest when the TFT-solution serves only as an initial start for a subsequent bundle-adjustment, since there already the linear solution might be sufficient.

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