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Contents

Preface 13

FIRST PART. FUNDAMENTAL CONCEPTS AND METHODS 17

Chapter 1. Time-Frequency Energy Distributions: An Introduction . . . 19
Patrick FLANDRIN

1.1. Introduction 19
1.2. Atoms 20
1.3. Energy 21
1.3.1. Distributions 22
1.3.2. Devices 22
1.3.3. Classes 23
1.4. Correlations 26
1.5. Probabilities 27
1.6. Mechanics 29
1.7. Measurements 29
1.8. Geometries 32
1.9. Conclusion 33
1.10. Bibliography 34

Chapter 2. Instantaneous Frequency of a Signal 37
Bernard PICINBONO

2.1. Introduction 37
2.2. Intuitive approaches 38
2.3. Mathematical definitions 40
2.3.1. Ambiguity of the problem 40
2.3.2. Analytic signal and Hilbert transform 40
2.3.3. Application to the definition of instantaneous frequency 42
2.3.4. Instantaneous methods 45
2.4. Critical comparison of the different definitions 46
2.4.1. Interest of linear filtering 46

2.4.2. Bounds of the quantities introduced 46
2.4.3. Instantaneous nature 47
2.4.4. Interpretation by the average 48
2.5. Canonical pairs 49
2.6. Phase signals 50
2.6.1. Blaschke factors 50
2.6.2. Oscillatory singularities 54
2.7. Asymptotic phase signals 57
2.7.1. Parabolic chirp 57
2.7.2. Cubic chirp 59
2.8. Conclusions 59
2.9. Bibliography 60

Chapter 3. Linear Time-Frequency Analysis I: Fourier-Type Representations 61
Rémi GRIBONVAL

3.1. Introduction 61
3.2. Short-time Fourier analysis 62
3.2.1. Short-time Fourier transform 63
3.2.2. Time-frequency energy maps 64
3.2.3. Role of the window 66
3.2.4. Reconstruction/synthesis 71
3.2.5. Redundancy 71
3.3. Gabor transform; Weyl-Heisenberg and Wilson frames 71
3.3.1. Sampling of the short-time Fourier transform 71
3.3.2. Weyl-Heisenberg frames 72
3.3.3. Zak transform and “critical” Weyl-Heisenberg frames 74
3.3.4. Balian-Low theorem 75
3.3.5. Wilson bases and frames, local cosine bases 75
3.4. Dictionaries of time-frequency atoms; adaptive representations 77
3.4.1. Multi-scale dictionaries of time-frequency atoms 77
3.4.2. Pursuit algorithm 78
3.4.3. Time-frequency representation 79
3.5. Applications to audio signals 80
3.5.1. Analysis of superimposed structures 80
3.5.2. Analysis of instantaneous frequency variations 80
3.5.3. Transposition of an audio signal 82
3.6. Discrete algorithms 82
3.6.1. Fast Fourier transform 83
3.6.2. Filter banks: fast convolution 83
3.6.3. Discrete short-time Fourier transform 85
3.6.4. Discrete Gabor transform 86
3.7. Conclusion 86

3.8. Acknowledgements 87
3.9. Bibliography 87

Chapter 4. Linear Time-Frequency Analysis II: Wavelet-Type Representations 93
Thierry BLU and Jérôme LEBRUN

4.1. Introduction: scale and frequency 94
4.2. Continuous wavelet transform 95
4.2.1. Analysis and synthesis 95
4.2.2. Multiscale properties 97
4.3. Discrete wavelet transform 98
4.3.1. Multi-resolution analysis 98
4.3.2. Mallat algorithm 104
4.3.3. Graphical representation 106
4.4. Filter banks and wavelets 107
4.4.1. Generation of regular scaling functions 108
4.4.2. Links with approximation theory 111
4.4.3. Orthonormality and bi-orthonormality/perfect reconstruction . . 112
4.4.4. Polyphase matrices and implementation 114
4.4.5. Design of wavelet filters with finite impulse response 114
4.5. Generalization: multi-wavelets 116
4.5.1. Multi-filter banks 116
4.5.2. Balancing and design of multi-filters 118
4.6. Other extensions 121
4.6.1. Wavelet packets 121
4.6.2. Redundant transformations: pyramids and frames 122
4.6.3. Multi-dimensional wavelets 123
4.7. Applications 124
4.7.1. Signal compression and denoising 124
4.7.2. Image alignment 125
4.8. Conclusion 125
4.9. Acknowledgments 126
4.10. Bibliography 126

Chapter 5. Quadratic Time-Frequency Analysis I: Cohen's Class 131
François AUGER and Éric CHASSANDE-MOTTIN

5.1. Introduction 131
5.2. Signal representation in time *or* in frequency 132
5.2.1. Notion of signal representation 132
5.2.2. Temporal representations 133
5.2.3. Frequency representations 134
5.2.4. Notion of stationarity 135
5.2.5. Inadequacy of monodimensional representations 136

5.3. Representations in time *and* frequency 137
5.3.1. “Ideal” time-frequency representations 137
5.3.2. Inadequacy of the spectrogram 140
5.3.3. Drawbacks and benefits of the Rihaczek distribution 142
5.4. Cohen’s class 142
5.4.1. Quadratic representations covariant under translation 142
5.4.2. Definition of Cohen’s class 143
5.4.3. Equivalent parametrizations 144
5.4.4. Additional properties 145
5.4.5. Existence and localization of interference terms 148
5.5. Main elements 155
5.5.1. Wigner-Ville and its smoothed versions 155
5.5.2. Rihaczek and its smoothed versions 157
5.5.3. Spectrogram and S transform 158
5.5.4. Choi-Williams and reduced interference distributions 158
5.6. Conclusion 159
5.7. Bibliography 159

Chapter 6. Quadratic Time-Frequency Analysis II: Discretization of Cohen’s Class 165
Stéphane GRASSIN

6.1. Quadratic TFRs of discrete signals 165
6.1.1. TFRs of continuous-time deterministic signals 167
6.1.2. Sampling equation 167
6.1.3. The autocorrelation functions of the discrete signal 168
6.1.4. TFR of a discrete signal as a function of its generalized ACF . . 169
6.1.5. Discussion 171
6.1.6. Corollary: ambiguity function of a discrete signal 172
6.2. Temporal support of TFRs 173
6.2.1. The characteristic temporal supports 173
6.2.2. Observations 175
6.3. Discretization of the TFR 176
6.3.1. Meaning of the frequency discretization of the TFR 176
6.3.2. Meaning of the temporal discretization of the TFR 176
6.3.3. Aliased discretization 177
6.3.4. “Non-aliased” discretization 179
6.4. Properties of discrete-time TFRs 180
6.4.1. Discrete-time TFRs 181
6.4.2. Effect of the discretization of the kernel 182
6.4.3. Temporal inversion 182
6.4.4. Complex conjugation 183
6.4.5. Real-valued TFR 183
6.4.6. Temporal moment 183

6.4.7. Frequency moment . . . 184
6.5. Relevance of the discretization to spectral analysis . . . 185
6.5.1. Formulation of the problem . . . 185
6.5.2. Trivial case of a sinusoid . . . 187
6.5.3. Signal with linear frequency modulation . . . 187
6.5.4. Spectral analysis with discretized TFRs . . . 188
6.6. Conclusion . . . 189
6.7. Bibliography . . . 189

Chapter 7. Quadratic Time-Frequency Analysis III: The Affine Class and Other Covariant Classes . . . 193

Paulo GONÇALVÈS, Jean-Philippe OVARLEZ and Richard BARANIUK

7.1. Introduction . . . 193
7.2. General construction of the affine class . . . 194
7.2.1. Bilinearity of distributions . . . 194
7.2.2. Covariance principle . . . 195
7.2.3. Affine class of time-frequency representations . . . 198
7.3. Properties of the affine class . . . 201
7.3.1. Energy . . . 201
7.3.2. Marginals . . . 202
7.3.3. Unitarity . . . 202
7.3.4. Localization . . . 203
7.4. Affine Wigner distributions . . . 206
7.4.1. Diagonal form of kernels . . . 206
7.4.2. Covariance to the three-parameter affine group . . . 209
7.4.3. Smoothed affine pseudo-Wigner distributions . . . 211
7.5. Advanced considerations . . . 216
7.5.1. Principle of tomography . . . 216
7.5.2. Operators and groups . . . 217
7.6. Conclusions . . . 222
7.7. Bibliography . . . 223

SECOND PART. ADVANCED CONCEPTS AND METHODS . . . 227

Chapter 8. Higher-Order Time-Frequency Representations . . . 229

Pierre-Olivier AMBLARD

8.1. Motivations . . . 229
8.2. Construction of time-multifrequency representations . . . 230
8.2.1. General form and desirable properties . . . 230
8.2.2. General classes in the symmetric even case . . . 231
8.2.3. Examples and interpretation . . . 236
8.2.4. Desired properties and constraints on the kernel . . . 237
8.2.5. Discussion . . . 239

8.3. Multilinear time-frequency representations 240
8.3.1. Polynomial phase and perfect concentration 240
8.3.2. Multilinear time-frequency representations: general class 242
8.4. Towards affine multilinear representations 243
8.5. Conclusion 246
8.6. Bibliography 247

Chapter 9. Reassignment 249
Éric CHASSANDE-MOTTIN, François AUGER, and Patrick FLANDRIN

9.1. Introduction 249
9.2. The reassignment principle 250
9.2.1. Classical tradeoff in time-frequency and time-scale analysis . . . 250
9.2.2. Spectrograms and scalograms re-examined and corrected by mechanics 252
9.2.3. Generalization to other representations 254
9.2.4. Link to similar approaches 257
9.3. Reassignment at work 257
9.3.1. Fast algorithms 258
9.3.2. Analysis of a few simple examples 259
9.4. Characterization of the reassignment vector fields 265
9.4.1. Statistics of the reassignment vectors of the spectrogram 265
9.4.2. Geometrical phase and gradient field 267
9.5. Two variations 269
9.5.1. Supervised reassignment 269
9.5.2. Differential reassignment 270
9.6. An application: partitioning the time-frequency plane 271
9.7. Conclusion 274
9.8. Bibliography 274

Chapter 10. Time-Frequency Methods for Non-stationary Statistical Signal Processing 279
Franz HLAWATSCH and Gerald MATZ

