Nonadditivity of Quantum Phases for mixed States

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Abstract — *In a neutron polarimetry experiment mixed state relative phases between spin eigenstates are determined. We consider evolutions leading to purely geometric, purely dynamical and combined phases. It is experimentally demonstrated that the sum of the geometric and dynamical phases - both obtained in separate measurements - is not equal to the associated total phase which is obtained from a single measurement, unless the system is in a pure state. Therefore, surprisingly, mixed state phases are not additive.*

I. Introduction

Evolving quantum systems acquire two kinds of phase factors: (*i*) the dynamical phase which depends on the dynamical properties of the system - like energy or time - during a particular evolution, and (*ii*) the geometric phase which only depends on the evolution path the system takes in state space on its way from the initial to the final state [1, 2]. Due to its robustness against noise [3] the geometric phase is an excellent candidate to be utilized for logic gate operations in quantum information science [4]. Thus, a rigorous investigation of all its properties is of great importance.

In addition to an approach by Uhlmann [5] a new concept of phase for mixed input states based on interferometry was developed by Sjöqvist *et al.* [6]. Here, each eigenvector of the initial density matrix independently acquires a geometric phase. The total mixed state phase is a weighted average of the individual phase factors. This concept is of great significance for such experimental situations or technical applications where pure state theories may imply strong idealizations. Theoretical predictions have been tested using NMR and single-photon interferometry [7, 8]. Here, we report on measurements of nonadiabatic and noncyclic geometric, dynamical and combined phases. These depend on noise strength in state preparation, defining the degree of polarization, the purity, of the neutron input state. In particular, our experiment demonstrates for the first time that the geometric and dynamical mixed state phases Φ_g and Φ_d, resulting from separate measurements, are not additive [9], because the phase resulting from a single, cumulative, measurement differs from $\Phi_g + \Phi_d$. These striking results are published in [10].

II. Theoretical Considerations

A polarized neutron beam propagating in y-direction, interacting with static magnetic fields $\vec{B}(y)$, undergoes Zeeman splitting. This results in a momentum shift $k_\pm \simeq k_0 \mp \Delta k$, where k_0 is the momentum of the free particle and $\Delta k = m\mu|\vec{B}(y)|/\hbar^2 k$. Δk can be detected from spinor precession. We focus on the resulting evolution of superposed spin eigenstates sometimes described as Larmor precession of the polarization vector $\vec{r} = \langle\varphi|\vec{\sigma}|\varphi\rangle$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli vector operator.

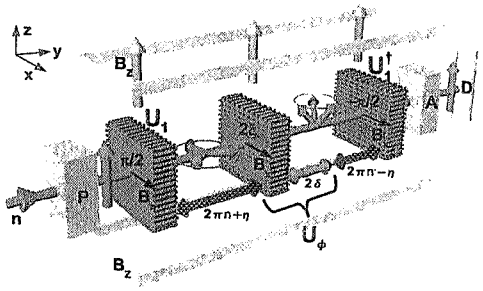

Figure 1: Schematic view of the measurement setup with overall guide field B_z, polarizer P, three DC-coils to implement unitary operations U_1, $U_1^\dagger$, U_ϕ, analyzer A and detector D. Greek letters denote spin rotation angles. Shifting the second coil induces an additional dynamical phase η resulting in intensity oscillations. The desired phase ϕ is determined from their minima and maxima.

Consider the experimental setup shown in Figure 1. In [11] it is stated that with such an apparatus one can obtain the pure state Pancharatnam (total) phase [1] between spin eigenstates of neutrons, induced by a SU(2) transformation $U_\phi(\xi, \delta, \zeta)$. The phase can be written as a function of the maximum I_{max} and minimum I_{min} of the intensity oscillations, exhibited by applying an auxiliary phase shift η. The intensity only depends on the SU(2) parameters ξ, δ and ζ - set by choosing the spin rotation angles in the second coil and the additional propagation distance within the guide field B_z, respectively. η is varied by stepwise translation of the second coil.

