High-Reynolds-Number-Asymptotics of Turbulent Boundary Layers: Novel Aspects with Emphasis on Flow Separation

Bernhard Scheichl
bernhard.scheichl@tuwien.ac.at

Institute of Fluid Mechanics and Heat Transfer

Technische Universität Wien
Vienna University of Technology

Colloquium About Applied Mathematics

Institute of Numerical and Applied Mathematics, University of Göttingen
January 20th, 2009
Overview

1. Turbulent wall-bounded flows
2. Classical theory of turbulent small-defect BLs
3. Turbulent BLs with moderately large velocity defect
4. Turbulent BLs with large velocity defect
5. Problem of (turbulent) break-away separation
6. Asymptotic structure of bluff-body flows
7. Separation for high turbulence intensities
8. Conclusions and outlook
Wall-bounded high-Reynolds-number flows: \( Re := \frac{\bar{U}\bar{L}}{\bar{v}} \gg 1 \)

Navier–Stokes eqs (\( \rho = \text{const} \))

\[
\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad \mathbf{u}|_{y=0} = 0
\]

Reynolds-averaging, ergod hypothesis (nominally steady 2D flow)

\[
\mathbf{Q}(x, t; Re) = \langle \mathbf{Q} \rangle(x, y; Re) + \mathbf{Q}'(x, t; Re)
\]

\[
\langle \mathbf{Q} \rangle := \lim_{t_{av} \to \infty} \frac{1}{t_{av}} \int_{-t_{av}/2}^{t_{av}/2} \mathbf{Q}(x, t + t'; Re) \, dt'
\]
Wall-bounded high-Reynolds-number flows: \( Re := \tilde{U}\tilde{L}/\tilde{v} \gg 1 \)

**Navier–Stokes eqs (\( \rho = \text{const} \))**

\[
\nabla \cdot \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \rho + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad \mathbf{u}|_{y=0} = 0
\]

**Reynolds-averaging, ergod hypothesis (nominally steady 2D flow)**

\[
Q(x, t; Re) = \langle Q \rangle(x, y; Re) + Q'(x, t; Re)
\]

\[
\langle Q \rangle := \lim_{t_{av} \to \infty} \frac{1}{t_{av}} \int_{-t_{av}/2}^{t_{av}/2} Q(x, t + t'; Re) \, dt'
\]

B. Scheichl (TU Vienna)  
Asymptotic theory of turbulent boundary layers
Outline

1. Turbulent wall-bounded flows
   Numerically-based methods
   Analytically-based methods
Numerically-based methods

- **DNS**: resolution of all scales $\Rightarrow$ no modelling needed but at present restricted to moderately large values of $Re$

- **LES**: modelling of short scales $\Rightarrow$ reduction of computational efforts, which, however, are still too massive to be useful for the solution of engineering problems

- **RANS**: modelling of all scales $\Rightarrow$ engineering problems can be solved with an acceptable amount of computational efforts, which, however, increase with increasing values of $Re$
Numerically-based methods

- **DNS**: resolution of all scales $\Rightarrow$ no modelling needed but at present restricted to moderately large values of $Re$

- **LES**: modelling of short scales $\Rightarrow$ reduction of computational efforts, which, however, are still too massive to be useful for the solution of engineering problems

- **RANS**: modelling of all scales $\Rightarrow$ engineering problems can be solved with an acceptable amount of computational efforts, which, however, increase with increasing values of $Re$
Numerically-based methods

- **DNS**: resolution of all scales $\Rightarrow$ no modelling needed but at present restricted to moderately large values of $Re$

- **LES**: modelling of short scales $\Rightarrow$ reduction of computational efforts, which, however, are still too massive to be useful for the solution of engineering problems

- **RANS**: modelling of all scales $\Rightarrow$ engineering problems can be solved with an acceptable amount of computational efforts, which, however, increase with increasing values of $Re$
Turbulent wall-bounded flows
Numerically-based methods
Analytically-based methods
Analytically-based methods

- full NS eqs: starting efforts, but still no complete theory exists
  Scheichl & Kluwick (JFS, 2008)

- non-dimensional Reynolds-averaged NS eqs
  (nominally 2D, curvature effects on BL flow of higher order):

  \[
  \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
  \]

  \[
  u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} - \frac{\partial \langle u'^2 \rangle}{\partial x} - \frac{\partial \langle u' v' \rangle}{\partial y} + \frac{1}{Re} \nabla^2 u
  \]

  \[
  u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} - \frac{\partial \langle u' v' \rangle}{\partial x} - \frac{\partial \langle v'^2 \rangle}{\partial y} + \frac{1}{Re} \nabla^2 v
  \]

  \[
  y = 0: \quad u = v = u' = v' = 0, \quad y \sim \delta(x; Re): \quad u \sim U_e(x), \quad \tau \sim 0
  \]

  \Rightarrow \text{asymptotic theory faces closure problem for } \tau
Analytically-based methods

- full NS eqs: starting efforts, but still no complete theory exists
  Scheichl & Kluwick (JFS, 2008)

- non-dimensional Reynolds-averaged NS eqs
  (nominally 2D, curvature effects on BL flow of higher order):

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} - \frac{\partial \langle u' \rangle^2}{\partial y} - \frac{\partial \langle u' v' \rangle}{\partial y} + \frac{1}{Re} \nabla^2 u \\
\frac{u}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} - \frac{\partial \langle u' \rangle^2}{\partial x} - \frac{\partial \langle v' \rangle^2}{\partial x} + \frac{1}{Re} \nabla^2 v
\end{align*}
\]

\[y = 0: \ u = v = u' = v' = 0, \quad y \sim \delta(x; Re): \ u \sim U_0(x), \quad \tau \sim 0\]

⇒ asymptotic theory faces closure problem for \(\tau\)
Outline

2. Classical theory of turbulent small-defect BLs
   General framework
   Self-similar flows
Classical theory of turbulent small-defect BLs: $u_\tau := \tau_w^{1/2} \to 0$


$$\begin{align*}
\tau_w &= Re^{-1}(\partial u/\partial y)_{y=0} \\
\delta &= O(u_\tau)
\end{align*}$$

outer, predominately inviscid region

overlap region: $y^+ := y u_\tau Re \to \infty$

$$\begin{align*}
\frac{u}{u_\tau} &\sim \frac{1}{\kappa} \ln y^+ + C_i
\end{align*}$$

assumptions


(a) locally isotropic turbulence $\Rightarrow \langle u'^2 \rangle, \langle u'v' \rangle, \langle v'^2 \rangle$ of same magnitude

(b) wall layer: total shear stress essentially unaffected by pressure gradient

(c) direct match with outer fully turbulent region $\Rightarrow$ two-tiered BL
Classical theory of turbulent small-defect BLs: \[ u_\tau := \frac{\tau_w}{2} \to 0 \]


\[
y^+ := y u_\tau Re \to \infty
\]

\[
\frac{u}{u_\tau} \sim \frac{1}{\kappa} \ln y^+ + C_i
\]

\[
\tau_w = Re^{-1} (\partial u / \partial y)_{y=0}
\]

assumptions


(a) locally isotropic turbulence \( \Rightarrow \langle u'^2 \rangle, \langle u'v' \rangle, \langle v'^2 \rangle \) of same magnitude

(b) wall layer: total shear stress essentially unaffected by pressure gradient

(c) direct match with outer fully turbulent region \( \Rightarrow \) two-tiered BL
Classical theory of turbulent small-defect BLs:  \( u_\tau := \tau_w^{1/2} \rightarrow 0 \)


\( \tau_w = Re^{-1}(\partial u/\partial y)_{y=0} \)

assumptions  

(a) locally isotropic turbulence  \( \Rightarrow \langle u'^2 \rangle, \langle u'v' \rangle, \langle v'^2 \rangle \) of same magnitude

(b) wall layer: total shear stress essentially unaffected by pressure gradient

(c) direct match with outer fully turbulent region  \( \Rightarrow \) two-tiered BL
Distinguished limit \( \gamma := \frac{u_\tau}{U_e} \to 0, \quad \text{Re} \to \infty \)

viscous wall layer: \( y^+ = y u_\tau \text{Re} \)

\[
\frac{u}{U_e} \sim \gamma(x, \text{Re}) u^+(x, y^+) + \cdots
\]

\[
\frac{\tau}{U_e^2} \sim \gamma^2(x, \text{Re}) t^+(x, y^+) + \cdots
\]

\[
p \sim p_0(x) + \cdots
\]

outer defect layer: \( \eta = y/\delta \)

\[
\frac{u}{U_e} \sim 1 - \gamma \frac{\partial F_1(x, \eta)}{\partial \eta} + O(\gamma^2)
\]

\[
\frac{\tau}{U_e^2} \sim \gamma^2 T_1(x, \eta) + O(\gamma^3)
\]

\[
p \sim p_e(x) + O(\gamma^2)
\]

---

Experimental observation / Result from first principles (?.)


\[
u^+(y^+) \sim \kappa^{-1} \ln y^+ + C_i, \quad y^+ \to \infty, \quad \kappa \approx 0.384, \quad C_i \approx 4.1
\]

---

Provides expansion in outer layer and matching with wall layer

\[
\eta \to 0: \quad \partial F_1/\partial \eta \sim -\kappa^{-1} \ln \eta + C_o(x), \quad T_1 \to 1; \quad p_0(x) = p_e(x)
\]

Skin-friction law \( \kappa/\gamma \sim \ln(\text{Re} \gamma \delta U_e) + \kappa(C_i + C_o) \sim \ln \text{Re} \)
Distinguished limit \( \gamma := \frac{u_\tau}{U_e} \to 0, \quad Re \to \infty \)

**viscous wall layer:** \( y^+ = y u_\tau Re \)

\[
\frac{u}{U_e} \sim \gamma(x, Re) u^+(x, y^+) + \cdots \\
\frac{\tau}{U_e^2} \sim \gamma^2(x, Re) t^+(x, y^+) + \cdots \\
p \sim p_0(x) + \cdots
\]

**outer defect layer:** \( \eta = y/\delta \)

\[
\frac{u}{U_e} \sim 1 - \gamma \frac{\partial F_1(x, \eta)}{\partial \eta} + O(\gamma^2) \\
\frac{\tau}{U_e^2} \sim \gamma^2 T_1(x, \eta) + O(\gamma^3) \\
p \sim p_e(x) + O(\gamma^2)
\]

---

Experimental observation / Result from first principles (\(?\))

\( \text{Österlund et al. (Phys Fluids, 2000)} \quad \text{Walker (AIAA J, 1989, CISM, 1998)} \)

\[ u^+(y^+) \sim \kappa^{-1} \ln y^+ + C_i, \quad y^+ \to \infty, \quad \kappa \approx 0.384, \quad C_i \approx 4.1 \]

---

Provides expansion in outer layer and matching with wall layer

\( \eta \to 0 : \quad \partial F_1/\partial \eta \sim -\kappa^{-1} \ln \eta + C_o(x), \quad T_1 \to 1; \quad p_0(x) = p_e(x) \)

Skin-friction law \( \kappa/\gamma \sim \ln(Re \gamma \delta U_e) + \kappa(C_i + C_o) \sim \ln Re \)
Substitution into Reynolds-averaged NS eqs

viscous wall layer: \( du^+ / dy^+ + t^+ = 1 \) . . . constant total shear stress

outer defect layer: \( \delta \sim \gamma \Delta_1(x) + O(\gamma^2) \)

**Leading-order equation**

\[
(E + 2\beta_0)\eta F'_1 - E F_1 - \Delta_1 F_{1,e} \partial F_1 / \partial x = F_{1,e} (T_1 - 1), \quad F_{1,e} = F_1(x, 1)
\]

\[
E := 1 - \Delta_1 df_{1,e} / dx, \quad \beta_0 := - (\Delta_1 F_{1,e} dU_e / dx) / U_e
\]

layer thickness ratio exponentially small

\[
\frac{\delta^+}{\delta} = \frac{(u^- R_e)^{-1}}{\delta} \sim \frac{1}{Re \gamma^2 \Delta_1 U_e} \sim \frac{1}{\Delta_1 U_e} \frac{\exp(-\kappa / \gamma)}{\gamma^2}
\]

consequences

classical theory not capable of describing BL separation - Sykes (JFM, 1980)

consider BL's having a velocity defect \( \epsilon \gg \gamma \)

Scheichl & Kluwick

B. Scheichl (TU Vienna)  Asymptotic theory of turbulent boundary layers
Substitution into Reynolds-averaged NS eqs

viscous wall layer: \[ \frac{du^+}{dy^+} + t^+ = 1 \ldots \] constant total shear stress

outer defect layer: \[ \delta \sim \gamma \Delta_1(x) + O(\gamma^2) \]

leading-order equation

\[
(E + 2\beta_0)\eta F'_1 - EF_1 - \Delta_1 F_{1,e} \frac{\partial F_1}{\partial x} = F_{1,e}(T_1 - 1), \quad F_{1,e} = F_1(x, 1)
\]

\[
E := 1 - \Delta_1 \frac{dF_{1,e}}{dx}, \quad \beta_0 := -\left(\Delta_1 F_{1,e} \frac{dU_e}{dx}\right) / U_e
\]

layer thickness ratio exponentially small

\[
\frac{\delta^+}{\delta} = \frac{(u_T Re)^{-1}}{\delta} \sim \frac{1}{Re \gamma^2 \Delta_1 U_e} \sim \frac{1}{\Delta_1 U_e} \frac{\exp(-\kappa/\gamma)}{\gamma^2}
\]

consequences

classical theory not capable of describing BL separation

consider BLs having a velocity defect \( \epsilon \gg \gamma \)

Sykes (JFM, 1980)

Scheichl & Kluwick
Substitution into Reynolds-averaged NS eqs

viscous wall layer: \[ \frac{du^+}{dy^+} + t^+ = 1 \ldots \text{constant total shear stress} \]

outer defect layer: \[ \delta \sim \gamma \Delta_1(x) + O(\gamma^2) \]

leading-order equation

\[
(E + 2\beta_0)\eta F_1' - EF_1 - \Delta_1 F_{1,e} \frac{\partial F_1}{\partial x} = F_{1,e}(T_1 - 1), \quad F_{1,e} = F_1(x, 1)
\]

\[ E := 1 - \Delta_1 \frac{dF_{1,e}}{dx}, \quad \beta_0 := -\left(\Delta_1 F_{1,e} \frac{dU_e}{dx}\right) / U_e \]

layer thickness ratio exponentially small

\[
\frac{\delta^+}{\delta} = \left(\frac{u_\tau Re}{\delta}\right)^{-1} \sim \frac{1}{Re \gamma^2 \Delta_1 U_e} \sim \frac{1}{\Delta_1 U_e} \exp(-\kappa/\gamma) \gamma^2
\]

consequences

classical theory not capable of describing BL separation

consider BLs having a velocity defect \( \epsilon \gg \gamma \)
Outline

2 Classical theory of turbulent small-defect BLs
   General framework
   Self-similar flows
Self-similar flows: \( \beta_0 = \text{const} \)

- require \( \partial[F_1, T_1]/\partial x \equiv 0 \iff [F_1, T_1] = [F_1, T_1](\eta; \beta_0) \)

- Rotta–Clauser parameter \( \beta := -\frac{\delta^* U_e U_{ex}}{\tau_w} \sim \beta_0 + O(\gamma) \)

**leading-order equation**

\[
(1 + 2\beta_0)\eta F_1' - F_1 = F_{1,e} (T_1 - 1), \quad \beta_0 = -\frac{\Delta_1 F_{1,e} U_{ex}}{U_e}
\]

\( U_e \propto (x - x_v)^m, \quad m = -\beta_0/(1 + 3\beta_0), \quad \Delta_1 F_{1,e} = (1 + 3\beta_0)(x - x_v) \)

**large values of \( \beta_0 \) \( \iff \) \( m \to -1/3 \)

Mellor & Gibson (JFM, 1966)

\( F_1 \sim \beta_0^{1/2} \hat{F}(\hat{\eta}), \quad T_1 \sim \beta_0 \hat{T}(\hat{\eta}), \quad \eta = \beta_0^{1/2} \hat{\eta} \)

\( 2\hat{\eta} \hat{F}' = \hat{F}_e \hat{T}, \quad \hat{F}_e = \hat{F}(1), \quad \text{note:} \quad \hat{T}(0) = 0 \)
Self-similar flows: \( \beta_0 = \text{const} \)

- require \( \partial[F_1, T_1]/\partial x \equiv 0 \iff [F_1, T_1] = [F_1, T_1](\eta; \beta_0) \)
- Rotta–Clauser parameter \( \beta := -\frac{\delta^* U_e U_{ex}}{\tau_w} \sim \beta_0 + O(\gamma) \)

**leading-order equation**

\[
(1 + 2\beta_0)\eta F_1' - F_1 = F_{1,e} (T_1 - 1), \quad \beta_0 = -\frac{\Delta_1 F_{1,e} U_{ex}}{U_e}
\]

\( U_e \propto (x - x_v)^m \), \quad m = -\beta_0/(1 + 3\beta_0), \quad \Delta_1 F_{1,e} = (1 + 3\beta_0)(x - x_v) \)

**large values of \( \beta_0 \) \iff \( m \to -1/3_+ \)**

\[
F_1 \sim \beta_0^{1/2} \hat{F}(\hat{\eta}), \quad T_1 \sim \beta_0 \hat{T}(\hat{\eta}), \quad \eta = \beta_0^{1/2} \hat{\eta}
\]

\( 2\hat{\eta} F' = \hat{F}_e \hat{T}, \quad \hat{F}_e = \hat{F}(1), \quad \text{note:} \quad \hat{T}(0) = 0 \)
Self-similar flows: $\beta_0 = \text{const}$

- require $\partial[F_1, T_1]/\partial x \equiv 0 \iff [F_1, T_1] = [F_1, T_1](\eta; \beta_0)$

- Rotta–Clauser parameter $\beta := -\frac{\delta^* U_e U_{ex}}{\tau_w} \sim \beta_0 + O(\gamma)$

**leading-order equation**

$$(1 + 2\beta_0)\eta F_1' - F_1 = F_{1,e}(T_1 - 1), \quad \beta_0 = -\frac{\Delta_1 F_{1,e} U_{ex}}{U_e}$$

$U_e \propto (x - x_V)^m, \quad m = -\beta_0/(1 + 3\beta_0), \quad \Delta_1 F_{1,e} = (1 + 3\beta_0)(x - x_V)$

**large values of $\beta_0$** $\Rightarrow$ $m \to -1/3_+$

$F_1 \sim \beta_0^{1/2} \hat{F}(\hat{\eta}), \quad T_1 \sim \beta_0 \hat{T}(\hat{\eta}), \quad \eta = \beta_0^{1/2} \hat{\eta}$

$2\hat{\eta} \hat{F}' = \hat{F}_e \hat{T}, \quad \hat{F}_e = \hat{F}(1), \quad \text{note: } \hat{T}(0) = 0$
Self-similar flows: \( \beta_0 = \text{const} \)

- require \( \partial[F_1, T_1]/\partial x \equiv 0 \iff [F_1, T_1] = [F_1, T_1](\eta; \beta_0) \)

- Rotta–Clauser parameter \( \beta := -\frac{\delta^* U_e U_{ex}}{\tau_w} \sim \beta_0 + O(\gamma) \)

**leading-order equation**

\[
(1 + 2\beta_0)\eta F'_1 - F_1 = F_{1,e}(T_1 - 1), \quad \beta_0 = -\frac{\Delta_1 F_{1,e} U_{ex}}{U_e}
\]

\( U_e \propto (x - x_v)^m, \quad m = -\beta_0/(1 + 3\beta_0), \quad \Delta_1 F_{1,e} = (1 + 3\beta_0)(x - x_v) \)

**large values of \( \beta_0 \) \iff \( m \rightarrow -1/3^+ \) \quad \text{Mellor & Gibson (JFM, 1966)}

\[
F_1 \sim \beta_0^{1/2} \hat{F}(\hat{\eta}), \quad T_1 \sim \beta_0 \hat{T}(\hat{\eta}), \quad \eta = \beta_0^{1/2} \hat{\eta}
\]

\[
2\hat{\eta} \hat{F}' = \hat{F}_e \hat{T}, \quad \hat{F}_e = \hat{F}(1), \quad \text{note:} \quad \hat{T}(0) = 0
\]
Outline

3 Turbulent BLs with moderately large velocity defect
   General framework
   Quasi-equilibrium flows
Moderately large velocity defect: $\gamma \ll \epsilon \ll 1$


Outer main layer

$$\frac{u}{U_e} = 1 - \epsilon F'(x, \eta; \epsilon, \gamma), \quad \frac{\tau}{U_e^2} = \epsilon^2 T(x, \eta; \epsilon, \gamma), \quad \eta = \frac{y}{\delta}$$

Flow behaviour as $\eta \to 0$: $\tau_w/U_e^2 \sim \gamma^2 \sim \epsilon^2 T(x, 0; \ldots)$

$T(x, 0; 0, 0) = 0$
Intermediate layer

\[ \frac{u}{U_e} \sim 1 - \epsilon W - \gamma \hat{u} + \cdots \]

estimate of layer thickness \( \delta_i \) and velocity defect \( \epsilon \)

\[ u \frac{\partial u}{\partial x} + \cdots \sim -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \quad \Rightarrow \quad \tau \sim \tau_w - \epsilon y \frac{\partial (U_e^2 W)}{\partial x} \]

\[ \delta_i \sim \tau_w \frac{\tau_w}{-U_e U_{ex}} \]

\[ \frac{\tau_w}{U_e^2} = \gamma^2 \sim \epsilon \frac{\epsilon}{\beta} \quad \Rightarrow \quad \epsilon := \gamma \beta^{1/2} \]
Intermediate layer

\[
\frac{u}{U_e} \sim 1 - \epsilon W - \gamma \hat{u} + \cdots
\]

estimate of layer thickness \(\delta_i\) and velocity defect \(\epsilon\)

\[
u \frac{\partial u}{\partial x} + \cdots \sim -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \Rightarrow \tau \sim \tau_w - \epsilon y \frac{\partial (U_e^2 W)}{\partial x}
\]

- \[
\frac{\tau_w}{-U_e U_{ex}} \sim \delta_i \epsilon \quad \Rightarrow \quad \delta_i \sim \frac{\tau_w}{\epsilon (-U_e U_{ex})} \quad \Rightarrow \quad \frac{\delta_i}{\delta} \sim \frac{\tau_w}{\delta^* (-U_e U_{ex})} = \frac{1}{\beta}
\]

- \[
\frac{\tau_w}{U_e^2} = \gamma^2 \sim \epsilon \frac{\epsilon}{\beta} \quad \Rightarrow \quad \epsilon := \gamma \beta^{1/2}
\]
formal expansions

\[ \frac{u}{U_e} \sim 1 - \epsilon W(x; \epsilon, \gamma) - \gamma \hat{u}(x, \zeta) + \cdots, \quad \zeta = \frac{y}{\Delta \gamma^2} \sim \frac{y \beta}{\delta} \sim \frac{y}{\delta_i} \]

\[ \frac{\tau}{U_e^2 \gamma^2} \sim T_1(x, \zeta; \epsilon, \gamma) = 1 + \lambda \zeta, \quad \lambda = -\frac{\Delta}{U_e^2} \frac{d(U_e^2 W)}{dx} \]

matching with adjacent layers \( \Rightarrow \) mixing length \( \ell \sim \kappa y \)

\[ T_1 = (\kappa \zeta)^2 \left( \frac{\partial \hat{u}}{\partial \zeta} \right)^2 = 1 + \lambda \zeta \Rightarrow \]

\[ \kappa \hat{u} = -\ln \zeta + 2 \ln \left( (1 + \lambda \zeta)^{1/2} + 1 \right) - 2(1 + \lambda \zeta)^{1/2} \]

\[ \zeta \to \infty : \quad \kappa \hat{u} \sim -2(\lambda \zeta)^{1/2} + \cdots \quad \text{square-root law} \quad \text{Townsend (JFM, 1961)} \]

\[ \zeta \to 0 : \quad \kappa \hat{u} \sim -\ln(\lambda \zeta/4) - 2 + \cdots \quad \text{log law} \]

matching with classical wall layer \( \Rightarrow \) generalised skin-friction law

\[ \frac{\kappa}{\gamma} \sim \ln \left[ \frac{Re \gamma^2 U_e^3}{\beta_0^{1/2}} \right] + \beta_0^{1/2} \kappa W + O(\gamma \beta_0) \sim \ln Re \]
Outer layer — cont’d

2nd-order analysis includes surface curvature \(k(x)\)

\[
\eta = \frac{y}{\delta(x; \epsilon, \gamma)}
\]

\[
p = p_0(x) + U_e^2[k(x) y + \epsilon^2 P(x, \eta; \ldots)] , \quad \frac{dp_0}{dx} = -U_e \frac{dU_e}{dx}
\]

\[
\frac{\psi}{U_e} = y - k(x) \frac{y^2}{2} - \delta \epsilon F(x, \eta; \ldots) , \quad -\left[ \frac{\langle u'^2 \rangle, \langle v'^2 \rangle, \langle u' v \rangle}{U_e^2} \right] = \epsilon^2 [R, S, T](x, \eta; \ldots)
\]

expansions

\[
Q = Q_1 + \epsilon Q_2 + \cdots , \quad Q = F, P, R, S, T, W , \quad \delta/\epsilon = \Delta_1 + \epsilon \delta_2 + \cdots
\]

\[
\beta/\beta_0(x_0) = B_1(x) + \epsilon B_2(x) + \cdots , \quad B_1(x_0) = 1 , \quad B_2(x_0) = \cdots = 0
\]

1st-order problem

\[
\frac{1}{U_e} \frac{d(U_e \Delta_1)}{dx} \eta F'_1 - \frac{1}{U_e^3} \frac{\partial(U_e^3 \Delta_1 F_1)}{\partial x} = T_1 , \quad F_1(x, 0) = F'_1(x, 0) = T_1(x, 1) = 0
\]

\[\eta \to 0 : \quad T_1 \sim (\kappa \eta F'_1'')^2 , \quad F'_1 \sim W_1(x) - (2/\kappa)(\lambda \eta)^{1/2}\]
Turbulent BLs with moderately large velocity defect

General framework

Quasi-equilibrium flows
Quasi-equilibrium flows: self-similar up to 2nd order

**classical theory**: $\beta \sim \beta_0 = O(1)$

$$B_1 \equiv 1, \quad \Delta_1 F_{1,e} = (1 + 3\beta_0)(x - x_v), \quad U_e \propto (x - x_v)^m$$

$m = -\beta_0/(1 + 3\beta_0)$ with $-1/3 < m < \infty$

limit of large $\beta_0$: $m \sim -1/3 + 1/(9\beta_0) + \cdots$

**1st-order problem**: $\partial F_1/\partial x \equiv \partial T_1/\partial x \equiv 0, \quad \beta_0 \to \infty \iff m \to -1/3$+

$$U_e = \hat{U}(x), \quad F_1 = \hat{F}(\eta), \quad T_1 = \hat{T}(\eta), \quad \Delta_1 = \hat{\Delta}(x)$$

$$B_1 \equiv 1, \quad \hat{\Delta}\hat{F}_e = 3(x - x_v), \quad \hat{U} = (C/3)^{1/3} (x - x_v)^{-1/3}$$

$$2\eta\hat{F}' = \hat{F}_e \hat{T}, \quad \hat{F}'(1) = \hat{F}''(1) = \hat{T}(1) = \hat{F}(0) = \hat{T}(0) = 0$$

mixing length model: $\hat{T} = l(\eta)^2 \hat{F}''(\eta)^2$

$$\hat{F}'(\eta) = \frac{1}{2\hat{F}_e} \left( \int_\eta^1 \frac{z^{1/2}}{l(z)} \, dz \right)^2, \quad \hat{F}_e = \left[ \frac{1}{2} \int_0^1 \left( \int_\eta^1 \frac{z^{1/2}}{l(z)} \, dz \right)^2 \, d\eta \right]^{1/2}$$
Quasi-equilibrium flows: self-similar up to 2nd order

classical theory: $\beta \sim \beta_0 = O(1)$

$B_1 \equiv 1$, $\Delta_1 F_{1,e} = (1 + 3\beta_0)(x - x_v)$, $U_e \propto (x - x_v)^m$

$m = -\beta_0/(1 + 3\beta_0)$ with $-1/3 < m < \infty$

limit of large $\beta_0$: $m \sim -1/3 + 1/(9\beta_0) + \cdots$

1st-order problem: $\partial F_1/\partial x \equiv \partial T_1/\partial x \equiv 0$, $\beta_0 \to \infty \Leftrightarrow m \to -1/3$

$U_e = \hat{U}(x)$, $F_1 = \hat{F}(\eta)$, $T_1 = \hat{T}(\eta)$, $\Delta_1 = \hat{\Delta}(x)$

$B_1 \equiv 1$, $\hat{\Delta}\hat{F}_e = 3(x - x_v)$, $\hat{U} = (C/3)^{1/3}(x - x_v)^{-1/3}$

$2\eta\hat{F}' = \hat{F}_e \hat{T}$, $\hat{F}'(1) = \hat{F}''(1) = \hat{T}(1) = \hat{F}(0) = \hat{T}(0) = 0$

mixing length model: $\hat{T} = l(\eta)^2 \hat{F}''(\eta)^2$

$\hat{F}'(\eta) = \frac{1}{2\hat{F}_e} \left( \int_{\eta}^{1} \frac{z^{1/2}}{l(z)} \, dz \right)^2$, $\hat{F}_e = \left[ \frac{1}{2} \int_{0}^{1} \left( \int_{\eta}^{1} \frac{z^{1/2}}{l(z)} \, dz \right)^2 \, d\eta \right]^{1/2}$
mixing length closure

\[ \ell(\eta) = c_\ell I(\eta)^{1/2} \tanh(\kappa \eta / c_\ell), \quad c_\ell = 0.085, \quad I(\eta) = 1/(1 + 5.5 \eta^6) \]

\[ W_1(x) = \hat{F}'(0) \overset{\sim}{=} 13.868, \quad \hat{F}(1) = 5.682, \quad d\Delta_1/dx \overset{\sim}{=} 0.528 \]
2nd-order problem: \( \partial F_2/\partial x \equiv \partial T_2/\partial x \equiv 0, \ \beta_0 \to \infty \iff m \to -1/3 \)

least-degenerate limit
\[
\tau_w/U_e^2 \equiv \gamma^2 = \epsilon^3 T_2(0) \Rightarrow T_2(0) = 1/\Gamma, \ \epsilon^3 = \Gamma \gamma^2
\]
\[
\epsilon \equiv \gamma \beta_0^{1/2}
\]

consistency with 1st-order results requires
\[
\hat{U}(x) = (C/3)^{1/3}(x - x_v)^m, \ m \sim -1/3 + \gamma^{2/3} \mu_1 + \cdots
\]

solvability condition
\[
9\Gamma^{2/3}\mu_1 = 1 + r\Gamma
\]
\[
r = \frac{1}{\hat{F}_e} \int_0^1 (\hat{F}'^2 + \hat{S} - \hat{R})(\eta) \, d\eta - 12 \frac{\hat{K}}{\hat{F}_e^2} \int_0^1 \eta \hat{F}'(\eta) \, d\eta, \ \ k = \frac{\hat{K}}{x - x_v}
\]

canonical form for plane or concave surface: \( \hat{K} \leq 0 \Rightarrow r > 0 \)
\[
9\hat{D}^{2/3} \hat{\mu} = 1 + \hat{D}^{3} \quad \hat{D} = r^{1/3} \Gamma^{1/3}, \ \ \hat{\mu} = r^{-2/3} \mu_1
\]
2nd-order problem:  \( \frac{\partial F_2}{\partial x} \equiv \frac{\partial T_2}{\partial x} \equiv 0 \),  \( \beta_0 \to \infty \Leftrightarrow m \to -1/3^+ \)

least-degenerate limit

\[ \tau_w/U_e^2 \equiv \gamma^2 = \epsilon^3 T_2(0) \Rightarrow T_2(0) = 1/\Gamma, \quad \epsilon^3 = \Gamma \gamma^2 \]

\[ \epsilon \equiv \gamma \beta_0^{1/2} \quad \Gamma := \epsilon \beta_0 = O(1) \]

consistency with 1st-order results requires

\[ \hat{U}(x) = (C/3)^{1/3}(x - x_v)^m, \quad m \sim -1/3 + \gamma^{2/3} \mu_1 + \cdots \]

solvability condition

\[ 9 \Gamma^{2/3} \mu_1 = 1 + r \Gamma \]

\[ r = \frac{1}{\hat{F}_e} \int_0^1 (\hat{F}'^2 + \hat{S} - \hat{R})(\eta) \, d\eta - 12 \frac{\hat{K}}{\hat{F}_e^2} \int_0^1 \eta \hat{F}'(\eta) \, d\eta, \quad k = \frac{\hat{K}}{x - x_v} \]

canonical form for plane or concave surface:  \( \hat{K} \leq 0 \Rightarrow r > 0 \)

\[ 9 \hat{D}^{2/3} \hat{\mu} = 1 + \hat{D}^3 \]

\[ \hat{D} = r^{1/3} \Gamma^{1/3}, \quad \hat{\mu} = r^{-2/3} \mu_1 \]
non-uniqueness supported experimentally

Clauser (Adv Appl Mech, 1956)
Turbulent BLs with large velocity defect

General framework

Turbulent marginal separation
Large velocity defect: \[ 1 - \frac{u}{U_e} = O(1) \]

**nonlinear momentum balance**

\[
u \frac{\partial u}{\partial x} + \cdots \sim \frac{\partial \tau}{\partial y} = O(1) \iff \tau = -\langle u' v' \rangle = O(\delta) \iff u', v' = O(\delta^{1/2})
\]

**consequences**

- \( \delta \ll 1 \Rightarrow u', v' \ll 1 \)
- \( u_\tau \) inappropriate turbulent velocity scale

**question**: why is BL still slender (lacking a small turbulent velocity scale)?

**answer**: outer part of BL essentially behaves as turbulent free shear layer (wake), which is (empirically) known to be slender

**strategy**: combine asymptotic theory of turbulent free shear layers where slenderness is enforced by a slenderness (entrainment) parameter \( \alpha \ll 1 \) Schneider (ZFW, 1991) with asymptotic description of turbulent wall-bounded flows
Large velocity defect: \( 1 - u/U_e = O(1) \)

**nonlinear momentum balance**

\[
u \frac{\partial u}{\partial x} + \cdots \sim \frac{\partial \tau}{\partial y} = O(1) \quad \Leftrightarrow \quad \tau = -\langle u'v' \rangle = O(\delta) \quad \Leftrightarrow \quad u', v' = O(\delta^{1/2})
\]

**consequences**

- \( \delta \ll 1 \quad \Rightarrow \quad u', v' \ll 1 \)
- \( u_\tau \) inappropriate turbulent velocity scale

**question:** why is BL still slender (lacking a small turbulent velocity scale)?

**answer:** outer part of BL essentially behaves as turbulent free shear layer (wake), which is (empirically) known to be slender

**strategy:** combine asymptotic theory of turbulent free shear layers where slenderness is enforced by a slenderness (entrainment) parameter \( \alpha \ll 1 \) Schneider (ZFW, 1991) with asymptotic description of turbulent wall-bounded flows
Large velocity defect: \( 1 - u/U_e = O(1) \)

**nonlinear momentum balance**

\[
\begin{align*}
    u \frac{\partial u}{\partial x} + \cdots \sim \frac{\partial \tau}{\partial y} &= O(1) \quad \Leftrightarrow \quad \tau = -\left\langle u'v' \right\rangle = O(\delta) \quad \Leftrightarrow \quad u', v' = O(\delta^{1/2})
\end{align*}
\]

**consequences**

- \( \delta \ll 1 \quad \Rightarrow \quad u', v' \ll 1 \)
- \( u_T \) inappropriate turbulent velocity scale

**question:** why is BL still slender (lacking a small turbulent velocity scale)?

**answer:** outer part of BL essentially behaves as turbulent free shear layer (wake), which is (empirically) known to be slender

**strategy:** combine asymptotic theory of turbulent free shear layers where slenderness is enforced by a slenderness (entrainment) parameter \( \alpha \ll 1 \) \( \text{Schneider (ZFW, 1991)} \) with asymptotic description of turbulent wall-bounded flows
Large velocity defect: \(1 - u/U_e = O(1)\)

**nonlinear momentum balance**

\[
\frac{\partial u}{\partial x} + \cdots \sim \frac{\partial \tau}{\partial y} = O(1) \iff \tau = -\langle u'v' \rangle = O(\delta) \iff u', v' = O(\delta^{1/2})
\]

**consequences**

- \(\delta \ll 1 \Rightarrow u', v' \ll 1\)
- \(u_\tau\): inappropriate turbulent velocity scale

**question**: why is BL still slender (lacking a small turbulent velocity scale)?

**answer**: outer part of BL essentially behaves as turbulent free shear layer (wake), which is (empirically) known to be slender

**strategy**: combine asymptotic theory of turbulent free shear layers where slenderness is enforced by a slenderness (entrainment) parameter \(\alpha \ll 1\) Schneider (ZFW, 1991) with asymptotic description of turbulent wall-bounded flows
Outer wake region: \( \bar{y} := y/\alpha = O(1) \)

expansions

\[
p \sim p_0(x) + O(\alpha), \quad q \sim \alpha q_0(x, \bar{y}) + O(\alpha^2), \quad q = \delta, \psi, \tau, \sigma_x, \sigma_y
\]

leading-order problem

\[
\frac{\partial \psi_0}{\partial \bar{y}} \frac{\partial^2 \psi_0}{\partial \bar{y} \partial x} - \frac{\partial \psi_0}{\partial x} \frac{\partial^2 \psi_0}{\partial \bar{y}^2} = - \frac{\partial p_0}{\partial x} + \frac{\partial \tau_0}{\partial \bar{y}}, \quad \frac{\partial p}{\partial x} = -U_e \frac{\partial U_e}{\partial x}, \quad \frac{\partial p_0}{\partial \bar{y}} = 0
\]

\( \bar{y} = 0: \quad \psi_0 = \tau_0 = 0, \quad \bar{y} = \delta_0(x): \quad \partial \psi_0 / \partial \bar{y} = U_e, \quad \tau_0 = 0 \)

flow behaviour as \( \bar{y} \to 0 \)

\[
\frac{\partial \psi_0}{\partial \bar{y}} \sim U_s(x) + O(\bar{y}^{3/2})
\]

\[
\tau_0 \sim \Lambda_0(x) \bar{y} + O(\bar{y}^{3/2})
\]

\[
\Lambda_0 := U_s U_{sx} - U_e U_{ex}
\]
Interpretation of mixing length $\ell$

$-\langle u'v' \rangle \equiv \ell^2 \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} = O(\alpha)$, \quad $\frac{\partial u}{\partial y} = O\left(\frac{1}{\alpha}\right)$ \quad $\Rightarrow$ \quad $\delta \gg \ell = O(\alpha^{3/2})$

$\ell = \text{distance where outer wake flow "begins to feel" presence of confining wall}$

$\tau \sim \alpha \tau_0 = \alpha \Lambda_0 \bar{y} = O(\alpha^{3/2})$, \quad $u - U_s = O(\alpha^{3/4})$

Inner wake region: \quad $Y := y/\alpha^{3/2} = \bar{y}/\alpha^{1/2}$

expansions

$p \sim p_0(x) + O(\alpha)$, \quad $\psi \sim \alpha^{3/2} U_s(x) + \alpha^{9/4} \Psi(x, Y) + \cdots$

$[\tau, \ell] \sim \alpha^{3/2} [T, L](x, Y) + \cdots$, \quad $L_\infty := \lim_{Y \to \infty} L = O(1)$

leading-order problem

$T = \Lambda_0 Y$, \quad $T(x, 0) = \Psi(x, 0) = 0$, \quad $Y \to \infty$: \quad $\frac{\partial \psi}{\partial Y} \sim \frac{2}{3} \frac{\Lambda_0^{1/2}}{L_\infty} Y^{3/2} + \cdots$
solution of leading-order problem

\[
\frac{\partial \Psi}{\partial Y} = \frac{2}{3} \frac{\Lambda_0^{1/2}}{L_\infty} Y^{3/2} - J(x, Y), \quad J := \int_Y^\infty \left[ \frac{1}{L(x, z)} - \frac{1}{L_\infty} \right] (\Lambda_0 z)^{1/2} \, dz
\]

flow behaviour as \( Y \to 0 \ldots \)

overlap with intermediate layer \( \Rightarrow \) square-root law

\[
\frac{\partial u}{\partial y} \sim \frac{u_\tau}{u_\tau^2} \Pi \left( x, \frac{y}{u_\tau^2} \right) \quad \Rightarrow \quad \frac{\partial \Psi}{\partial Y} \sim U_s(x) + 2 \frac{(\Lambda_0 Y)^{1/2}}{\chi(x)} + \cdots, \quad L \sim \chi Y
\]

slip velocity

\[
u_s \sim U_s(x) + \alpha^{3/4} U_s(x) + \cdots
\]

\( U_s = -J(x, 0) < 0 \)

\( \ldots \) completes wake region analysis
Asymptotic investigation of shear stress closures

... confirms that $\alpha$ is independent of $Re$

specific algebraic closures

- mixing length
  \[
  \alpha^{1/2} = c_\ell \approx 0.085
  \]

- eddy viscosity
  \[
  \alpha \approx 0.0168
  \]
Asymptotic investigation of shear stress closures

...confirms that $\alpha$ is independent of $Re$

specific algebraic closures

- mixing length
  
  $\alpha^{1/2} = c_\ell \approx 0.085$

- eddy viscosity
  
  $\alpha \approx 0.0168$

Schlichting & Gersten (2000)

Michel, Quémard & Durant (Proc Stanford Conf, 1969)

Asymptotic investigation of shear stress closures

... confirms that $\alpha$ is independent of $Re$

specific algebraic closures

- mixing length
  \[ \alpha^{1/2} = c_\ell \approx 0.085 \]

- eddy viscosity
  \[ \alpha \approx 0.0168 \]
4 Turbulent BLs with large velocity defect

General framework

Turbulent marginal separation
Turbulent marginal separation

numerical solution of leading-order outer-wake problem

- algebraic closure: mixing length $\ell \sim \ell_0 + \cdots$

\[
\tau_0 = \ell_0^2 \left. \frac{\partial^2 \psi_0}{\partial \bar{y}^2} \right|_{\bar{y}} \ell_0 = l(\xi)^{1/2} \Delta_0(x), \quad l(\xi) = \frac{1}{1 + 5.5 \xi^6}, \quad \xi := \frac{\bar{y}}{\Delta_0(x)}
\]

- 2-parameter family of retarded external flows

\[
U_e(x; m_s, \sigma) = (1 + x)^m x; m_s, \sigma) \\
\frac{m}{m_s} = 1 + \frac{\sigma}{1 - \sigma} H(2 - x)[1 - (1 - x)^2]^3, \quad m_s < 0, \quad 0 \leq \sigma < 1
\]

- similarity solutions: $\sigma = 0$, $m \equiv m_s > -1/3$

\[
\psi_0 = \Delta_0(x) f(\xi), \quad \Delta_0 = b(1 + x)
\]

specific choice for initial conditions at $x = 0$:

\[
f'(0) = 0.95 \Rightarrow m_s \approx -0.3292, \quad b \approx 0.3656
\]
Numerical solution of leading-order outer-wake problem

- Algebraic closure: mixing length \( \ell \sim \ell_0 + \cdots \)
  \[
  \tau_0 = \ell_0^2 \frac{\partial^2 \psi_0}{\partial \tilde{y}^2} \left| \frac{\partial^2 \psi_0}{\partial \tilde{y}^2} \right|, \quad \ell_0 = I(\xi)^{1/2} \Delta_0(x), \quad I(\xi) = \frac{1}{1 + 5.5 \xi^6}, \quad \xi := \frac{\tilde{y}}{\Delta_0(x)}
  \]

- 2-parameter family of retarded external flows
  \[
  U_e(x; m_s, \sigma) = (1 + x)^m(x; m_s, \sigma)
  \]
  \[
  \frac{m}{m_s} = 1 + \frac{\sigma}{1 - \sigma} \left( 2 - x \right) \left[ 1 - (1 - x)^2 \right]^3, \quad m_s < 0, \quad 0 \leq \sigma < 1
  \]

- Similarity solutions: \( \sigma = 0, \quad m \equiv m_s > -1/3 \)
  \[
  \psi_0 = \Delta_0(x)f(\xi), \quad \Delta_0 = b(1 + x)
  \]
  Specific choice for initial conditions at \( x = 0 \):
  \[
  f'(0) = 0.95 \Rightarrow m_s \approx -0.3292, \quad b \approx 0.3656
  \]
Turbulent marginal separation

Scheichl & Kluwick (AIAA J, IJCSM, 2007)

Numerical solution of leading-order outer-wake problem

- Algebraic closure: mixing length \( \ell \sim \ell_0 + \cdots \)

\[
\tau_0 = \ell_0^2 \frac{\partial^2 \psi_0}{\partial \tilde{y}^2} \bigg|_{\partial^2 \psi_0 \over \partial \tilde{y}^2}, \quad \ell_0 = I(\xi)^{1/2} \Delta_0(x), \quad I(\xi) = \frac{1}{1 + 5.5 \xi^6}, \quad \xi := \frac{\tilde{y}}{\Delta_0(x)}
\]

- 2-parameter family of retarded external flows

\[
U_e(x; m_s, \sigma) = (1 + x)^m(x; m_s, \sigma)
\]

\[
\frac{m}{m_s} = 1 + \frac{\sigma}{1 - \sigma} H(2 - x)[1 - (1 - x)^2]^3, \quad m_s < 0, \quad 0 \leq \sigma < 1
\]

- Similarity solutions: \( \sigma = 0, \quad m \equiv m_s > -1/3 \)

\[
\psi_0 = \Delta_0(x)f(\xi), \quad \Delta_0 = b(1 + x)
\]

Specific choice for initial conditions at \( x = 0 \):

\[
f'(0) = 0.95 \Rightarrow m_s \approx -0.3292, \quad b \approx 0.3656
\]
\[ P_M = \frac{\partial p_0}{\partial x} \text{ at } x = x_M \]

critical conditions \( \sigma = \sigma_M \)

\( s := x - x_M \)

marginal separation

\( s \rightarrow 0_- : \quad \frac{U_s}{P_M^{1/2}} \sim B(-s) \)

\( s \rightarrow 0_+ : \quad \frac{U_s}{P_M^{1/2}} \sim U_+ s^{1/2} \)

\( B > 0, \quad U_+ \approx 1.1835 \)

supercritical conditions \( \sigma > \sigma_M \)

\( s := x - x_G \)

“turbulent” Goldstein singularity

\( s \rightarrow 0_- : \quad \frac{U_s}{P_M^{1/2}} \sim U_-( -s )^{1/2} \)
marginal separation

breakdown of classical “hierarchical” BL theory in regions where BL thickness changes so rapidly that pressure response in external-flow region is large enough to affect the lowest-order BL approximation

⇒ local interaction strategy

“turbulent” (novel) triple-deck structure
“turbulent” triple-deck problem

similarity parameter $\hat{\Gamma}$

induced pressure gradient of strength $\hat{\Lambda}$

fundamental lower-deck problem

$$\frac{\partial \hat{\Psi}}{\partial \hat{Y}} \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y} \partial \hat{X}} - \frac{\partial \hat{\Psi}}{\partial \hat{X}} \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2} = -1 - \hat{\Lambda}(\hat{\Gamma}) \hat{P}'(\hat{X}) + \frac{\partial \hat{T}}{\partial \hat{Y}}, \quad \hat{T} = \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2} \left| \frac{\partial^2 \hat{\Psi}}{\partial \hat{Y}^2} \right|$$

$\hat{Y} = 0: \quad \hat{\Psi} = \hat{T} = 0, \quad \hat{Y} \to \infty: \quad \hat{T} \to \hat{Y} + \hat{A}(\hat{X})$

$\hat{X} \to -\infty: \quad \hat{\Psi} \to (4/15) \hat{Y}^{5/2} + \hat{\Gamma} \hat{Y}, \quad \hat{X} \to \infty: \quad \hat{\Psi}/\hat{X}^{5/6} \to \hat{F}_+(\hat{Y}/\hat{X}^{1/3})$

$$\hat{P}(\hat{X}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{A}'(\hat{S})}{\hat{X} - \hat{S}} \, d\hat{S}$$
slip velocity
\[ \hat{U}_s = \partial \hat{\psi} / \partial \hat{Y} \text{ at } \hat{Y} = 0 \]

numerical solution
\[ \hat{\Gamma} = 0.019 \]
\[ \hat{\Lambda} = 3 \]
Problem of (turbulent) break-away separation

Motivation

Global flow picture

Separation — experimental findings
Sandborn & Liu (JFM, 1968)

“Turbulent boundary-layer separation is normally listed as one of the most important unsolved problems in fluid mechanics...”
Problem formulation — motivation

Local description of turbulent break-away separation …

… requires answers to three basic questions:

(1) Global topology of flow for ‘vanishing viscosity’?
(2) Characteristics of incident boundary layer flow?
(3) How do these issues interdepend?

How turbulent are the BL and the separated SL?

Problem formulation — motivation
Local description of turbulent break-away separation . . .

. . . requires answers to three basic questions:

(1) **Global** topology of flow for ‘vanishing viscosity’?
(2) Characteristics of *incident* boundary layer flow?
(3) How do these issues *interdepend*?

basic assumptions

• flow incompressible, nominally steady, 2D
• free-stream turbulence disregarded
• \( Re := \frac{U \theta}{v} \rightarrow \infty \)


How *turbulent* are the BL and the separated SL ?

Problem formulation — motivation
Local description of turbulent break-away separation ... 

... requires answers to three basic questions:

(1) Global topology of flow for ‘vanishing viscosity’?
(2) Characteristics of incident boundary layer flow?
(3) How do these issues interdepend?

basic assumptions

- flow incompressible, nominally steady, 2D
- free-stream turbulence disregarded
- \( Re := \bar{U} \bar{L} / \bar{v} \rightarrow \infty \)

How turbulent are the BL and the separated SL?
Problem formulation — motivation
Local description of turbulent break-away separation . . .

. . . requires answers to three basic questions:

(1) **Global** topology of flow for ‘vanishing viscosity’?
(2) Characteristics of **incident** boundary layer flow?
(3) How do these issues **interdepend**?

basic assumptions

- flow incompressible, nominally steady, 2D
- free-stream turbulence disregarded
- \[ Re := \frac{\tilde{U} \tilde{L}}{\tilde{v}} \to \infty \]

How **turbulent** are the BL and the separated SL?

Problem formulation — motivation
Local description of turbulent break-away separation . . .

. . . requires answers to three basic questions:

1. **Global** topology of flow for ‘vanishing viscosity’?
2. Characteristics of **incident** boundary layer flow?
3. How do these issues **interdepend**?

**Basic assumptions**

- Flow incompressible, nominally steady, 2D
- Free-stream turbulence disregarded
- \( Re := \frac{\tilde{U}L}{\tilde{v}} \rightarrow \infty \)

**Canonical example: Circular-cylinder flow**

\( u_\infty = 1 \)

How **turbulent** are the **BL** and the separated **SL**?

5 Problem of (turbulent) break-away separation
   Motivation
   Global flow picture
   Separation — experimental findings
Global flow picture

\[ Re \gtrsim 3 \times 10^6 : \text{postcritical regime}, \quad Re \to \infty : \text{transition point approximates } F \]

\[ Re^{-1} = 0 : \text{ultimate or } T\text{-state of flow, transition in } F \]

Transition in flow

\[ u_\infty = 1 \]

\[ \theta_S \approx 115^\circ \]

Neish \& Smith (JFM, 1992)

unlikely

Scheichl \& Kluwick (JFS, 2008)

corroborated experimentally up to \( Re \approx 4 \times 10^7 \)

Global flow picture

$Re \gtrsim 3 \times 10^6$: postcritical regime, $Re \rightarrow \infty$: transition point approximates $F$

$Re^{-1} = 0$: ultimate or $T$-state of flow, transition in $F$

$u_\infty = 1$

transition

$\theta_S \approx 115^\circ$

$u_S \lesssim u_\infty$

$p_S \lesssim p_\infty$

$R \sim S$?

Neish & Smith (JFM, 1992)

unlikely

Scheichl & Kluwick (JFS, 2008)

corroborated experimentally up to $Re \approx 4 \times 10^7$

Global flow picture

\[ Re \gtrsim 3 \times 10^6 : \text{postcritical regime, } Re \to \infty : \text{transition point approximates } F \]

\[ Re^{-1} = 0 : \text{ultimate or } T-\text{state of flow, transition in } F \]

\[ u_\infty = 1 \]

transition

\[ \theta_S \approx 115^\circ \]

\[ u_S \lesssim u_\infty \quad p_S \lesssim p_\infty \]

\[ R \sim S ? \]

\[ \text{Neish & Smith (JFM, 1992)} \]

\[ \text{Scheichl & Kluwick (JFS, 2008)} \]

\[ \text{unlikely} \]

corroborated experimentally up to \( Re \approx 4 \times 10^7 \)

Outline

Problem of (turbulent) break-away separation
  Motivation
  Global flow picture
  Separation — experimental findings
Separation

Is there really a fully developed turbulent BL as $Re \to \infty$?

Oil-flow measurements:

$$Re_C := \frac{\bar{U} \bar{C}}{\bar{\nu}}, \quad \bar{C} = \text{airfoil chord length}$$

by courtesy of G. Schewe (2001)
Separation

Is there really a fully developed turbulent BL as $Re \to \infty$?

oil-flow measurements: $Re_C := \tilde{U}\tilde{C}/\tilde{\nu}$, \(\tilde{C}\) = airfoil chord length

\[ \tilde{W} \approx 6 \times \tilde{C} \]
\[ Re_C = 7.4 \times 10^5 : \quad c_D \approx 0.11, \quad c_L \approx 1.1 \]
\[ Re_C = 7.7 \times 10^6 : \quad c_D \approx 0.14, \quad c_L \approx 0.65 \]

by courtesy of G. Schewe (2001)
Asymptotic structure of bluff-body flows

‘Ideal-fluid limit’: Kirchhoff-type potential flow

Boundary layer
‘Ideal-fluid limit’
Hierarchy of external flow – shear layer: potential flow approached as $Re \to \infty$
Oseen (1927), Lamb (1932), Batchelor (1956), Prandtl (1961)

$Re^{-1} = 0$: external potential flows

Imai (J Phys Soc Japan, 1953),
Birkhoff & Zarantonello (1957), Gurevich (1966)

- free streamlines confine (open) dead-water region
- class of flows / $\theta_S$ controlled by
  Brillouin–Villat parameter $k \geq 0$, free-stream velocity $u_S \leq u_\infty$

$$
\begin{align*}
  u_e & \sim b(k)\theta + O(\theta^2), & \theta & \to 0_+ \\
  & \sim u_S[1 + 2k(-s)^{1/2} + O(-s)], & s & \to 0_- \\
  & \equiv u_S, & s & \geq 0 \\
  \kappa & \sim -k s^{-1/2} + \kappa_B(0) + O(s^{1/2}), & s & \to 0_+
\end{align*}
$$
‘Ideal-fluid limit’

Hierarchy of external flow – shear layer: potential flow approached as $Re \to \infty$

Oseen (1927), Lamb (1932), Batchelor (1956), Prandtl (1961)

$Re^{-1} = 0$: external potential flows

Imai (J Phys Soc Japan, 1953), Birkhoff & Zarantonello (1957), Gurevich (1966)

- free streamlines confine (open) dead-water region
- class of flows / $\theta_S$ controlled by

Brillouin–Villat parameter $k \geq 0$, free-stream velocity $u_S \leq u_\infty$

$$u_e \left\{ \begin{array}{l}
\sim b(k)\theta + O(\theta^2), \quad \theta \to 0_+ \\
\sim u_S \left[ 1 + 2k(-s)^{1/2} + O(-s) \right], \quad s \to 0_- \\
\equiv u_S, \quad s \geq 0 \\
\end{array} \right.$$  

$$\kappa \left\{ \begin{array}{l}
= \kappa_B(s), \quad s < 0 \\
\sim -k s^{-1/2} + \kappa_B(0) + O(s^{1/2}), \quad s \to 0_+ \\
\end{array} \right.$$
‘Ideal-fluid limit’

Hierarchy of external flow – shear layer: potential flow approached as \( Re \to \infty \)

Oseen (1927), Lamb (1932), Batchelor (1956), Prandtl (1961)

\( Re^{-1} = 0 \) : external potential flows

Imai (J Phys Soc Japan, 1953),
Birkhoff & Zarantonello (1957), Gurevich (1966)

- free streamlines confine (open) dead-water region
- class of flows / \( \theta_S \) controlled by
  Brillouin–Villat parameter \( k \geq 0 \), free-stream velocity \( u_S \leq u_\infty \)

\[
\begin{align*}
\begin{aligned}
u_e & \sim b(k) \theta + O(\theta^2), & \theta \to 0_+ \\
& \sim u_S \left[1 + 2k(-s)^{1/2} + O(-s)\right], & s \to 0_- \\
& \equiv u_S, & s \geq 0 \\
\kappa & \equiv \kappa_B(s), & s < 0 \\
& \sim -k s^{-1/2} + \kappa_B(0) + O(s^{1/2}), & s \to 0_+ 
\end{aligned}
\end{align*}
\]
Kirchhoff-type flows — numerical solutions

Open cavity: \( u_S \equiv u_\infty = 1 \)

surface velocity \( u_e(\theta, k) \)

\[
k = 0, \ 0.05, \ 0.1, \ \ldots, \ 0.5
\]

\( 55^\circ \ 2' \ 30'' \ \leq \ \theta_S \ \leq \ 126^\circ \)

\( k = 0: \)

**Brillouin–Villat condition**

free streamlines

\( 126^\circ \ \leq \ \theta_S \ \leq \ 180^\circ \)

\( 0.5 \ \leq \ k \ < \ \infty \)

\( 1 \ \geq \ u_S \ \geq \ 0 \)
Kirchhoff-type flows — numerical solutions

Open cavity: \( u_S \equiv u_\infty = 1 \)

surface velocity \( u_e(\theta, k) \)

\( k = 0, 0.05, 0.1, \ldots, 0.5 \)

\( 55^\circ \, 2' \, 30'' \leq \theta_S \leq 126^\circ \)

\( k = 0: \)

Brillouin–Villat condition

free streamlines

\( 126^\circ \leq \theta_S \leq 180^\circ \)

\( 0.5 \leq k < \infty \)

\( 1 \geq u_S \geq 0 \)
(1) **Global** topology of flow for ‘vanishing viscosity’?

(2) Characteristics of **incident** boundary layer flow?

(3) How do these issues **interdepend**?
Outline

- Asymptotic structure of bluff-body flows
  - 'Ideal-fluid limit': Kirchhoff-type potential flow
  - Boundary layer
Finite turbulence intensity level


asymptotic analysis of Reynolds-averaged NS eqs

\[ \epsilon := Re^{-1/2} \to 0, \quad Tu := \sqrt{\langle u'^2 + v'^2 + w'^2 \rangle}/3 \ll 1 \]

hypothesis of *locally isotropic* turbulence

\[ \tau, \sigma_s, \sigma_n, \sigma_z] := -[\langle u'v' \rangle, \langle u'^2 \rangle, \langle v'^2 \rangle, \langle w'^2 \rangle] = O(Tu^2) \]

classical inner expansions (Prandtl-type BL)

\[ p - p_F \sim -u_e^2(\theta; k)/2 + O(\epsilon), \quad dp_e/d\theta = -u_e du_e/d\theta \]

\[ [\psi, r] \sim \epsilon [\psi_0, Tr_0](\theta, N; k, T) + O(\epsilon^2), \quad r := \tau, \sigma_s, \sigma_n, \sigma_z, \quad N := Re^{1/2}n \]

time-mean flow governed by three parameters

Brillouin–Villat parameter \( k \)

Reynolds number \( Re \)

turbulence level gauge factor \( