Gravity in lower dimensions

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Center for Theoretical Physics, Massachusetts Institute of Technology, December 2008
Outline

Why lower-dimensional gravity?

Which 2D theory?

Which 3D theory?

How to quantize 3D gravity?

What next?
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What next?
Quantum gravity
The Holy Grail of theoretical physics

There is a lot we do know about quantum gravity already

- It should exist in some form
- String theory: (perturbative) theory of quantum gravity
- Microscopic understanding of extremal BH entropy
- Conceptual insight — information loss problem resolved

There is a lot we still do not know about quantum gravity

- Reasonable alternatives to string theory?
- Non-perturbative understanding of quantum gravity?
- Microscopic understanding of non-extremal BH entropy?
- Experimental signatures? Data?
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Gravity in lower dimensions

Riemann-tensor $\frac{D^2(D^2-1)}{12}$ components in $D$ dimensions:

- 11D: 1210 (1144 Weyl and 66 Ricci)
- 10D: 825 (770 Weyl and 55 Ricci)
- 5D: 50 (35 Weyl and 15 Ricci)
- 4D: 20 (10 Weyl and 10 Ricci)

Simplest gravitational theories with BHs in 2D

Simplest gravitational theories with BHs and gravitons in 3D
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What next?
Attempt 1: Einstein–Hilbert in and near two dimensions

Let us start with the simplest attempt. Einstein–Hilbert action in 2 dimensions:

\[ I_{EH} = \frac{1}{16\pi G} \int d^2 x \sqrt{|g|} R = \frac{1}{2G} (1 - \gamma) \]

- Action is topological
- No equations of motion
- Formal counting of number of gravitons: -1
 Attempt 1: Einstein–Hilbert in and near two dimensions

Let us continue with the next simplest attempt. Einstein–Hilbert action in $2 + \epsilon$ dimensions:

$$ I_{EH}^\epsilon = \frac{1}{16\pi G} \int d^{2+\epsilon} x \sqrt{|g|} \, R $$

- **Weinberg**: theory is asymptotically safe
- **Mann**: limit $\epsilon \to 0$ should be possible and lead to 2D dilaton gravity
- **DG, Jackiw**: limit $\epsilon \to 0$ yields Liouville gravity

$$ \lim_{\epsilon \to 0} I_{EH}^\epsilon = \frac{1}{16\pi G_2} \int d^2 x \sqrt{|g|} \left[ XR - (\nabla X)^2 + \lambda e^{-2X} \right] $$
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**Result of attempt 1:**

A specific 2D dilaton gravity model
Attempt 2: Gravity as a gauge theory and the Jackiw-Teitelboim model

Jackiw, Teitelboim (Bunster): (A)dS_2 gauge theory

\[ [P_a, P_b] = \Lambda \epsilon_{ab} J \quad [P_a, J] = \epsilon_a^b P_b \]

describes constant curvature gravity in 2D.

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- Construct non-abelian BF theory

$$I = \int X_A F^A = \int \left[ X_a (de^a + \epsilon^a_b \omega \wedge e^b) + X d\omega + \epsilon_{ab} e^a \wedge e^b \Lambda X \right]$$
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- Eliminate $X_a$ (Torsion constraint) and $\omega$ (Levi-Civita connection)
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- Obtain the second order action

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Result of attempt 2:
A specific 2D dilaton gravity model
Attempt 3: Dimensional reduction
For example: spherical reduction from $D$ dimensions

Line element adapted to spherical symmetry:

$$ds^2 = \underbrace{g^{(D)}_{\mu\nu}}_{\text{full metric}} \, dx^\mu \, dx^\nu = \underbrace{g_{\alpha\beta}(x^\gamma)}_{\text{2D metric}} \, dx^\alpha \, dx^\beta - \underbrace{\phi^2(x^\alpha)}_{\text{surface area}} \, d\Omega_{S_{D-2}}^2,$$
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Insert into $D$-dimensional EH action $I_{EH} = \kappa \int d^Dx \sqrt{-g(D)} R(D)$:
\[
I_{EH} = \kappa \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \int d^2x \sqrt{-g} \phi^{D-2} \left[ R + \frac{(D-2)(D-3)}{\phi^2} \left( (\nabla \phi)^2 - 1 \right) \right]
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\]

Cosmetic redefinition $X \propto (\lambda \phi)^{D-2}$:

\[
I_{EH} = \frac{1}{16\pi G_2} \int \text{d}^2 x \sqrt{-g} \left[ X R + \frac{D-3}{(D-2)X} (\nabla X)^2 - \lambda^2 X^{(D-4)/(D-2)} \right]
\]

Result of attempt 3:

A specific class of 2D dilaton gravity models
Attempt 4: Poincare gauge theory and higher power curvature theories

Basic idea: since EH is trivial consider $f(R)$ theories or/and theories with torsion or/and theories with non-metricity
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- Example: Katanaev–Volovich model (Poincare gauge theory)

$$ I_{KV} \sim \int d^2 x \sqrt{-g} \left[ \alpha T^2 + \beta R^2 \right] $$

- Kummer, Schwarz: bring into first order form:

$$ I_{KV} \sim \int \left[ X_a (de^a + \epsilon_{ab}^a \omega \wedge e^b) + X d\omega + \epsilon_{ab}^a e^a \wedge e^b (\alpha X^a X_a + \beta X^2) \right] $$

- Use same algorithm as before to convert into second order action:

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Result of attempt 4:

A specific 2D dilaton gravity model
Attempt 5: Strings in two dimensions

Conformal invariance of the $\sigma$ model

$$I_\sigma \propto \int d^2 \xi \sqrt{|h|} \left[ g_{\mu\nu} h^{ij} \partial_i x^\mu \partial_j x^\nu + \alpha' \phi R + \ldots \right]$$

requires vanishing of $\beta$-functions

$$\beta^\phi \propto -4b^2 - 4(\nabla \phi)^2 + 4\Box \phi + R + \ldots$$

$$\beta^{g}_{\mu\nu} \propto R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi + \ldots$$

Conditions $\beta^\phi = \beta^{g}_{\mu\nu} = 0$ follow from target space action

$$I_{\text{target}} = \frac{1}{16\pi G_2} \int d^2 x \sqrt{-g} \left[ X R + \frac{1}{X} (\nabla X)^2 - 4b^2 \right]$$

where $X = e^{-2\phi}$
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Result of attempt 5:

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Synthesis of all attempts: Dilaton gravity in two dimensions

Second order action:

\[ I = \frac{1}{16\pi G_2} \int_\mathcal{M} d^2x \sqrt{|g|} \left[ X R - U(X)(\nabla X)^2 - V(X) \right] \]

\[ - \frac{1}{8\pi G_2} \int_{\partial \mathcal{M}} dx \sqrt{|\gamma|} \left[ X K - S(X) \right] + I^{(m)} \]
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- Dilaton $X$ defined by its coupling to curvature $R$
- Kinetic term $(\nabla X)^2$ contains coupling function $U(X)$
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- Hamilton–Jacobi counterterm contains superpotential $S(X)$

\[ S(X)^2 = e^{-\int_X U(y) \, dy} \int_X^X V(y)e^{\int_y^z U(z) \, dz} \, dy \]

and guarantees well-defined variational principle $\delta I = 0$
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- **Interesting option**: couple 2D dilaton gravity to matter
Recent example: AdS$_2$ holography
Two dimensions supposed to be the simplest dimension with geometry, and yet...

- extremal black holes universally include AdS$_2$ factor
- funnily, AdS$_3$ holography more straightforward
- study charged Jackiw–Teitelboim model as example

\[
I_{JT} = \frac{\alpha}{2\pi} \int d^2 x \sqrt{-g} \left[ e^{-2\phi} \left( R + \frac{8}{L^2} \right) - \frac{L^2}{4} F^2 \right]
\]

Metric $g$ has signature $- , +$ and Ricci-scalar $R < 0$ for AdS
Maxwell field strength $F_{\mu\nu} = 2E \varepsilon_{\mu\nu}$ dual to electric field
Dilaton $\phi$ has no kinetic term and no coupling to gauge field
Cosmological constant $\Lambda = -\frac{8}{L^2}$ parameterized by AdS radius $L$
Coupling constant $\alpha$ usually is positive
$\delta\phi$ EOM: $R = -\frac{8}{L^2} \Rightarrow$ AdS$_2$!
$\delta A$ EOM: $\nabla_\mu F_{\mu\nu} = 0 \Rightarrow E = \text{constant}$
$\delta g$ EOM:
\[
\nabla_\mu \nabla_\nu e^{-2\phi} - g_{\mu\nu} \nabla^2 e^{-2\phi} + \frac{4}{L^2} g_{\mu\nu} - \frac{L^2}{4} F_{\mu\lambda} F_{\nu\lambda} - \frac{L^2}{8} g_{\mu\nu} F^2 = 0
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- Metric $g$ has signature $- , +$ and Ricci-scalar $R < 0$ for AdS
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- $\delta g$ EOM: complicated for non-constant dilaton...

\[ \nabla_\mu \nabla_\nu e^{-2\phi} - g_{\mu\nu} \nabla^2 e^{-2\phi} + \frac{4}{L^2} e^{-2\phi} g_{\mu\nu} + \frac{L^2}{2} F_{\mu}^\lambda F_{\nu\lambda} - \frac{L^2}{8} g_{\mu\nu} F^2 = 0 \]
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- $\delta g$ EOM: ...but simple for constant dilaton: $e^{-2\phi} = \frac{L^4}{4} E^2$

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Some surprising results
Hartman, Strominger = HS     Castro, DG, Larsen, McNees = CGLM

- Holographic renormalization leads to boundary mass term (CGLM)

\[ I \sim \int dx \sqrt{|\gamma|} mA^2 \]

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- Boundary stress tensor transforms anomalously (HS)

\[ (\delta \xi + \delta \lambda) T_{tt} = 2T_{tt} \partial_t \xi + \xi \partial_t T_{tt} - \frac{c}{24\pi} L \partial_t^3 \xi \]

where \( \delta \xi + \delta \lambda \) is combination of diffeo- and gauge trasfos that preserve the boundary conditions (similarly: \( \delta \lambda J_t = -\frac{k}{4\pi} L \partial_t \lambda \))
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- Positive central charge only for negative coupling constant \(\alpha\) (CGLM)

  \[ \alpha < 0 \]
Outline

Why lower-dimensional gravity?

Which 2D theory?

Which 3D theory?

How to quantize 3D gravity?

What next?
Attempt 1: Einstein–Hilbert

As simple as possible... but not simpler!

Let us start with the simplest attempt. Einstein–Hilbert action:

\[ I_{EH} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \, R \]

Equations of motion:

\[ R_{\mu\nu} = 0 \]

Ricci-flat and therefore Riemann-flat – locally trivial!
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Properties of Einstein–Hilbert

- No gravitons (recall: in $D$ dimensions $D(D - 3)/2$ gravitons)
- No BHs
- Einstein–Hilbert in 3D is too simple for us!
Attempt 2: Topologically massive gravity
Deser, Jackiw and Templeton found a way to introduce gravitons!

Let us now add a gravitational Chern–Simons term. TMG action:

\[ I_{TMG} = I_{EH} + \frac{1}{16\pi G} \int d^3 x \sqrt{-g} \frac{1}{2\mu} \varepsilon^{\lambda \mu \nu} \Gamma^\rho_{\lambda \sigma} \left( \partial_\mu \Gamma^\sigma_{\nu \rho} + \frac{2}{3} \Gamma^\sigma_{\mu \tau} \Gamma^\tau_{\nu \rho} \right) \]

Equations of motion:

\[ R_{\mu \nu} + \frac{1}{\mu} C_{\mu \nu} = 0 \]

with the Cotton tensor defined as

\[ C_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu}^{\alpha \beta} \nabla_\alpha R_{\beta \nu} + (\mu \leftrightarrow \nu) \]
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\]

Properties of TMG

- Gravitons! Reason: third derivatives in Cotton tensor!
- No BHs
- TMG is slightly too simple for us!
Attempt 3: Einstein–Hilbert–AdS
Bañados, Teitelboim and Zanelli (and Henneaux) taught us how to get 3D BHs

Add negative cosmological constant to Einstein–Hilbert action:

\[
I^{\Lambda_{EH}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right)
\]

Equations of motion:

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{\ell^2} g_{\mu\nu} = 0
\]

Particular solutions: BTZ BH with line-element

\[
ds^2_{\text{BTZ}} = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \, dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} \, dr^2 + r^2 \left( d\phi - \frac{r_+ + r_-}{\ell r^2} \, dt \right)^2
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Equations of motion:

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$$ds_{BTZ}^2 = -\frac{(r^2 - r_+^2)(r_+^2 - r_-^2)}{\ell^2 r^2} \, dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r_+^2 - r_-^2)} \, dr^2 + r^2 \left( d\phi - \frac{r+r_-}{\ell r^2} \, dt \right)^2$$

Properties of Einstein–Hilbert–AdS

- No gravitons
- Rotating BH solutions that asymptote to AdS$_3$!
- Adding a negative cosmological constant produces BH solutions!
Cosmological topologically massive gravity

CTMG is a 3D theory with BHs and gravitons!

We want a 3D theory with gravitons and BHs and therefore take CTMG action

\[
I_{\text{CTMG}} = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left[ R + \frac{2}{\ell^2} + \frac{1}{2\mu} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^\rho \left( \partial_\mu \Gamma_{\nu\rho}^\sigma + \frac{2}{3} \Gamma_{\sigma\mu\tau} \Gamma_{\nu\rho}^\tau \right) \right]
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Einstein sector of the classical theory

Solutions of Einstein’s equations

\[ G_{\mu\nu} = 0 \quad \leftrightarrow \quad R = -\frac{6}{\ell^2} \]

also have vanishing Cotton tensor

\[ C_{\mu\nu} = 0 \]

and therefore are solutions of CTMG.
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This sector of solutions contains

- BTZ BH
- Pure AdS
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Line-element of pure AdS:

\[ ds^2_{\text{AdS}} = \bar{g}_{\mu \nu} \, dx^\mu \, dx^\nu = \ell^2 \left( - \cosh^2 \rho \, d\tau^2 + \sinh^2 \rho \, d\phi^2 + d\rho^2 \right) \]

Isometry group: \( SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R \)

Useful to introduce light-cone coordinates \( u = \tau + \phi, \, v = \tau - \phi \)
AdS_3-algebra of Killing vectors

A technical reminder

The $SL(2, \mathbb{R})_L$ generators

$$L_0 = i \partial_u$$

$$L_{\pm 1} = i e^{\pm i u} \left[ \frac{\cosh 2\rho}{\sinh 2\rho} \partial_u - \frac{1}{\sinh 2\rho} \partial_v \mp \frac{i}{2} \partial_\rho \right]$$

obey the algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0$$

and have the quadratic Casimir

$$L^2 = \frac{1}{2} (L_1 L_{-1} + L_{-1} L_1) - L_0^2$$

The $SL(2, \mathbb{R})_R$ generators $\bar{L}_0, \bar{L}_{\pm 1}$ obey same algebra, but with

$$u \leftrightarrow v, \quad L \leftrightarrow \bar{L}$$
Cotton sector of the classical theory

Solutions of CTMG with

\[ G_{\mu\nu} \neq 0 \]

necessarily have also non-vanishing Cotton tensor

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Few exact solutions of this type are known.
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Perhaps most interesting solution:

▶ Warped AdS (stretched/squashed), see Bengtsson & Sandin

Line-element of space-like warped AdS:

\[
\text{d}s^2_{\text{warped AdS}} = \frac{\ell^2}{\nu^2 + 3} \left( -\cosh^2 \rho \, \text{d}\tau^2 + \frac{4\nu^2}{\nu^2 + 3} (\text{d}u + \sinh \rho \, \text{d}\tau)^2 + \text{d}\rho^2 \right)
\]

Sidenote: null-warped AdS in holographic duals of cold atoms:

\[
\text{d}s^2_{\text{null warped AdS}} = \ell^2 \left( \frac{\text{d}y^2 + 2 \, \text{d}x^+ \, \text{d}x^-}{y^2} \pm \frac{(\text{d}x^-)^2}{y^4} \right)
\]
CTMG as particle mechanics problem
Stationary and axi-symmetric solutions

Stationarity plus axi-symmetry:
  ▶ Two commuting Killing vectors
CTMG as particle mechanics problem
Stationary and axi-symmetric solutions

Stationarity plus axi-symmetry:

- Two commuting Killing vectors
- Effectively reduce 2+1 dimensions to 1+0 dimensions

Reduced action (Clement):

\[
\mathcal{I} \sim \int d\rho \left[ \dot{\zeta}^2 - \frac{\eta_{ij}}{2} \zeta^2 \right] - \zeta \epsilon^{ijk} \dot{X}^i \ddot{X}^j \dot{X}^k
\]

Here \( \zeta \) is a Lagrange-multiplier and \( X^i = (T, X, Y) \) a Lorentzian 3-vector.

It could be rewarding to investigate this mechanical problem systematically and numerically!
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\[ I_\text{C}[\zeta, X_i] \sim \int d\rho \left[ \zeta^2 \dot{X}_i \dot{X}_j \eta_{ij} - 2\zeta \ell^2 + \zeta^2 \frac{1}{2} \epsilon^{ijk} X_i \ddot{X}_j \dddot{X}_k \right] \]

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CTMG at the chiral point
...abbreviated as CCTMG

Definition: CTMG at the chiral point is CTMG with the tuning

\[ \mu \ell = 1 \]

between the cosmological constant and the Chern–Simons coupling.

Notes:
▶ Abbreviate “CTMG at the chiral point” as CCTMG
▶ CCTMG is also known as “chiral gravity”
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Why special?
Calculating the central charges of the dual boundary CFT yields

$$c_L = \frac{3}{2G} \left( 1 - \frac{1}{\mu \ell} \right), \quad c_R = \frac{3}{2G} \left( 1 + \frac{1}{\mu \ell} \right)$$

Thus, at the chiral point we get

$$c_L = 0, \quad c_R = \frac{3}{G}$$
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Gravitons around AdS$_3$ in CTMG

Linearization around AdS background

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \]
Gravitons around AdS$_3$ in CTMG

Linearization around AdS background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

leads to linearized EOM that are third order PDE

$$G^{(1)}_{\mu\nu} + \frac{1}{\mu} C^{(1)}_{\mu\nu} = (\mathcal{D}^R \mathcal{D}^L \mathcal{D}^M h)_{\mu\nu} = 0$$ (1)

with three mutually commuting first order operators

$$(\mathcal{D}^{L/R})_{\mu}{}^{\nu} = \delta_{\mu}^{\nu} \pm \ell \varepsilon_{\mu}{}^{\alpha\nu} \bar{\nabla}_{\alpha}, \quad (\mathcal{D}^M)_{\mu}{}^{\nu} = \delta_{\mu}^{\nu} + \frac{1}{\mu} \varepsilon_{\mu}{}^{\alpha\nu} \bar{\nabla}_{\alpha}$$
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\]

Three linearly independent solutions to (1):

\[
(\mathcal{D}^L h^L)_{\mu\nu} = 0, \quad (\mathcal{D}^R h^R)_{\mu\nu} = 0, \quad (\mathcal{D}^M h^M)_{\mu\nu} = 0
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\[ (\mathcal{D}^{L/R})_{\mu} {} ^{\nu} = \delta_{\mu} ^{\nu} \pm \ell \varepsilon_{\mu} {} ^{\alpha\nu} \bar{\nabla}^{\alpha}, \quad (\mathcal{D}^M)_{\mu} {} ^{\nu} = \delta_{\mu} ^{\nu} + \frac{1}{\mu} \varepsilon_{\mu} {} ^{\alpha\nu} \bar{\nabla}^{\alpha} \]

Three linearly independent solutions to (1):

\[ (\mathcal{D}^L h^L)_{\mu\nu} = 0, \quad (\mathcal{D}^R h^R)_{\mu\nu} = 0, \quad (\mathcal{D}^M h^M)_{\mu\nu} = 0 \]

At chiral point left (L) and massive (M) branches coincide!
Degeneracy at the chiral point
Will be quite important later!

Li, Song & Strominger found all solutions of linearized EOM.

- Primaries: $L_0, \bar{L}_0$ eigenstates $\psi^{L/R/M}$ with

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Linearized metric is then the real part of the wavefunction

At chiral point: $L$ and $M$ branches degenerate. Get new solution ($DG & Johansson$)

\[
\psi^{new}_{\mu\nu} = \lim_{\mu\ell \to 1} \psi^M_{\mu\nu}(\mu\ell) - \psi^L_{\mu\nu}(\mu\ell - 1)
\]
with property

\[
(D_L \psi^{new})_{\mu\nu} = (D_M \psi^{new})_{\mu\nu} \neq 0, \quad (D_L^2 \psi^{new})_{\mu\nu} = 0
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Sign oder nicht sign?
That is the question. Choosing between Skylla and Charybdis.

- With signs defined as in this talk: BHs positive energy, gravitons negative energy

- With signs as defined in Deser-Jackiw-Templeton paper: BHs negative energy, gravitons positive energy

- Either way need a mechanism to eliminate unwanted negative energy objects – either the gravitons or the BHs

- Even at chiral point the problem persists because of the logarithmic mode. See Figure.
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Why lower-dimensional gravity?

Which 2D theory?

Which 3D theory?

How to quantize 3D gravity?

What next?
Witten’s attempt

Different approach (without gravitons!):

- Naive remark 1: 3D gravity is trivial
- Naive remark 2: 3D gravity is non-renormalizable
- Synthesis of naive remarks: 3D quantum gravity may exist as non-trivial theory
- Positive cosmological constant: impossible?
- Vanishing cosmological constant: S-matrix, but no gravitons!
- Therefore introduce negative cosmological constant
- Define quantum gravity by its dual CFT at the AdS boundary
- Constructing this CFT still a “monstrous” effort...

Maloney & Witten: taking into account all known contributions to path integral leads to non-sensible result for partition function $Z$. In particular, no holomorphic factorization: $Z_{MW} \neq Z_L \cdot Z_R$. Various suggestions to interpret this problem: need cosmic strings, need sum over complex geometries, 3D quantum gravity does not exist by itself.
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Interesting observations:

1. If left-moving sector is trivial, \( Z_L = 1 \), then problem of holomorphic factorization

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But:

Disagrees with results by Carlip, Deser, Waldron & Wise!
Gravitons in CCTMG

Is CCTMG dual to a logarithmic CFT?

New mode resolves apparent contradiction between LSS and CDWW.

Interesting property: \( L_0(\psi_{\text{new}} \psi_{\bar{L}}) = (2^{-1/2} 0 2)(\psi_{\text{new}} \psi_{L}) \), \( \bar{L}_0(\psi_{\text{new}} \psi_{\bar{L}}) = (0 1 0 0)(\psi_{\text{new}} \psi_{L}) \).

Such a Jordan form of \( L_0, \bar{L}_0 \) is defining property of a logarithmic CFT!

Note: called "logarithmic CFT" because some correlators take the form \( \langle \psi_{\text{new}}(z) \psi_{\text{new}}(0) \rangle \sim \ln z + \ldots \).

▶ Logarithmic CFT: not unitary and not chiral!

Either logarithmic or chiral CFT dual (or none)

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Viability of the logarithmic mode, part 1
Explicit solution for logarithmic mode (DG & Johansson)

Is the logarithmic mode really there?
Collect in the following suggestions how the logarithmic mode could drop out of the physical spectrum and show that none of them is realized.

Before starting, here is the explicit form of the logarithmic mode:

\[
\begin{align*}
\tilde{h}_{\mu\nu} &= \sinh \rho \cosh^3 \rho \left( c\tau - s \ln \cosh \rho \right) \\
&\quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}_{\mu\nu} - \tanh^2 \rho \left( s\tau + c \ln \cosh \rho \right) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -a^2 \end{pmatrix}_{\mu\nu}
\end{align*}
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with \( c = \cos (2u) \), \( s = \sin (2u) \), \( a = \frac{1}{\sinh \rho \cosh \rho} \).
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Viability of the logarithmic mode, part 2

Physical mode with negative energy

**Suggestion 1**

The logarithmic mode is pure gauge?

No!

Note: confirmed by Sachs who considered logarithmic quasi-normal modes.

Logarithmic mode has infinite energy and thus must be discarded?

No!

**Suggestion 2**

$E_{\text{new}} = -\frac{47}{1152} G \ell^3$

Energy is finite and negative. Thus logarithmic mode leads to instability but cannot be discarded.
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Viability of the logarithmic mode, part 3
Boundary conditions beyond Brown–Henneaux

**Suggestion 3**

New mode is not a small perturbation?

It is!

Suggestion 4

Solution is asymptotically AdS but violates Brown-Henneaux boundary conditions! ($\gamma^{(1)}_{ij} |_{BH} = 0$)

Henneaux et al. showed precedents where this may happen in 3D.
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$h^{\text{new}}$ diverges asymptotically like $\rho$, but AdS background diverges asymptotically like $e^{2\rho}$. Thus $h^{\text{new}}$ is really a small perturbation.
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Solution is asymptotically AdS

\[ ds^2 = d\rho^2 + \left( \gamma^{(0)}_{ij} e^{2\rho/\ell} + \gamma^{(1)}_{ij} \rho + \gamma^{(0)}_{ij} + \gamma^{(2)}_{ij} e^{-2\rho/\ell} + \ldots \right) dx^i dx^j \]

but violates Brown–Henneaux boundary conditions! ($\gamma^{(1)}_{ij} \big|_{\text{BH}} = 0$)

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New boundary conditions replacing Brown–Henneaux (DG & Johansson)
Viability of the \textit{logarithmic} mode, part 4

Brown–York boundary stress tensor

\textbf{Suggestion 5}

New mode leads to ill-defined Brown–York boundary stress tensor?
Viability of the logarithmic mode, part 4

Brown–York boundary stress tensor

Suggestion 5

New mode leads to ill-defined Brown–York boundary stress tensor? No!

Total action including boundary terms (Kraus & Larsen)

\[ I_{\text{total}} = I_{\text{CTMG}} + \frac{1}{8\pi G} \int d^2 x \sqrt{-\gamma} \left( K - \frac{1}{\ell} \right) \]

Its first variation leads to Brown–York boundary stress-tensor:

\[ \delta I_{\text{total}} \bigg|_{\text{EOM}} = \frac{1}{32\pi G} \int d^2 x \sqrt{-\gamma^{(0)}} T^{ij} \delta \gamma^{(0)}_{ij} \]

DG & Johansson: \( T_{ij} \) is finite, traceless and chiral:

\[ T_{ij} = -\frac{\ell}{16\pi G} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{ij} \]

Note: coincides with Brown–York boundary stress-tensor of global AdS$_3$
Viability of the logarithmic mode, part 5
Artifact of linearization?

Suggestion 6

Maybe some non-linear “magic” kills the new mode?

\[
\begin{align*}
N &= \frac{1}{2} (2 \times 18 - 2 \times 14 - 6) = 1 \\
N_1(2) &= \text{number of linearly independent first (second) class constraints confirmed in more general calculation by Carlip}
\end{align*}
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Conclusion 1: logarithmic mode passed all tests so far
Conclusion 2: CCTMG is unstable; dual CFT probably logarithmic
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Maybe some non-linear “magic” kills the new mode? Unlikely!

DG, Jackiw & Johansson: classical phase space analysis of CCTMG

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N = \frac{1}{2} \left( 2 \times D - 2 \times N_1 - N_2 \right) = \frac{1}{2} \left( 2 \times 18 - 2 \times 14 - 6 \right) = 1
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- \(N\): number of physical degrees of freedom (per point)
- \(D\): number of canonical pairs in full phase space
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Chiral vs. logarithmic

Pivotal open question: does dual CFT exist? is it chiral or logarithmic?

To Do

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- **Logarithmic**: must show consistency of 2nd order perturbations!
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ad chiral:

- restricting to Brown–Henneaux boundary conditions does not help
- Giribet, Kleban & Porrati showed that descendent of new mode

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\bar{L}_{-1} \psi_\mu^\text{new} = Y_{\mu\nu} = X_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu}
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after a diffeomorphism $\xi$ obeys Brown–Henneaux boundary conditions

- Descendants of logarithmic mode are there even when boundary conditions are restricted beyond requiring variational principle!
Chiral vs. logarithmic

Pivotal open question: does dual CFT exist? is it chiral or logarithmic?

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after a diffeomorphism $\xi$ obeys Brown-Henneaux boundary conditions

- Descendants of logarithmic mode are there even when boundary conditions are restricted beyond requiring variational principle!
- Need different mechanism of truncation!
Chiral vs. logarithmic

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To Do

- **Chiral** route: must show consistency of truncation!
- **Logarithmic**: must show consistency of 2\textsuperscript{nd} order perturbations!

ad logarithmic:

- straightforward but somewhat lengthy calculation
- expand metric around AdS background up to second order:

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}^{\text{new}} + h_{\mu\nu}^{(2)}
\]

EOM lead to linear PDE for \(h_{\mu\nu}^{(2)}\):

\[
\mathcal{D}^{(3)} h^{(2)} = f \left( (h_{\mu\nu}^{\text{new}})^2 \right)
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- Check if \(h^{(2)}\) really is smaller than \(h_{\mu\nu}^{\text{new}}\)
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- Might be rewarding exercise for a student
Which groundstate?

Two observations:
- Global AdS$_3$ has mass and angular momentum in (C)CTMG

$$M_{\text{AdS}_3} = \mu J_{\text{AdS}_3} = -\frac{1}{8G}$$

If AdS$_3$ is unstable in CCTMG because of mode, where does it run to?

Both observations suggest that there might be a ground state different from pure AdS$_3$ in (C)CTMG.

Consider other possible ground states with less symmetry
- Example: warped AdS has four Killing vectors with $\text{U}(1)_L \times \text{SL}(2,R) \times \text{RS}$

Suggestive to consider warped AdS as possible groundstate of (C)CTMG

Strominger et al.:
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Which groundstate?

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Example: warped $AdS$ has four Killing vectors with $U(1)_L \times SL(2, \mathbb{R})_R$

Strominger et al. : Suggestive to consider warped $AdS$ as possible groundstate of (C)CTMG
Most crucial question we would like to answer

Does 3D quantum gravity exist with no strings attached?

If yes: we would have an interesting quantum theory of gravity with BHs and gravitons to get conceptual insight into quantum gravity

if no: potentially exciting news for string theory

Perhaps a win-win situation!

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Thank you for your attention!
Some literature


D. Grumiller, R. Jackiw and N. Johansson, 0806.4185.

Thanks to Bob McNees for providing the \LaTeX{} beamerclass!