A neutron beam with incident purity $r = |\vec{r}|$ along the $+z$-axis ($\vec{r} = (0, 0, r)$) is described by the density operator $\rho_{in}(r) = 1/2(\mathbb{1} + r\sigma_z)$. For mixed input states, $0 \leq r < 1$. In this case we find the intensity to be proportional to

$$\frac{1-r}{2} + r\left(\cos^2\xi\cos^2\delta + \sin^2\xi\cos^2(\zeta-\eta)\right). \quad (1)$$

Considering again the maxima and minima of the inten-

sity, one obtains the mixed state phase [12]. For $r = 1$,

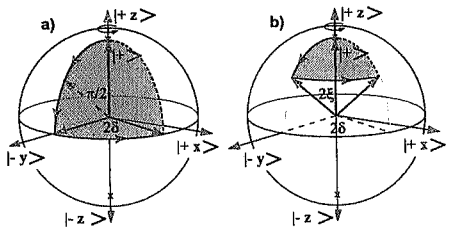

Figure 2: Evolutions U_ϕ associated to: a) Purely (noncyclic) geometric phase ($2\xi = \pi/2$). b) Combinations of dynamical and geometric phase ($0 < 2\xi < \pi/2$).

Equation (1) reduces to the pure state intensity.

Generally, the noncyclic geometric phase is given by $\phi_g = -\Omega/2$, where Ω is the solid angle enclosed by a geodesic path and its shortest geodesic closure on the Bloch sphere [13]: ϕ_g and the total phase ϕ are related to the path by the polar and azimuthal angles 2ξ and 2δ, so that the pure state geometric phase becomes

$$\phi_g = \phi - \phi_d = \delta[1 - \cos(2\xi)]. \quad (2)$$

ϕ_d is the dynamical phase. By proper choice of 2ξ and 2δ, U_ϕ can be set to generate purely geometric, purely dynamical, or arbitrary combinations of both phases, as is shown in Figure 2.

The theoretical prediction for the mixed state phase is [6, 12]

$$\Phi(r) = \arctan(r \tan\delta) \quad (3)$$

III. Experiment

To access (3) experimentally r needs to be varied. In addition to the DC current, which effects the transformation U_1, random noise is applied to the first coil, thereby changing B_x in time. Neutrons, which are part of the ensemble $\rho_{in}(r)$, arrive at different times at the coil and experience different magnetic field strengths. The system ends up in a mixed state, with $r < 1$.

The experiment was carried out at the research reactor facility of the Vienna University of Technology. The experimental data reproduce well the r-dependence predicted by (3) [10].

IV. Nonadditivity

Furthermore, our experiment focuses on a special property of the mixed state phase: its nonadditivity. The Sjöqvist mixed state phase is defined as a weighted average of phase factors rather then one of phases, so it is true only for pure states that phases of separate measurements can be added up to the usual total phase. Suppose we carry out two measurements on a pure state system: the input state is subjected to a unitary transformation U_g in the first and to U_d in the second measurement, inducing the pure state phases ϕ_g and ϕ_d, respectively. Applying (2) we can also choose a combination of angles 2ξ and 2δ leading to a transformation U_{tot}, so that we measure the total pure state phase $\phi_g + \phi_d$ (note that the three evolution paths induced by U_g, U_d and U_{tot} differ from each other). However, the result of the latter experiment for the system in a mixed state is $\Phi_{tot}(r) = \arctan[r \tan(\phi_g + \phi_d)]$. The total phase is then *not* given by $\Phi_g(r) + \Phi_d(r)$, with $\Phi_g(r) = \arctan(r \tan\phi_g)$ and $\Phi_d(r) = \arctan(r \tan\phi_d)$. Two examples of related data clearly exhibiting this effect are shown in Figure 3.

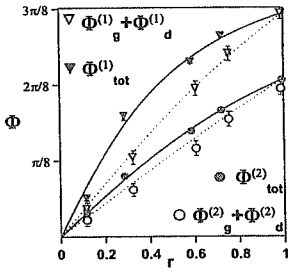

Figure 3: Filled markers: Measured total mixed state phase Φ_{tot} versus purity r for two examples of U_{tot} associated to the total pure state phases $\phi_g^{(1,2)} + \phi_d^{(1,2)}$ (see text). Open markers: $\Phi_g^{(1,2)} + \Phi_d^{(1,2)}$ as calculated from measured data. The solid and dotted theory curves assume either nonadditivity or additivity, respectively.

This surprising feature is not expected by straightforward extrapolation of phase concepts from familiar pure state behaviour. Since, in real experiments, absolute pure states do not exist, nonadditivity is crucial for possible applications of quantum phases.

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