10.1. Introduction 279
10.2. Time-varying systems 281
10.3. Non-stationary processes 283
10.4. TF analysis of non-stationary processes – type I spectra 285
10.4.1. Generalized Wigner-Ville spectrum 285
10.4.2. TF correlations and statistical cross-terms 286
10.4.3. TF smoothing and type I spectra 287
10.4.4. Properties of type I spectra 289
10.5. TF analysis of non-stationary processes – type II spectra 289
10.5.1. Generalized evolutionary spectrum 289
10.5.2. TF smoothing and type II spectra 291

10.6. Properties of the spectra of underspread processes 291
10.6.1. Approximate equivalences 292
10.6.2. Approximate properties 295
10.7. Estimation of time-varying spectra 296
10.7.1. A class of estimators 296
10.7.2. Bias-variance analysis 297
10.7.3. Designing an estimator 299
10.7.4. Numerical results 300
10.8. Estimation of non-stationary processes 302
10.8.1. TF formulation of the optimum filter 303
10.8.2. TF design of a quasi-optimum filter 304
10.8.3. Numerical results 305
10.9. Detection of non-stationary processes 306
10.9.1. TF formulation of the optimum detector 309
10.9.2. TF design of a quasi-optimum detector 310
10.9.3. Numerical results 311
10.10. Conclusion 313
10.11. Acknowledgements 315
10.12. Bibliography 315

Chapter 11. Non-stationary Parametric Modeling 321
Corinne MAILHES and Francis CASTANIÉ

11.1. Introduction 321
11.2. Evolutionary spectra 322
11.2.1. Definition of the “evolutionary spectrum” 322
11.2.2. Properties of the evolutionary spectrum 324
11.3. Postulate of local stationarity 325
11.3.1. Sliding methods 325
11.3.2. Adaptive and recursive methods 326
11.3.3. Application to time-frequency analysis 328
11.4. Suppression of a stationarity condition 329
11.4.1. Unstable models 329
11.4.2. Models with time-varying parameters 332
11.4.3. Models with non-stationary input 340
11.4.4. Application to time-frequency analysis 346
11.5. Conclusion 348
11.6. Bibliography 349

Chapter 12. Time-Frequency Representations in Biomedical Signal Processing 353
Lotfi SENHADJI and Mohammad Bagher SHAMSOLLAHI

12.1. Introduction 353
12.2. Physiological signals linked to cerebral activity 356

12.2.1. Electroencephalographic (EEG) signals 356
12.2.2. Electrocorticographic (ECoG) signals 359
12.2.3. Stereoelectroencephalographic (SEEG) signals 359
12.2.4. Evoked potentials (EP) 362
12.3. Physiological signals related to the cardiac system 363
12.3.1. Electrocardiographic (ECG) signals 363
12.3.2. R-R sequences 365
12.3.3. Late ventricular potentials (LVP) 367
12.3.4. Phonocardiographic (PCG) signals 369
12.3.5. Doppler signals 372
12.4. Other physiological signals 372
12.4.1. Electrogastrographic (EGG) signals 372
12.4.2. Electromyographic (EMG) signals 373
12.4.3. Signals related to respiratory sounds (RS) 374
12.4.4. Signals related to muscle vibrations 374
12.5. Conclusion 375
12.6. Bibliography 376

Chapter 13. Application of Time-Frequency Techniques to Sound Signals: Recognition and Diagnosis 383
Manuel DAVY

13.1. Introduction 383
13.1.1. Decision 384
13.1.2. Sound signals 384
13.1.3. Time-frequency analysis as a privileged decision-making tool . 384
13.2. Loudspeaker fault detection 386
13.2.1. Existing tests 386
13.2.2. A test signal 388
13.2.3. A processing procedure 389
13.2.4. Application and results 391
13.2.5. Use of optimized kernels 395
13.2.6. Conclusion 399
13.3. Speaker verification 399
13.3.1. Speaker identification: the standard approach 399
13.3.2. Speaker verification: a time-frequency approach 403
13.4. Conclusion 405
13.5. Bibliography 406

List of Authors 409

Index 413

Preface

Is time-frequency a mathematical utopia or, on the contrary, a concept imposed by the observation of physical phenomena? Various “archetypal” situations demonstrate the validity of this concept: musical notes, a linear chirp, a frequency shift keying signal, or the signal analysis performed by our auditory system. These examples show that “frequencies” can have a temporal localization, even though this is not immediately suggested by the Fourier transform. In fact, very often the analyzed phenomena manifest themselves by oscillating signals *evolving with time*: to the examples mentioned above, we may add physiological signals, radar or sonar signals, acoustic signals, astrophysical signals, etc. In such cases, the time-domain representation of the signal does not provide a good view of multiple oscillating components, whereas the frequency-domain representation (Fourier transform) does not clearly show the temporal localization of these components. We may conjecture that these limitations can be overcome by a *time-frequency analysis* where the signal is represented as a joint function of time *and* frequency – i.e., over a “time-frequency plane” – rather than as a function of time *or* frequency. Such an analysis should constitute an important tool for the understanding of many processes and phenomena within problems of estimation, detection or classification.

We thus have to find the mathematical transformation that allows us to map the analyzed signal into its time-frequency representation. Which “generalized Fourier transform” establishes this mapping? At this point, we find ourselves confronted with a fundamental limitation, known as the *uncertainty principle*, that excludes any *precise* temporal localization of a frequency. This negative result introduces some degree of uncertainty, or even of arbitrariness, into time-frequency analysis. One of its consequences is that we can never consider a transformation as *the only* correct time-frequency transformation, since time-frequency localization cannot be verified in an exact manner.

Is time-frequency an ill-posed problem then? Maybe, since it does not have a unique solution. However, this ambiguity and mathematical freedom have led to the definition of a great diversity of time-frequency transformations. Today, the chimeric concept of time-frequency analysis is materialized by a multitude of different transformations (or representations) that are based on principles even more diverse than the domains from which they originated (signal processing, mathematics, quantum mechanics, etc.). These principles and signal analysis or processing methods are just as useful in real-life applications as they are interesting theoretically.

Thus, is time-frequency a reality today? This is what we attempt to demonstrate in this book, in which we describe the principles and methods that make this field an everyday fact in industry and research. Written at the end of a period of approximately 25 years in which the discipline of time-frequency analysis witnessed an intensive development, this tutorial-style presentation is addressed mainly to researchers and engineers interested in the analysis and processing of non-stationary signals. The book is organized into two parts and consists of 13 chapters written by recognized experts in the field of time-frequency analysis. The first part describes the fundamental notions and methods, whereas the second part deals with more recent extensions and applications.

The diversity of viewpoints from which time-frequency analysis can be approached is demonstrated in Chapter 1, *"Time-Frequency Energy Distributions: An Introduction"*. Several of these approaches – originating from quantum mechanics, pseudo-differential operator theory or statistics – lead to the same set of fundamental solutions, for which they provide complementary interpretations. Most of the concepts and methods discussed in this introductory chapter will be developed in the following chapters.

Chapter 2, entitled *"Instantaneous Frequency of a Signal",* studies the concept of a "time-dependent frequency", which corresponds to a simplified and restricted form of time-frequency analysis. Several definitions of an instantaneous frequency are compared, and the one appearing most rigorous and coherent is discussed in detail. Finally, an in-depth study is dedicated to the special case of phase signals.

The two following chapters deal with *linear* time-frequency methods. Chapter 3, *"Linear Time-Frequency Analysis I: Fourier-Type Representations",* presents methods that are centered about the short-time Fourier transform. This chapter also describes signal decompositions into time-frequency "atoms" constructed through time and frequency translations of an elementary atom, such as the Gabor and Wilson decompositions. Subsequently, adaptive decompositions using redundant dictionaries of multi-scale time-frequency atoms are discussed.

Chapter 4, *"Linear Time-Frequency Analysis II: Wavelet-Type Representations",* discusses "multi-resolution" or "multi-scale" methods that are based on the notion of scale rather than frequency. Starting with the continuous wavelet transform, the chapter presents orthogonal wavelet decompositions and multi-resolution analyses. It also studies generalizations such as multi-wavelets and wavelet packets, and presents some applications (compression and noise reduction, image alignment).

Quadratic (or bilinear) time-frequency methods are the subject of the three following chapters. Chapter 5, *"Quadratic Time-Frequency Analysis I: Cohen's Class",* provides a unified treatment of the principal elements of Cohen's class and their main characteristics. This discussion is helpful for selecting the Cohen's class time-frequency representation best suited for a given application. The characteristics studied concern theoretical properties as well as interference terms that may cause practical problems. This chapter constitutes an important basis for several of the methods described in subsequent chapters.

Chapter 6, *"Quadratic Time-Frequency Analysis II: Discretization of Cohen's Class",* considers the time-frequency analysis of sampled signals and presents algorithms allowing a discrete-time implementation of Cohen's class representations. An approach based on the signal's sampling equation is developed and compared to other discretization methods. Subsequently, some properties of the discrete-time version of Cohen's class are studied.

The first part of this book ends with Chapter 7, *"Quadratic Time-Frequency Analysis III: The Affine Class and Other Covariant Classes".* This chapter studies quadratic time-frequency representations with covariance properties different from those of Cohen's class. Its emphasis is placed on the affine class, which is covariant to time translations and contractions-dilations, similarly to the wavelet transform in the linear domain. Other covariant classes (hyperbolic class, power classes) are then considered, and the role of certain mathematical concepts (groups, operators, unitary equivalence) is highlighted.

The second part of the book begins with Chapter 8, *"Higher-Order Time-Frequency Representations",* which explores multilinear time-frequency analysis. The class of time-*multifrequency* representations that are covariant to time and frequency translations is presented. Time-(mono)frequency representations ideally concentrated on polynomial modulation laws are studied, and the corresponding covariant class is presented. Finally, an opening towards multilinear affine representations is proposed.

Chapter 9, *"Reassignment",* describes a technique that is aimed at improving the localization of time-frequency representations, in order to enable a better interpretation by a human operator or a better use in an automated processing scheme. The reassignment technique is formulated for Cohen's class and for the affine class, and its properties and results are studied. Two recent extensions – supervised reassignment and differential reassignment – are then presented and applied to noise reduction and component extraction problems.

The two following chapters adopt a statistial approach to non-stationarity and time-frequency analysis. Various definitions of a non-parametric "time-frequency spectrum" for non-stationary random processes are presented in Chapter 10, *"Time-Frequency Methods for Non-stationary Statistical Signal Processing".* It is demonstrated that these different spectra are effectively equivalent for a subclass of processes referred to as "underspread". Subsequently, a method for the estimation of time-frequency spectra is proposed, and finally the use of these spectra for the estimation and detection of underspread processes is discussed.

Chapter 11, *"Non-stationary Parametric Modeling"*, considers non-stationary random processes within a parametric framework. Several different methods for non-stationary parametric modeling are presented, and a classification of these methods is proposed. The development of such a method is usually based on a parametric model for stationary processes, whose extension to the non-stationary case is obtained by means of a sliding window, adaptivity, parameter evolution or non-stationarity of a filter input.

The two chapters concluding this book are dedicated to the application of time-frequency analysis to measurement, detection, and classification tasks. Chapter 12, *"Time-Frequency Representations in Biomedical Signal Processing"*, provides a well-documented review of the contribution of time-frequency methods to the analysis of neurological, cardiovascular and muscular signals. This review demonstrates the high potential of time-frequency analysis in the biomedical domain. This potential can be explained by the fact that diagnostically relevant information is often carried by the non-stationarities of biomedical signals.

Finally, Chapter 13, *"Application of Time-Frequency Techniques to Sound Signals: Recognition and Diagnosis"*, proposes a time-frequency technique for supervised non-parametric decision. Two different applications are considered, i.e., the classification of loudspeakers and speaker verification. The decision is obtained by minimizing a distance between a time-frequency representation of the observed signal and a reference time-frequency function. The kernel of the time-frequency representation and the distance are optimized during the training phase.

As the above outline shows, this book provides a fairly extensive survey of the theoretical and practical aspects of time-frequency analysis. We hope that it will contribute to a deepened understanding and appreciation of this fascinating subject, which is still witnessing considerable developments.

We would like to thank J.-P. Ovarlez for important contributions during the initial phase of this work, G. Matz for helpful assistance and advice, and, above all, P. Flandrin for his eminent role in animating research on time-frequency analysis.

Vienna and Saint Nazaire, June 2008.

Chapter 5

Quadratic Time-Frequency Analysis I: Cohen's Class

Abstract: Cohen's class gathers some of the time-frequency representations which are most often used in practice. After a succinct reminder of the needs to which it tries to respond and the difficulty of responding to these needs with simple solutions, this chapter justifies the definition of Cohen's class based on two particular covariance properties. The further theoretical properties that the elements of this class can satisfy and the constraints that they impose on the parametrization kernel of these elements are listed and the practical problems due to the existence of interference terms are analyzed. The chapter ends with a discussion of the main members of Cohen's class.

Keywords: time-frequency representation, energy distributions, covariance principles, short-time Fourier transform, spectrogram, Cohen's class, Wigner-Ville distribution.

5.1. Introduction

Presenting in a single chapter all the knowledge acquired on Cohen's class of time-frequency representations would be a foolish challenge. The diversity of theoretical as well as practical viewpoints (some of which have been mentioned in Chapter 1) from which they could be approached and the numerous known results would make a complete synthesis of this subject a voluminous document. Moreover, succinct [HLA 92b,

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Chapter written by François AUGER and Éric CHASSANDE-MOTTIN.

COH 89, BOU 95, CLA 80] or detailed [FLA 98, FLA 99, COH 95] presentations already exist, and the reader is invited to refer to them.

Thus, without claiming to be exhaustive, this chapter aims at presenting the main characteristics of Cohen's class in a homogeneous and unified fashion. To this end, a simple presentation of the needs and concepts leading to quadratic representations will be given in Section 5.2, supported by the notions of instantaneous frequency and linear time-frequency representation studied in earlier chapters of this book. Section 5.4 will then present the general framework of Cohen's class of quadratic time-frequency representations, followed by a more detailed review of its main elements in Section 5.5.

5.2. Signal representation in time *or* in frequency

5.2.1. *Notion of signal representation*

Quite often, the purpose of the design and implementation of an acquisition chain is to enable the *observation* of a physical phenomenon through the measurement of some physical quantities which it influences. Under the condition that they be qualitatively and quantitatively sufficient, the information so supplied could then be used to deduce from it an *action* that could range from the simplest to the most complex, from the simple estimation of characteristic parameters to more complex applications of event detection, diagnosis or process control.

However, as the complexity of these actions grows, it becomes more and more necessary to carefully choose a *representation space* for the received signals, in which the information carried by them can be clearly perceived, structured and processed. Indeed, once this choice is made, the implementation of a *transformation* phase then makes it possible to associate with the received signal a *descriptor* belonging to this representation space, from which the appropriate actions can be easily deduced.

The choice of a representation space may be guided by different principles. If the designer has *a priori* knowledge about the structure of the observed signal, this knowledge can be used to devise a signal evolution model. The unknown parameters of this model will then be deduced from the observed signal and considered as its characteristics. This procedure leads to a *parametric representation*, of a generally lower dimension[1] than the number of acquired signal samples. However, when no *a priori* information about the signal's composition is available, it is strongly recommended not to try to conform the signal to any model, and to use a *non-parametric* representation. Its necessarily larger dimension is equal to the signal dimension in the case of "usual" temporal and frequency representations. However, it can also be greater if the resulting redundancy enables a better perception of the information carried by the sig-

1. The term "dimension" here designates the amount of data describing the signal in the representation space. Chapter 11 presents parametric representations of signals in detail.

nal, and offers a better trade-off between the pertinence of the representation and the complexity of its implementation. Unlike the parametric approach, such a procedure does not aim to compress a signal into a minimum number of descriptors.

5.2.2. *Temporal representations*

The temporal representation is, obviously, the most natural mode of observing a signal. For instance, it provides a precise indication of the order and spacing of the different events perceived within a signal. As a consequence, it constitutes an appropriate workspace for parameter estimation (for example, for the identification of linear systems) or transmission (coding, compression) problems.

Many temporal representations can be deduced from energy principles: for a finite-energy signal $x(t)$, its *instantaneous power* equals $|x(t)|^2$ (if $x(t)$ can be considered as a voltage, $|x(t)|^2$ corresponds to the power that would be dissipated in a resistance of $1\,\Omega$). Just as for probability distributions, this value can then be considered as a distribution of the energy of the signal in the temporal domain, since by definition its integral over the entire signal history is equal to its energy:

$$E_x := \int_{-\infty}^{\infty} |x(t)|^2\, dt\,.$$

Consequently, the average time instant where the signal's energy is localized (temporal center of gravity) and the signal's dispersion around this time instant constitute two elementary descriptors. These descriptors can be defined using first and second order moments of this distribution (when they exist):

$$t_x := \frac{1}{E_x}\int_{-\infty}^{\infty} t\, |x(t)|^2\, dt$$

and

$$\begin{aligned}(\Delta t_x)^2 &:= \frac{1}{E_x}\int_{-\infty}^{\infty} (t-t_x)^2\, |x(t)|^2\, dt\\ &= \frac{1}{E_x}\int_{-\infty}^{\infty} t^2\, |x(t)|^2\, dt \;-\; t_x^2\,.\end{aligned}$$

Although this example is purely academic and does not illustrate the possibilities of practical use of these definitions, we may consider the case of a sinusoidal signal with an amplitude modulated by a one-sided exponential:

$$x(t) = \begin{cases} e^{j2\pi f_0 t}\, e^{-\frac{t}{2T}} & \text{if} \quad t \geq 0\\ 0 & \text{if} \quad t < 0\,.\end{cases}$$

The average time and average duration of the signal are then equal to

$$t_x = T \qquad \text{and} \qquad \Delta t_x = T\,.$$

This is quite logical: the smaller T is, the more the signal's center of gravity tends towards the initial time instant $t = 0\,\mathrm{s}$, and the more the signal is concentrated around this time instant. These temporal values thus provide an indication of the localization and concentration of the signal's energy, which sometimes are enough to obtain the desired information about the observed signal.

5.2.3. *Frequency representations*

In a very large number of cases, temporal representations unfortunately do not allow a good perception of the information contained in the signal, notably when this information concerns the regular repetition of an event. Frequency representations then constitute an interesting alternative. They are based on the notion of frequency, which in turn is based on a very particular family of undamped waves: the sinusoids. Their pertinence is probably due to the abundance of wave phenomena and the fact that they are the eigenfunctions of time-invariant linear systems. Motivated by this relevance to the understanding of physical phenomena, mathematical analysis tools have been created, notably the Fourier representation. This representation corresponds to a signal expansion into the orthogonal family of complex exponentials:

$$x(t) \;=\; \int_{-\infty}^{\infty} \widehat{x}(f)\, e^{j2\pi f t}\, df\,, \qquad \text{with} \qquad \widehat{x}(f) \;:=\; \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt\,, \tag{5.1}$$

since

$$\int_{-\infty}^{\infty} e^{j2\pi(f_1-f_2)t}\, dt \;=\; \delta(f_1-f_2)\,.$$

The remarkable covariance properties of this representation with respect to the differentiation and convolution operators also constitute a major incitement to use it. Moreover, its squared modulus, called the *spectral energy density*, constitutes a signal energy distribution in the frequency domain, since *Parseval's theorem* shows that its integral over the entire frequency axis is equal to the signal's energy:

$$\int_{-\infty}^{\infty} |\widehat{x}(f)|^2\, df \;=\; \int_{-\infty}^{\infty} |x(t)|^2\, dt \;=\; E_x\,. \tag{5.2}$$

As in the previous section, this allows us to characterize a signal by its average localization in frequency and the frequency band that it occupies:

$$f_x \;:=\; \frac{1}{E_x} \int_{-\infty}^{\infty} f\, |\widehat{x}(f)|^2\, df$$

and

$$\begin{aligned}
(\Delta f_x)^2 \;&:=\; \frac{1}{E_x} \int_{-\infty}^{\infty} (f-f_x)^2\, |\widehat{x}(f)|^2\, df \\
&=\; \frac{1}{E_x} \int_{-\infty}^{\infty} f^2\, |\widehat{x}(f)|^2\, df \;-\; f_x^2\,.
\end{aligned}$$

5.2.4. *Notion of stationarity*

For all these frequency characterizations to be pertinent, it is however necessary for the signal to satisfy a stationarity assumption, which expresses the fact that the signal always bears the same information during the entire duration of observation. This assumption is made necessary by the definition via equation (5.1) of a frequency variable f through an integration over all the time from $-\infty$ to $+\infty$. In the *deterministic* framework, this assumption corresponds to a signal equal to a discrete sum of sinusoids,

$$x(t) = \sum_i A_i \cos(2\pi f_i t + \varphi_i)\,,$$

where A_i, f_i, and φ_i are real coefficients. In the *stochastic* framework, the stationarity assumption corresponds to a covariance under time translation of all the statistical moments of the signal. However, this strict stationarity assumption of random signals is often extended to a (more easily satisfied) assumption of covariance under temporal translation of only the first two statistical moments. The resulting wide-sense stationarity assumes that the mean value is independent of time and that the autocorrelation function only depends on the difference between two time instants:

$$\forall\, t, T, \quad \mathrm{E}\,[x(t)] = \mathrm{E}\,[x(t+T)] = m_x = \text{constant}$$

and

$$\begin{aligned}\forall\, t_1, t_2, T, \quad \mathrm{E}\,[x_c(t_1)\, x_c^*(t_2)] &= \mathrm{E}\,[x_c(t_1+T)\, x_c^*(t_2+T)] \\ &= \gamma_x(t_1 - t_2)\,,\end{aligned}$$

with $x_c(t) := x(t) - m_x$.

The *temporal* orthogonality of the family of complex exponentials is then complemented by a *statistical* orthogonality between the values of the Fourier transform of the signal at different frequencies, showing the decorrelation between the values of the signal's spectral representation and, thus, the absence of redundancy between these values: if

$$\widehat{x}_c(f) = \int_{-\infty}^{\infty} x_c(t)\, e^{-j2\pi f t}\, dt\,,$$

then

$$\begin{aligned}&\mathrm{E}\,[\widehat{x}_c(f_1)\, \widehat{x}_c^*(f_2)] \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{E}\,[x_c(t_1)\, x_c^*(t_2)]\; e^{-j2\pi(f_1 t_1 - f_2 t_2)}\, dt_1\, dt_2 \\ &\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{E}\left[x_c\Big(t + \frac{\tau}{2}\Big)\, x_c^*\Big(t - \frac{\tau}{2}\Big)\right] e^{-j2\pi\left[(f_1 - f_2)t + \frac{f_1+f_2}{2}\tau\right]}\, dt\, d\tau \\ &\quad = \widehat{\gamma}_x\!\left(\frac{f_1 + f_2}{2}\right) \delta(f_1 - f_2)\,,\end{aligned}$$

with

$$\widehat{\gamma}_x(f) := \int_{-\infty}^{\infty} \gamma_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{5.3}$$

where

$$\gamma_x(\tau) := \mathrm{E}\left[x_c\Big(t+\frac{\tau}{2}\Big)\, x_c^*\Big(t-\frac{\tau}{2}\Big)\right].$$

These expressions are written under the assumption that $x(t)$ has finite power and that the covariance function is square-integrable. This result, which constitutes the *Wiener-Khinchine theorem*, shows that if the random signal $x(t)$ may be considered as stationary, and *only* in this case, then the value of its Fourier transform at a given frequency carries information that cannot be found at any other frequency.

5.2.5. *Inadequacy of monodimensional representations*

When the observed signal cannot be considered as stationary, frequency representations lose part of their pertinence. To clearly demonstrate this, we may, for instance, consider the case of a signal consisting of two fragments of complex sinusoids with different frequencies, like two successive musical notes of the same duration:

$$x(t) = \begin{cases} e^{j2\pi f_1 t} & \text{for} \quad 0 \le t < T \\ e^{j2\pi f_2 t} & \text{for} \quad T \le t < 2T \\ 0 & \text{for} \quad t < 0 \quad \text{and} \quad t \ge 2T. \end{cases}$$

Its Fourier transform, which is equal to

$$\begin{aligned} \widehat{x}(f) = {} & T \operatorname{sinc}\big(\pi(f-f_1)T\big)\, e^{-j2\pi(f-f_1)T/2} \\ & + T \operatorname{sinc}\big(\pi(f-f_2)T\big)\, e^{-j2\pi(f-f_2)3T/2}, \end{aligned}$$

with $\operatorname{sinc}(x) = \sin(x)/x$, clearly indicates the presence of two frequencies. However, the temporal order of these two components and the absence of signal components outside the interval $[0, 2T]$ constitute information that is indicated in the phases of the Fourier transforms of the two components and are thus difficult to perceive and to use. Mathematically, these phases combine to reconstruct, via the inverse Fourier transform, a signal that is equal to segments of sinusoids between $[0, T]$ and $[T, 2T]$ and zero everywhere else. For this signal, the classical monodimensional representations $x(t)$ and $\widehat{x}(f)$ do not provide good signal descriptions, since the signal's instantaneous power does not highlight the frequency change and its spectral energy density does not clearly show if the two frequencies are present simultaneously or successively. Common physical sense would rather lead us to believe that frequency f_1 is only present in the interval $[0, T]$ and frequency f_2 only in the interval $[T, 2T]$. However, to give meaning to such an assertion, we must first define a notion of *time-dependent* frequency and then look for an analysis tool built on the basis of (or compatible with) this notion.

5.3. Representations in time *and* frequency

There are, indeed, numerous physical situations where the information carried by the signal resides in the evolution of its frequency content over time. For example, this is the case for speech signals and for some natural phenomena, such as the transition of light from yellow to red during sunset, or the variations of waves emitted by moving objects that is caused by the Doppler effect. Many signals observed in astrophysics are also of this type. Among artificial phenomena, we also mention signals received by radar or sonar systems, modulation signals used for data transmission, and the transitory phenomena of oscillating systems.

5.3.1. *"Ideal" time-frequency representations*

In such situations, it does not seem reasonable to consider that the observed signals carry the same information throughout the entire duration of their observation and, thus, to expand them as a sum of sinusoids. These situations rather incite us to search for analysis methods which allow the time and frequency variables to jointly co-exist, just as a musical score describing a succession of notes or chords over time. But for that, the notion of frequency should not be based on the sinusoidal waves, which make any temporal localization disappear. A different notion of frequency can be obtained with the help of the *analytic projection* of a signal, which assigns to any signal $x(t)$, with Fourier transform $\widehat{x}(f)$, a signal $x_a(t)$ called its *analytic signal*, whose Fourier transform is given by

$$\widehat{x}_a(f) \;=\; \big[\widehat{x}(f) + \widehat{x}^*(-f)\big]\,\Gamma(f)\,, \qquad \text{with } \Gamma(f) \;:=\; \begin{cases} 1 & \text{if} \quad f > 0 \\ 1/2 & \text{if} \quad f = 0 \\ 0 & \text{if} \quad f < 0\,. \end{cases}$$

For a real-valued signal $x(t)$, this linear projection suppresses the redundancy within $\widehat{x}(f)$ between positive and negative frequencies; it keeps only the positive frequencies, which are considered as physically more significant. Thus, to any real signal $x(t)$, the analytic projection assigns a complex signal $x_a(t)$ with the same energy ($E_{x_a} = E_x$), whose real part equals $x(t)$ and whose imaginary part equals the Hilbert transform of $x(t)$. Based on the analytic signal $x_a(t)$, we can then define *in a unique manner* the signal's *envelope* $a_x(t)$, equal to the modulus of $x_a(t)$, and its *instantaneous frequency*[2] [BOA 92] $f_x(t)$, deduced from the derivative of its phase:

$$a_x(t) \;:=\; |x_a(t)| \qquad \text{and} \qquad f_x(t) \;:=\; \frac{1}{2\pi}\,\frac{d}{dt}\,\phi_x(t)$$

with

$$x_a(t) \;=\; a_x(t)\, e^{j\phi_x(t)}.$$

2. Chapter 2 presents this notion in more detail.

To verify this concept of instantaneous frequency, we consider the example of a sinusoidal signal $x(t) = A\cos(2\pi f_0 t + \phi)$, whose Fourier transform is equal to $\widehat{x}(f) = \frac{A}{2}\, e^{j\phi}\, \delta(f - f_0) + \frac{A}{2}\, e^{-j\phi}\, \delta(f + f_0)$. The Fourier transform of the analytic signal is $\widehat{x}_a(f) = A\, e^{j\phi}\, \delta(f - f_0)$, and therefore the analytic signal is obtained as $x_a(t) = A\, e^{j(2\pi f_0 t + \phi)}$. The instantaneous frequency of this signal is thus constant and equals $f_x(t) = f_0$. On the one hand, this shows that the analytic projection operator generalizes to the set of all signals the classical Fresnel projection, which associates with a real signal $A\cos(2\pi f_0 t + \phi)$ a complex signal $A\, e^{j(2\pi f_0 t + \phi)}$. On the other hand, this example also demonstrates that this definition of instantaneous frequency is consistent with the usual concept of frequency associated with a sinusoidal wave. This consistency is corroborated by a supplementary result showing that the average value of the instantaneous frequency is equal to the average frequency of a signal:

$$\frac{1}{E_x} \int_0^{\infty} f\, |\widehat{x}_a(f)|^2\, df \;=\; \frac{1}{E_x} \int_{-\infty}^{\infty} f_x(t)\, |x_a(t)|^2\, dt \;=\; f_{x_a}\,.$$

By duality, a similar reasoning can be performed in the frequency domain, based on the polar form of the Fourier transform of a signal:

$$\widehat{x}(f) \;=\; |\widehat{x}(f)|\, e^{j\psi_x(f)}.$$

Its modulus $|\widehat{x}(f)|$ is then called the *spectral amplitude* of the signal and its phase $\psi_x(f)$ is called the *spectral phase*, from which the *group delay* can be deduced by differentiation:

$$t_x(f) \;:=\; -\frac{1}{2\pi}\, \frac{d}{df}\, \psi_x(f)\,.$$

This group delay can be considered as the main time instant where the frequency f appears[3]. As for the instantaneous frequency, the average value of the group delay equals the average time of the signal:

$$\frac{1}{E_x} \int_{-\infty}^{\infty} t\, |x(t)|^2\, dt \;=\; \frac{1}{E_x} \int_{-\infty}^{\infty} t_x(f)\, |\widehat{x}(f)|^2\, df \;=\; t_x\,.$$

To illustrate these results, we may verify that for a signal equal to an impulse localized at time t_1, $x(t) = \delta(t - t_1)$, the Fourier transform equals $\widehat{x}(f) = e^{-j2\pi f t_1}$, and the group delay is therefore a constant function $t_x(f) = t_1$. Thus, an impulse is a signal where all the frequencies are present at the same time ($t_x(f) = t_1$) with the same energy ($|\widehat{x}(f)|^2 = 1$).

These two notions pave the way for the notion of time-frequency representation, since they allow a local characterization of a signal by its instantaneous frequency at a

3. If $x(t)$ is the impulse response of a system, $t_x(f)$ is the delay induced by the system at frequency f.

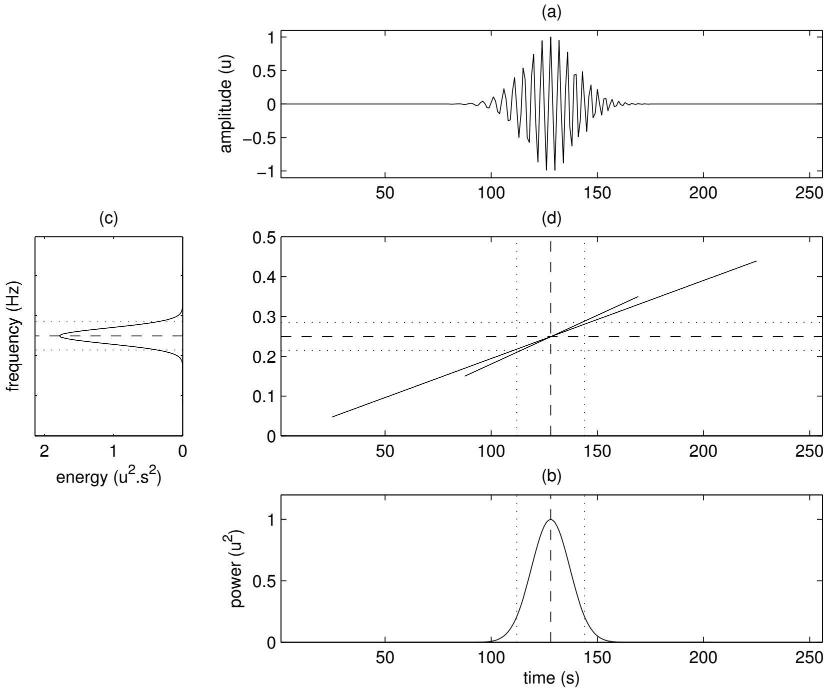

Figure 5.1. *First localizations of a signal in the time-frequency plane. The analyzed signal, whose real part is represented in (a), is a linearly modulated sinusoid with a Gaussian amplitude. Its physical unit is written as* u. *Its instantaneous power (b) and its spectral energy density (c) are localized around the average time* t_x *and the average frequency* f_x, *with measurable dispersions* Δt_x *and* Δf_x. *The signal energy can thus be localized in a time-frequency rectangle (d) and, more specifically, on the curves of the instantaneous frequency (calculated between the time instants* 25 *and* 225 *s) and of the group delay (calculated between the frequencies* 0.15 *and* 0.35 *Hz). For this signal, which is strongly concentrated in the time-frequency plane, the two curves of instantaneous frequency and of group delay are not reciprocal*

given time or by the main time where a given frequency appears. In fact, they suggest the definition of time-"instantaneous frequency" and "group delay"-frequency representations, which localize the local energy measure at points $\big(t, f_x(t)\big)$ or $\big(t_x(f), f\big)$ [BER 83] (see Figure 5.1):

$$\mathrm{TIF}_x(t,f) \;=\; |x_a(t)|^2\,\delta\big(f - f_x(t)\big)$$
$$\mathrm{GDF}_x(t,f) \;=\; |\widehat{x}_a(f)|^2\,\delta\big(t - t_x(f)\big)\,.$$

However, these distributions can seldom be used in practice, since they do not provide pertinent characterizations when the signal is a sum of several components localized at the same time or at the same frequency [COH 95, FLA 99]. It is thus necessary to find other tools for time-frequency representation that will have to yield pertinent results in the case of multi-component signals while approaching as closely as possible the "ideal" representations in the case of mono-component signals.

5.3.2. *Inadequacy of the spectrogram*

Explaining first the limitations of representations deduced from simple arguments allows a better justification of the complex tools that will follow. In fact, a natural approach to the construction of a time-frequency representation is to incorporate a temporal dependence in the classical frequency representation, by simply applying a Fourier transform to a signal segment centered about the analysis time t. To some extent, however, this implies that this neighborhood, delimited by an analysis window $h(t)$, corresponds to a time interval within which the signal can be considered as stationary. The result constitutes the *short-time Fourier transform* of the signal, whose squared modulus is called the *spectrogram* [POR 80, NAW 88]:

$$\begin{aligned} \mathrm{STFT}_x^h(t,f) &:= \int_{-\infty}^{\infty} x(u)\, h^*(u-t)\, e^{-j2\pi f u}\, du \\ S_x^h(t,f) &:= \left|\mathrm{STFT}_x^h(t,f)\right|^2. \end{aligned} \tag{5.4}$$

The latter constitutes a signal energy distribution in the time-frequency plane, provided that the analysis window $h(t)$ is correctly normalized to obtain unit energy:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x^h(t,f)\, dt\, df \;=\; E_x \quad\Longleftrightarrow\quad \int_{-\infty}^{\infty} |h(t)|^2\, dt \;=\; 1\,.$$

Due to the simplicity of both its construction and interpretation, the spectrogram constitutes one of the most widely used methods of non-stationary signal analysis, and the one which signal analysis "experts" know best how to decipher. In some application fields such as speech analysis [DAL 92], the resulting signal description clearly highlights the structure and evolution of the analyzed signal. Nonetheless, the spectrogram is unsuitable for many contexts for two main reasons. The first reason, of a practical nature, is linked to the antagonism between the spectrogram's temporal and frequency resolutions. A simple way of highlighting this problem is to write the short-time Fourier transform as a function of the signal's time-domain and frequency-domain representations:

$$\begin{aligned} \mathrm{STFT}_x^h(t,f) &= \int_{-\infty}^{\infty} x(u)\, h^*(u-t)\, e^{-j2\pi f u}\, du \\ &= \int_{-\infty}^{\infty} \widehat{x}(\nu)\, \widehat{h}^*(\nu-f)\, e^{j2\pi(\nu-f)t}\, d\nu\,. \end{aligned}$$

These expressions show that the short-time Fourier transform can be simultaneously interpreted as the Fourier transform of a signal segment delimited by $h^*(u-t)$, or as the output of a band-pass filter with transmittance $\widehat{h}^*(\nu-f)$. Consequently, its temporal resolution, i.e., its ability to distinguish between two successive events, is linked

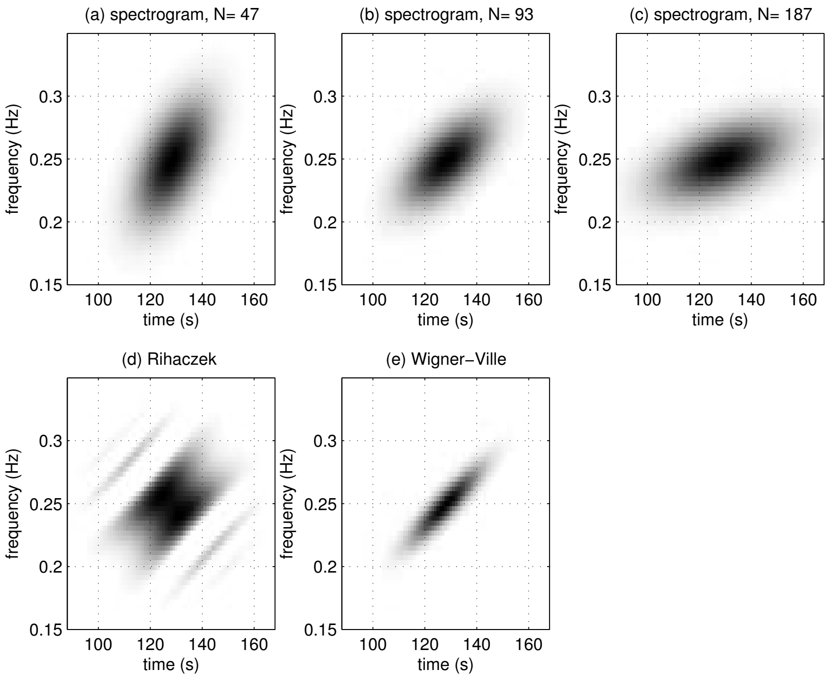

Figure 5.2. *Difficulties of use of the spectrogram and the Rihaczek distribution. The figure shows the following time-frequency representations of the signal from Figure 5.1: (a) spectrogram using a window that is too short (bad frequency localization), (b) spectrogram using a correct window, (c) spectrogram using a window that is too long (bad temporal localization), (d) real part of the Rihaczek distribution, and (e) Wigner-Ville distribution*

to the average duration Δt_h of the analysis window. Similarly, its frequency resolution, that is, its ability to distinguish between two sinusoids with close frequencies, is linked to the average bandwidth Δf_h of the filter $\widehat{h}(\nu)$. However, these quantities are linked by the Heisenberg-Gabor uncertainty principle [GAB 46], which shows that their product is bounded from below:

$$\Delta t_h \, \Delta f_h \geq \frac{1}{4\pi} \, .$$

It is therefore not possible to simultaneously obtain a very good time resolution and a very good frequency resolution. Consequently, the user must correctly choose the characteristics of the analysis window depending on the signal structure (see Figure 5.2), taking into account especially the proximity and evolution of the signal components in time and in frequency.

The second argument against the use of the spectrogram is of theoretical nature. It is related to the bias of the time and frequency marginals of the spectrogram, which correspond to *smoothed* versions of the signal's instantaneous power and spectral energy density:

$$\int_{-\infty}^{\infty} S_x^h(t,f)\,df \;=\; \int_{-\infty}^{\infty} |x(u)|^2\,|h(u-t)|^2\,du \;\neq\; |x(t)|^2 \tag{5.5}$$

$$\int_{-\infty}^{\infty} S_x^h(t,f)\,dt \;=\; \int_{-\infty}^{\infty} |\widehat{x}(\nu)|^2\,\left|\widehat{h}(\nu-f)\right|^2 d\nu \;\neq\; |\widehat{x}(f)|^2. \tag{5.6}$$

Therefore, apart from the two extreme (and mutually exclusive) cases where $h(t)$ is an impulse or a constant, a spectrogram does not play the same role with respect to instantaneous power and spectral energy density as a joint probability density does with respect to its marginals.

5.3.3. *Drawbacks and benefits of the Rihaczek distribution*

There exist, however, energy distributions whose marginals are unbiased. In fact, another time-frequency representation is given simply by the product of the temporal and frequency representations of the signal, multiplied by a complex exponential of the product ft. This representation is called the *Rihaczek distribution* [RIH 68]:

$$R_x(t,f) \;:=\; x(t)\,\widehat{x}^*(f)\,e^{-j2\pi ft}. \tag{5.7}$$

By construction, this representation has the desired marginals,

$$\int_{-\infty}^{\infty} R_x(t,f)\,df \;=\; |x(t)|^2 \quad \text{and} \quad \int_{-\infty}^{\infty} R_x(t,f)\,dt \;=\; |\widehat{x}(f)|^2,$$

but it also has two major drawbacks:

– its values are not always positive real numbers, which prevents an interpretation of $R_x(t,f)$ as a measure of the signal energy contained in an infinitesimal rectangle centered around the point (t,f);

– even worse, its values are most often complex numbers. The graphical representation of this distribution is thus difficult, since its real and imaginary parts or its modulus and phase have to be extracted almost arbitrarily.

5.4. Cohen's class

5.4.1. *Quadratic representations covariant under translation*

The unsatisfactory results obtained with the spectrogram and the Rihaczek distribution justify the search for better tools, which have, for example, strictly real values and correct marginals. Our aim is to pertinently distribute the signal energy in the time-frequency plane. One way of achieving this is to start from the general form of quadratic representations, obtained by application of Riesz's theorem [RIE 55]:

$$\begin{aligned} \mathrm{TFR}_x(t,f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t_1,t_2;t,f)\, x(t_1)\, x^*(t_2)\, dt_1 dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{K}(t,f;v,\tau)\, x\Big(v+\frac{\tau}{2}\Big)\, x^*\Big(v-\frac{\tau}{2}\Big)\, dv\, d\tau\,, \end{aligned} \tag{5.8}$$

with

$$\mathcal{K}(t,f;v,\tau) := K\Big(v+\frac{\tau}{2}, v-\frac{\tau}{2}; t, f\Big)\,,$$

and to look for a kernel $\mathcal{K}(t,f;v,\tau)$ that allows this representation to satisfy the desired properties. To guarantee the existence of the representation for finite-energy signals, this kernel must be square-integrable. Additional constraints may be obtained by imposing covariance properties. In fact, it seems natural to require that a simple shift of the signal produces an identical shift of its representation. If $y(t)$ is a signal derived from another signal $x(t)$ through a translation by time t_0, i.e., $y(t) = x(t-t_0)$, it is desirable that for any time t and for any frequency f, $\mathrm{TFR}_y(t,f) = \mathrm{TFR}_x(t-t_0, f)$. This covariance of the representation under time translations leads to a first constraint on the kernel:

$$\forall\ t,f,v,t_0,\tau, \qquad \mathcal{K}(t,f;v+t_0,\tau) = \mathcal{K}(t-t_0,f;v,\tau)\,.$$

This constraint must be satisfied for any choice of t, f, v, t_0 and τ. Taking $v = 0\,\mathrm{s}$ and $t_0 = v$, we obtain $\mathcal{K}(t,f;v,\tau) = \mathcal{K}(t-v,f;0,\tau)$. Expression (5.8) then becomes

$$\mathrm{TFR}_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(t-v,f,\tau)\, x\Big(v+\frac{\tau}{2}\Big)\, x^*\Big(v-\frac{\tau}{2}\Big)\, dv\, d\tau \tag{5.9}$$

where $k(t,f,\tau) := \mathcal{K}(t,f;0,\tau)$. This property thus imposes a rather weak constraint on the expression of the time-frequency representation.

5.4.2. *Definition of Cohen's class*

It may seem equally important that the modulation of a signal by a complex sinusoid with frequency f_0 results in a shift of the representation by f_0: if $y(t)$ is a signal derived from another signal $x(t)$ through a frequency modulation by f_0, i.e., $y(t) = x(t)\, e^{j2\pi f_0 t}$, then it is desirable that for any time t and for any frequency f, $\mathrm{TFR}_y(t,f) = \mathrm{TFR}_x(t, f-f_0)$. In particular, such a property is pertinent when the studied physical phenomenon naturally suggests the use of a linear scale (rather than a logarithmic one) for the frequency axis of the representation. It thus leads to an independence of the representation with respect to the origin of the frequency axis. This new covariance corresponds to an additional constraint on the kernel:

$$\forall\ t,f,\tau,f_0\,, \qquad k(t,f,\tau)\, e^{j2\pi f_0 \tau} = k(t,f-f_0,\tau)\,.$$

Taking $f = 0\,\mathrm{Hz}$ and $f_0 = -f$, we obtain $k(t,f,\tau) = k(t,0,\tau)\, e^{-j2\pi f\tau}$. Expression (5.9) then becomes

$$\mathrm{TFR}_x(t,f) = C_x(t,f) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{\text{t-d}}(t-v,\tau)\, x\Big(v+\frac{\tau}{2}\Big)\, x^*\Big(v-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau} dv\, d\tau, \tag{5.10}$$

with $\phi_{\text{t-d}}(t,\tau) := k(t,0,\tau) = \mathcal{K}(t,0;0,\tau)$.

To summarize, equation (5.10) characterizes the set of all bilinear representations covariant under time and frequency translations. This set is called *Cohen's class* [COH 66, ESC 76]. Equation (5.10) may be compared to equation (5.3), since it can also be written as the Fourier transform of an "instantaneous" autocorrelation function of the signal,

$$C_x(t,f) = \int_{-\infty}^{\infty} r_x(t,\tau)\, e^{-j2\pi f\tau}\, d\tau\,,$$

with

$$r_x(t,\tau) := \int_{-\infty}^{\infty} \phi_{\text{t-d}}(t-v,\tau)\, x\Big(v+\frac{\tau}{2}\Big)\, x^*\Big(v-\frac{\tau}{2}\Big)\, dv\,.$$

5.4.3. *Equivalent parametrizations*

At least two other expressions are equivalent to equation (5.10). The first is obtained by moving from the temporal convolution variable v to a frequency variable ξ using a Fourier transform:

$$C_x(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{\text{d-D}}(\tau,\xi)\, A_x(\tau,\xi)\, e^{j2\pi(t\xi - f\tau)}\, d\xi\, d\tau\,, \tag{5.11}$$

with

$$A_x(\tau,\xi) := \int_{-\infty}^{\infty} x\Big(t+\frac{\tau}{2}\Big)\, x^*\Big(t-\frac{\tau}{2}\Big)\, e^{-j2\pi\xi t}\, dt$$

and

$$\phi_{\text{d-D}}(\tau,\xi) := \int_{-\infty}^{\infty} \phi_{\text{t-d}}(t,\tau)\, e^{-j2\pi\xi t}\, dt\,.$$

The function $A_x(\tau,\xi)$ is called the *narrowband symmetric ambiguity function* [WOO 53]. It is also a quadratic time-frequency representation. However, its two variables τ and ξ (respectively called "*delay*" and "*Doppler frequency*") are not absolute values, but rather relative shift values. Therefore, this ambiguity function presents the particularity of having a modulus *invariant* under time and frequency translations: if $y(t) = x(t-t_1)\, e^{j2\pi f_1 t}$, then $\forall\, \tau, \xi,\ A_y(\tau,\xi) = A_x(\tau,\xi)\, e^{j2\pi(f_1\tau - t_1\xi)}$; therefore, $\forall\, \tau, \xi,\ |A_y(\tau,\xi)| = |A_x(\tau,\xi)|$. Due to this property, the ambiguity function is a signature of the signal that is invariant under translations, which makes it particularly interesting for radar localization systems [RIH 96].

The second expression is obtained from equation (5.10) by moving from the delay variable τ to a frequency convolution variable ν using a Fourier transform:

$$C_x(t,f) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{\text{t-f}}(t-v, f-\nu)\, W_x(v,\nu)\, dv\, d\nu\,, \tag{5.12}$$

with

$$W_x(t,f) := \int_{-\infty}^{\infty} x\Big(t+\frac{\tau}{2}\Big)\, x^*\Big(t-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, d\tau \tag{5.13}$$

and

$$\phi_{\text{t-f}}(t,f) := \int_{-\infty}^{\infty} \phi_{\text{t-d}}(t,\tau)\, e^{-j2\pi f\tau}\, d\tau\,.$$

The function $W_x(t,f)$ is called the *Wigner-Ville distribution* [VIL 48, WIG 32, CLA 80, MEC 97]. It is one of the most important members of Cohen's class, if not its central element. In fact, expression (5.12) presents Cohen's class as the set of all versions of the Wigner-Ville distribution that are doubly smoothed in time and frequency, provided that $\phi_{\text{t-f}}(t,f)$ can be interpreted as a two-dimensional low-pass filter.

Every representation in Cohen's class is thus perfectly characterized by one of the three kernels $\phi_{\text{t-d}}(t,\tau)$, $\phi_{\text{d-D}}(\tau,\xi)$, or $\phi_{\text{t-f}}(t,f)$, which can be derived from each other by the Fourier transformation. For the spectrogram for instance, $\phi_{\text{t-d}}(t,\tau) = h^*(-t+\tau/2)\, h(-t-\tau/2)$, $\phi_{\text{d-D}}(\tau,\xi) = A_h^*(\tau,\xi)$, and $\phi_{\text{t-f}}(t,f) = W_h(-t,-f)$. The first kernel is mainly used to calculate the representation from the time-domain representation of the signal. The second kernel allows an easy analysis of the properties of a representation (as will be seen in the next section). Finally, as will be seen in Section 5.4.5.3, the third kernel indicates the type of smoothing that has been applied to the Wigner-Ville distribution to obtain this representation, provided once more that $\phi_{\text{t-f}}(t,f)$ can be interpreted as a two-dimensional low-pass filter.

5.4.4. *Additional properties*

The constraints of covariance under time and frequency translations, which lead to the definition of Cohen's class, still leave room for a great diversity of possible representations. To restrict this choice, we require that other, more demanding properties be satisfied, and we analyze the resulting constraints on the characteristic kernel of the representation. Such a reasoning is satisfactory from a mathematical point of view, since it leads to the search for sufficient (and often necessary) conditions for these properties to be satisfied. It is also satisfactory from a practical point of view, since it allows us to specify the desirable properties for a given application, in order to determine the most suitable representation.

Research conducted in this domain [CLA 80, HLA 92b, FLA 99] has allowed the systematic translation of a property that a time-frequency representation (TFR) can satisfy into a constraint on its kernel. Tables 5.1 to 5.4 present a synthesis of the main existing pairs (properties, constraints). These properties can be divided into four groups:

– the first group (properties 1 to 5) gathers the properties of a practical nature, very similar to the covariance property defining Cohen's class. All these properties express the desire that the representation be modified in a logical manner when simple transformations are applied to the signal. For example, the real character of the TFR corresponds to the desire of having a tool that is easy to use and to represent graphically. Similarly, the property of causality expresses the desire to be able to perform an on-line calculation of the representation;

– the second group (properties 6 to 11) consists of some rather theoretical properties, which are related to the notions of energy distribution and energy density. If we desire, for example, that the TFR be an energy distribution, its moments and marginals must yield the elementary temporal and frequency representations of the signal. Wanting the representation to be an energy density is equivalent to imposing that it be positive or zero everywhere. This allows energy measurements in any sub-domain of the time-frequency plane. The constraint associated with this property forces the representation to be a linear combination of spectrograms with positive weighting coefficients. In view of expressions (5.5) and (5.6), this result shows that the two notions of energy distribution and energy density are incompatible;

– the third group (properties 12 to 15) corresponds to specific signal processing needs. Thus, we may wish that the application of filtering or modulation operators to a signal is translated into disjoint actions on the time and frequency variables, respectively. "Moyal's formula" (property 14) expresses the conservation of the scalar product and is the equivalent of Parseval's theorem (equation (5.2)) for bilinear time-frequency representations. It allows a time-frequency reformulation of the matched filters used in signal detection and pattern recognition [FLA 88]. Property 15 corresponds to the inverse problem of signal synthesis from a given representation [HLA 92a];

– finally, the fourth group (properties 16 to 18) concerns the ability to correctly interpret the results produced by a time-frequency representation. To reduce the risk of misinterpretation, it is indeed desirable that more or less strict conditions on support conservation be satisfied.

No.	Property	Mathematical formulation	Condition
0	covariance under time and frequency translations	$\forall\, t_1, f_1, y(t) = x(t-t_1)\, e^{j2\pi f_1 t}$ $\Rightarrow\ C_y(t,f) = C_x(t-t_1, f-f_1)$	none
1	covariance under scale changes	$\forall\, a > 0, y(t) = a^{-1/2}\, x(t/a)$ $\Rightarrow\ C_y(t,f) = C_x(t/a, af)$	$\forall\, \tau, \xi, \forall\, a > 0,$ $\phi_{\text{d-D}}(\tau,\xi) = \phi_{\text{d-D}}(\tau/a, a\xi)$
2	covariance under time inversion	$y(t) = x(-t),$ $\Rightarrow\ C_y(t,f) = C_x(-t,-f)$	$\forall\, \tau, \xi,$ $\phi_{\text{d-D}}(\tau,\xi) = \phi_{\text{d-D}}(-\tau,-\xi)$
3	covariance under complex conjugation	$y(t) = x^*(t),$ $\Rightarrow\ C_y(t,f) = C_x(t,-f)$	$\forall\, \tau, \xi,$ $\phi_{\text{d-D}}(\tau,\xi) = \phi_{\text{d-D}}(-\tau,\xi)$
4	real TFR	$\forall\, t, f,\ C_x(t,f) = C_x^*(t,f)$	$\forall\, \tau, \xi,$ $\phi^*_{\text{d-D}}(\tau,\xi) = \phi_{\text{d-D}}(-\tau,-\xi)$
5	causality	$C_x(t,f)$ only depends on $x(u)$ with $u < t$	$\forall\, t, \tau,$ $t < \lvert\tau\rvert/2\ \Rightarrow\ \phi_{\text{t-d}}(t,\tau) = 0$

Table 5.1. *First group of properties of Cohen's class representations*

No.	Property	Mathematical formulation	Condition
6	conservation of energy	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_x(t,f)\,dt\,df = \int_{-\infty}^{\infty}\|x(t)\|^2\,dt$	$\phi_{\text{d-D}}(0,0)=1$
7	conservation of instantaneous power	$\forall\, t,\ \int_{-\infty}^{\infty} C_x(t,f)\,df = \|x(t)\|^2$	$\forall\,\xi,\ \phi_{\text{d-D}}(0,\xi)=1$
8	conservation of spectral energy density	$\forall\, f,\ \int_{-\infty}^{\infty} C_x(t,f)\,dt = \|\widehat{x}(f)\|^2$	$\forall\,\tau,\ \phi_{\text{d-D}}(\tau,0)=1$
9	conservation of instantaneous frequency	$\forall\, t,\ \int_{-\infty}^{\infty} f\,C_x(t,f)\,df = f_x(t)\,\|x(t)\|^2$, $f_x(t)=\frac{1}{2\pi}\frac{d}{dt}\arg(x(t))$	$\forall\,\xi,\ \phi_{\text{d-D}}(0,\xi)=1$ and $\frac{\partial}{\partial\tau}\phi_{\text{d-D}}(0,\xi)=0$
10	conservation of group delay	$\forall\, f,\ \int_{-\infty}^{\infty} t\,C_x(t,f)\,dt = t_x(f)\,\|\widehat{x}(f)\|^2$, $t_x(f)=-\frac{1}{2\pi}\frac{d}{df}\arg(\widehat{x}(f))$	$\forall\,\tau,\ \phi_{\text{d-D}}(\tau,0)=1$ and $\frac{\partial}{\partial\xi}\phi_{\text{d-D}}(\tau,0)=0$
11	positivity	$\forall\, x(t),\ \forall\, t,f,\ C_x(t,f)\geq 0$	$\phi_{\text{d-D}}(\tau,\xi)=\int_{-\infty}^{\infty}c(\alpha)A_{h_\alpha}(\tau,\xi)\,d\alpha$ with $\forall\,\alpha,\ c(\alpha)\geq 0$

Table 5.2. *Second group of properties of Cohen's class representations*

No.	Property	Mathematical formulation	Condition
12	compatibility with linear filtering	$y(t)=\int_{-\infty}^{\infty} h(u)\,x(t-u)\,du \Rightarrow C_y(t,f)=\int_{-\infty}^{\infty} C_h(u,f)\,C_x(t-u,f)\,du$	$\forall\,\tau,\tau',\xi,$ $\phi_{\text{d-D}}(\tau,\xi)\,\phi_{\text{d-D}}(\tau',\xi)=\phi_{\text{d-D}}(\tau+\tau',\xi)$
13	compatibility with modulations	$y(t)=m(t)\,x(t) \Rightarrow C_y(t,f)=\int_{-\infty}^{\infty} C_m(t,\nu)\,C_x(t,f-\nu)\,d\nu$	$\forall\,\tau,\xi,\xi',$ $\phi_{\text{d-D}}(\tau,\xi)\,\phi_{\text{d-D}}(\tau,\xi')=\phi_{\text{d-D}}(\tau,\xi+\xi')$
14	conservation of scalar product (unitarity, "Moyal's formula")	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_x(t,f)\,C_y^*(t,f)\,dt\,df = \left\|\int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt\right\|^2$	$\forall\,\tau,\xi,\ \|\phi_{\text{d-D}}(\tau,\xi)\|^2=1$
15	invertibility	$x(t)$ can be recovered from the TFR up to a constant phase	$\forall\,\tau,\xi,\ \phi_{\text{d-D}}(\tau,\xi)\neq 0$

Table 5.3. *Third group of properties of Cohen's class representations*

No.	Property	Mathematical formulation	Condition
16	conservation of signal support	If $\forall\, t,\ \|t\|>T \Rightarrow x(t)=0$ then $\forall\, t,\ \|t\|>T \Rightarrow C_x(t,f)=0$	$\forall\, t,\tau,\ \|\tau\|<2\|t\| \Rightarrow \phi_{\text{t-d}}(t,\tau)=0$
17	conservation of zero values of signal	$x(t_0)=0 \Rightarrow \forall\, f,\ C_x(t_0,f)=0$	$\phi_{\text{d-D}}(\tau,\xi)=h_1(\tau)\,e^{-j2\pi\xi\tau/2}+h_2(\tau)\,e^{j2\pi\xi\tau/2}$
18	conservation of zero values of spectrum	$\widehat{x}(f_0)=0 \Rightarrow \forall\, t,\ C_x(t,f_0)=0$	$\phi_{\text{d-D}}(\tau,\xi)=G_1(\xi)\,e^{-j2\pi\xi\tau/2}+G_2(\xi)\,e^{j2\pi\xi\tau/2}$

Table 5.4. *Fourth group of properties of Cohen's class representations*

A global analysis of the constraints collected in these tables first shows that most of them find their simplest expression in the ambiguity plane (i.e., in the Doppler-delay plane). Thus, knowledge of the kernel $\phi_{\text{d-D}}(\tau,\xi)$ of a representation is the easiest way to deduce its properties. This analysis also shows that properties 1 to 10 are easily satisfied if $\phi_{\text{d-D}}(\tau,\xi)$ is a function of only the product of the two variables, i.e., $\phi_{\text{d-D}}(\tau,\xi) = P(\tau\xi)$, where $P(x)$ is an even real function satisfying $P(0) = 1$ and $P'(0) = 0$. These *product-type representations* [JEO 92, HLA 94], among which the Wigner-Ville distribution is a limit case since it corresponds to $P(x) = 1$, are also interesting because of the attractive properties of their interference terms, as will be seen in the next section.

5.4.5. *Existence and localization of interference terms*

5.4.5.1. *Existence and origin of interference terms*

Imposing that the time-frequency representation is a bilinear form of the signal (equation (5.8)) has the direct consequence that the representation of the sum of two signals does not equal the sum of the representations of the individual components: if $y(t) = x_1(t) + x_2(t)$, then

$$\text{TFR}_y(t,f) \;=\; \text{TFR}_{x_1}(t,f) + \text{TFR}_{x_2}(t,f) + \text{TFR}_{x_1x_2}(t,f) + \text{TFR}_{x_2x_1}(t,f)$$

with

$$\text{TFR}_{x_1x_2}(t,f) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{\text{t-d}}(t-v,\tau)\, x_1\Big(v+\frac{\tau}{2}\Big)\, x_2^*\Big(v-\frac{\tau}{2}\Big)\, e^{-j2\pi f\tau}\, dv\, d\tau\,.$$

The additional terms are called *interference terms* (by analogy with the interferences obtained in optics when two sources of light interact), while the terms due to the signal components taken independently are called *autoterms*. If the time-frequency representation conserves the signal energy (Table 5.2, property 6), these interference terms distribute the scalar product of the two signal components:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{TFR}_y(t,f)\, dt\, df \;=\; \int_{-\infty}^{\infty} |x_1(t)+x_2(t)|^2\, dt\,,$$

thus

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big[\text{TFR}_y(t,f) - \text{TFR}_{x_1}(t,f) - \text{TFR}_{x_2}(t,f)\big]\, dt\, df$$
$$=\; 2\,\text{Re}\left\{\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt\right\}.$$

Their presence is thus an indispensable condition for the energy conservation property to be satisfied [HLA 97a].

However, the localization and amplitude of these additional terms often make the use and interpretation of the representation difficult, or even impossible when the signal contains a large number of "elementary components". Since these interference terms distribute the real part of the scalar product in the time-frequency plane, they distribute negative values when the scalar product is negative. Therefore, it is not possible to use the time-frequency representation for local energy measurements. Moreover, when the scalar product is zero, the corresponding interference terms (which, unfortunately, are not uniformly zero) take on positive and negative values in an alternating manner, thus producing patterns that could be mistaken for autoterms.

5.4.5.2. *Janssen's formulae*

The difficulties due to the presence of interference terms are attenuated by the possibility of predicting their localization [HLA 97a]. *Janssen's formula*, in particular, shows that if $e_1(t)$ and $e_2(t)$ are localized around the points (t_1, f_1) and (t_2, f_2), respectively, the interference term of the Wigner-Ville distribution (defined by equation (5.13)) that is due to these components is localized at the center point $(\frac{t_1+t_2}{2}, \frac{f_1+f_2}{2})$:

$$\big|W_{e_1e_2}(t,f)\big|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{e_1}\Big(t+\frac{\tau}{2}, f+\frac{\xi}{2}\Big)\, W_{e_2}\Big(t-\frac{\tau}{2}, f-\frac{\xi}{2}\Big)\, d\tau\, d\xi\,. \quad (5.14)$$

Since this expression remains valid when $e_1(t) = e_2(t)$, it also shows the existence of a particular category of interference terms called *inner interferences*. When the time-frequency region where the signal energy is concentrated is not convex, there are at least two points (t_1, f_1) and (t_2, f_2) whose center point is outside that region. This generates an interference at that center point although both interfering points belong to the same signal component (see Figure 5.3).

Relations similar to Janssen's formula also exist for other elements of Cohen's class. For the Rihaczek distribution (defined by equation (5.7)), we simply obtain

$$\big|R_{e_1e_2}(t,f)\big|^2 = |e_1(t)|^2\, |\widehat{e}_2(f)|^2 \quad (5.15)$$

which shows that the interaction between two components is located where the time support of the first component and the frequency support of the second component intersect. The Rihaczek distribution of $e_1(t) + e_2(t)$ thus contains two interference terms centered around the points (t_1, f_2) and (t_2, f_1) (see Figure 5.3). Consequently, when $t_1 = t_2$, interference terms and autoterms have the same support (see Figure 5.4). Using this distribution for signals presenting multiple simultaneous components (or, by duality, multiple components in the same frequency band) is thus ill-advised.

The Wigner-Ville and Rihaczek distributions are special cases (obtained for $\alpha = 0$ and $\alpha = 1/2$, respectively) of the *generalized Wigner-Ville distribution* [JAN 82]

$$W_x^{\alpha}(t,f) = \int_{-\infty}^{\infty} x\Big(t+\Big(\frac{1}{2}-\alpha\Big)\tau\Big)\, x^*\Big(t-\Big(\frac{1}{2}+\alpha\Big)\tau\Big)\, e^{-j2\pi f\tau}\, d\tau\,,$$

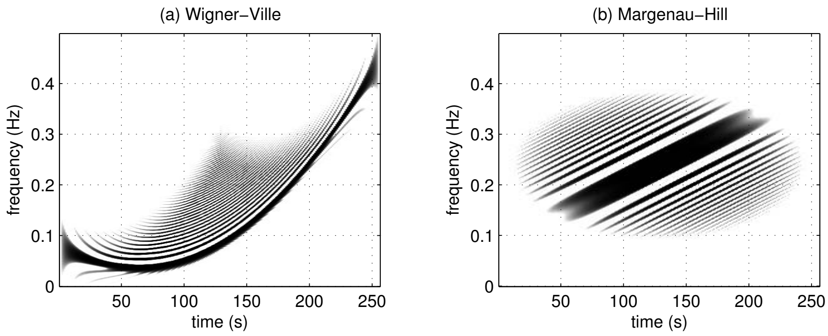

Figure 5.3. *Inner interferences of the Wigner-Ville distribution and of the Margenau-Hill distribution (equal to the real part of the Rihaczek distribution). The geometry of interferences of the Wigner-Ville distribution described by expression (5.14) shows that two points* (t_1, f_1) *and* (t_2, f_2) *of the same component produce an interference at the point* $\left(\frac{t_1+t_2}{2}, \frac{f_1+f_2}{2}\right)$*, which increases its time-frequency support. This result is illustrated in (a), which shows the representation of a signal with a constant envelope and a parabolic frequency. For the Margenau-Hill distribution, equation (5.15) shows that two points* (t_1, f_1) *and* (t_2, f_2) *of the same component produce an interference at* (t_1, f_2) *and* (t_2, f_1)*. This result is illustrated in (b), which shows the representation of a signal with a Gaussian envelope and a linearly modulated frequency*

for which $\phi_{\text{d-D}}(\tau, \xi) = e^{-j2\pi\alpha\xi\tau}$. When α is a real number between $-1/2$ and $1/2$, the generalized Wigner-Ville distribution corresponds to all the distributions satisfying properties 0, 1, 2, 6, 7, 8, 12, 13, 14, 15, and 16. For these distributions, there exists another relation analogous to Janssen's formula [HLA 97a]:

$$\left|W^{\alpha}_{e_1,e_2}(t,f)\right|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W^{\alpha}_{e_1}\left(t+\left(\frac{1}{2}-\alpha\right)\tau, f+\left(\frac{1}{2}+\alpha\right)\xi\right) \cdot W^{\alpha *}_{e_2}\left(t-\left(\frac{1}{2}+\alpha\right)\tau, f-\left(\frac{1}{2}-\alpha\right)\xi\right) d\tau\, d\xi\,.$$

This relation demonstrates that the interference terms of the signal $e_1(t) + e_2(t)$ are located at coordinates $\left(\frac{t_1+t_2}{2} + \alpha\,(t_1-t_2), \frac{f_1+f_2}{2} - \alpha\,(f_1-f_2)\right)$ and $\left(\frac{t_1+t_2}{2} - \alpha\,(t_1 - t_2), \frac{f_1+f_2}{2} + \alpha\,(f_1-f_2)\right)$. The main interest of this result is that it allows a characterization of the interference terms of all the product kernel representations. In fact, it is possible to express a representation of this type as a continuous sum of generalized Wigner-Ville distributions [HLA 94, HLA 97a]:

$$\phi_{\text{d-D}}(\tau,\xi) = P(\tau\xi) = \int_{-\infty}^{\infty} p(\alpha)\, e^{-j2\pi\alpha\xi\tau}\, d\alpha\,, \tag{5.16}$$

therefore

$$C_x(t,f) = \int_{-\infty}^{\infty} p(\alpha)\, W^{\alpha}_x(t,f)\, d\alpha\,.$$

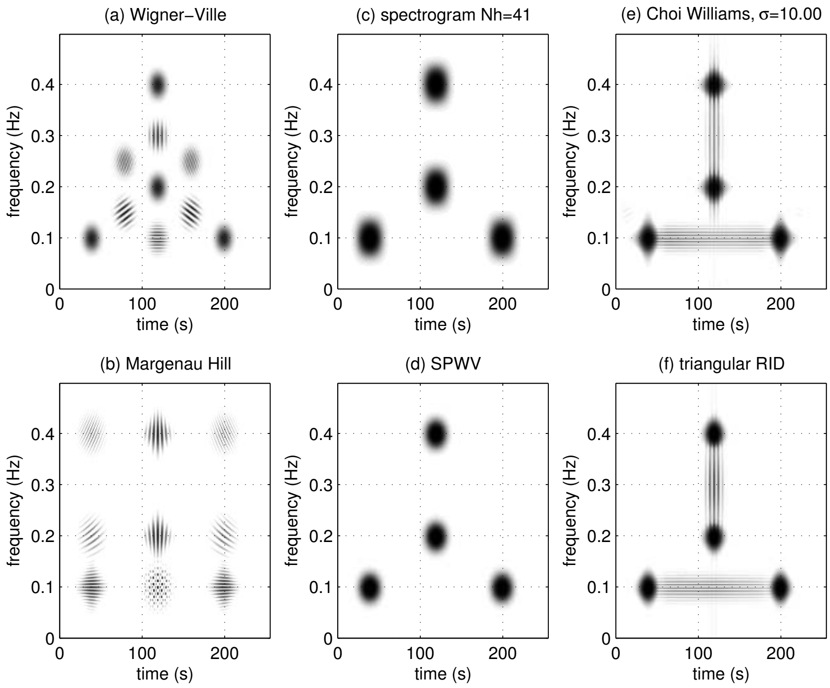

Figure 5.4. *Time-frequency representations of a signal consisting of four components with constant frequency and Gaussian amplitude. The Wigner-Ville distribution (a) localizes the components precisely but contains six interference terms. The Margenau-Hill distribution (b) contains five interference terms (in different locations than in the previous case) as well as additional interference terms superimposed with the autoterms. The spectrogram (c) does not present interferences (the components are too far apart for that), but the autoterms have a less precise localization. With the smoothed pseudo-Wigner-Ville distribution (d), a better compromise between the localization of autoterms and the reduction of interference terms is obtained. The Choi-Williams distribution (e) and the so-called "reduced interference" distribution (f) suppress a large part of the interference terms. However, due to the structure of their kernel, they do not suppress the interferences between two components located at the same frequency or at the same time instant. In this case, the interference terms in the Choi-Williams distribution can be greater than the autoterms, whereas this is not so for the reduced interference distribution*

It follows that for these representations, the support of the interference terms of the signal $e_1(t) + e_2(t)$ is on the straight line defined by

$$\left(\frac{t_1+t_2}{2} - \alpha\,(t_1-t_2), \frac{f_1+f_2}{2} + \alpha\,(f_1-f_2)\right),$$

and their amplitude and width are deduced from the values and support of $p(\alpha)$.