Spray Can Geometry

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Abstract — *A model of axiomatic projective geometry for geographic information systems (GIS) is proposed that incorporates positional uncertainty. We define mathematical models of extended points and lines that reflect the extended and uncertain character of geographic features. Geometric operations in this model exhibit tolerance for positional uncertainty. The axiomatic approach ensures the consistency of geometric reasoning despite the uncertainties in the locations.*

I. Introduction

Geometric functionality in current vector based geographic information systems (GIS) is based on infinitely small points and infinitely thin lines. This is in contrast to the fact that geographic features and their representation are extended and uncertain in location. Object representation in GIS is discrete and of limited computational accuracy and often destroys the consistency of geometric reasoning. A more realistic model of geometry is needed.

The present approach is based on Menger's axiomatic system for projective geometry [1]. We refer to a geometry as an algebra, consisting of a set of primitive objects and operations and a set of axioms. Every interpretation of primitives that complies with the axioms is considered to be a model for the geometry. In the proposed work interpretations of the primitive objects *point* and *line* are extended in all dimensions. Interpretations of the primitive operations *connection* and *intersection* exhibit tolerance for positional uncertainty in their input.

The advantage of the axiomatic approach over other more pragmatic efforts in the past is that the implementation of basic operations is sufficient to derive compound objects and operations, analytical functions, and tests for relations. It guarantees concistency in geometric reasoning.

II. Motivation

The idea for the proposed research emerged from a keynote speech given by Lotfi Zadeh: The name Spray Can Geometry refers to "a primitive world in which figures are drawn with a spray can, with no ruler or compass available" [2]. Many conventional approaches to geometric uncertainty modeling for GIS are built upon underlying exact operations and idealized object representation. By contrast, the present work aims at a flexible representation that tolerates uncertainty in location while preserving the consistency of the axiomatic system.

The Spray Can Geometry for GIS assumes interpretations of geometric primitives that reflect the extended character of geographic features and the positional uncertainty in their representation. These Spray Can primitives must meet a number of requirements in order to be viable for practical application in GIS. The most important requirement for the processing in a computer is closedness of operations. This is ensured by the axiomatic approach: Every set of interpretations that complies with geometric axioms automatically is operationally closed. Apart from this, two main criteria have been identified:

1. Spray Can Points and Spray Can Lines are uncertain in all dimensions.
2. Spray Can Operations exhibit tolerance for positional uncertainty in their input.

A model that violates criterion 1 does not account for the real character of geographic features. A model that violates criterion 2 is isomorphic to the exact model; the resulting geometric reasoning equals exact reasoning and produces similar problems in a descete representation.

III. Hypothesis

Our hypothesis is that an axiomatic model of 2D projective geometry can be defined that incorporates geometric operations with tolerance for positional uncertainty. These operations apply to objects that reflect the extended character of geographic features and the positional uncertainty in their representation in GIS.

IV. Building a Model

The proposed approach is based on Mengers axiomatic system for projective geometry [1]. Projective geometry provides a workable research domain for geometric uncertainty modeling and is preferable for GIS: Euclidean geometry can be naturally embedded in projective geometry; Menger's axiomatization is smaller and formulated in simple algebraic terms; It does not depend on dimension and can easily be extended to 3D.

In a first step of the modeling process, existing mathematical tools for geometric uncertainty modeling are inspected and tested for compliance with the criteria listed in chapter 2. Among these models are Rosenfeld's fuzzy geometry [3], fuzzy plane geometry introduced by Buckley and Eslami [3], fuzzy rough sets [4], probability metrics, and Poston's fuzzy geometry [5].

In a second step interpretations of primitive geometric objects and operations are defined based on the findings of step one. The primitive geometric objects of projective plane geometry are *points*, *lines*, *universe* ("everything"), and *vacuum* ("nothing"). Primitive operations comprise *intersection* ($\wedge$) and *connection* ($\vee$). In Menger's system the behavior of projective primitives is axiomized by two *projective laws* that are dual to each other:

$$X \vee ((X \vee Y) \wedge Z) = X \vee ((X \vee Z) \wedge Y) \quad (1)$$
$$X \wedge ((X \wedge Y) \vee Z) = X \wedge ((X \wedge Z) \vee Y) \quad (2)$$

where X, Y, Z are primitive objects. The special role of universe U and vacuum V is determined by four axioms:

$$U \vee X = U, \qquad U \wedge X = X \quad (3)$$
$$V \wedge X = V, \qquad V \vee X = X \quad (4)$$

for all primitive objects X. Initially three of the six primitives are defined. The remaining primitives must be chosen appropriately to ensure consistency of the axioms. Additionally, an approximate version of equality is defined. The set of interpretations is acceptable as a projective model, if Menger's axioms can be verified. As a result, consistency and operational closedness is ensured.

We applied the proposed procedure to a statistical interpretation of Spray Can Objects [6]. In the following section this interpretation is discussed to show the methodology of the research. The presented example complies with Menger's axioms. Primitive objects are uncertain in all dimensions, but primitive operations do not exhibit tolerance for positional uncertainty. Consequently, the model is not satisfactory for GIS applications. It is subject of ongoing research to prove or disprove the existence of a model that satisfies all criteria. It is expected that additionally a degree of uncertainty has to be assigned to the axioms in order to define a working model.

V. EXAMPLE

For a given exact projective point p the Spray Can Point $\tilde{p}$ is defined by the probability density function (pdf) of a Gaussian normal distribution on the homogeneous plane H_p in p (cf. Figure 1). The *Spray Can Dual* scd of $\tilde{p}$ is defined by $scd.\tilde{p} = \tilde{p}.d$, where d denotes the duality operation in the exact model and . denotes the composition of functions. The *Spray Can Connection* $\tilde{\vee}$ of two Spray Can Points $\tilde{p}$ and $\tilde{q}$ is specified by $\tilde{p}\tilde{\vee}\tilde{q} = scd.\big[d(p \vee q)\big]^{\sim}$, where $\vee$ denotes the connection operation in the exact model and p and q are the corresponding exact points. The definition of these primitives is sufficient to derive all remaining primitives. As we have proven in [6] the proposed set of interpretations of projective primitives complies with Mengers axioms.

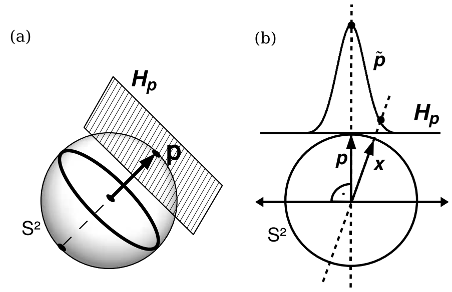

Figure 1: The projective plane is isomorphic to the unit sphere S^2 modulo opposite points. (a) The homogeneous plane H_p in $p \in S^2$. (b) A Spray Can Point $\tilde{p}$ in p.

VI. FUTURE RESEARCH

The main types of positional uncertainty in GIS are measurement errors and vaguely defined locations. Uncertain geometric primitives showing these types of uncertainty can be expressed in terms of probabilistic and possibilistic constraints [7]. The proposed work contributes to the long term goal of describing geometric constructions of objects under different types of uncertainty by generalized constraint propagation rules.

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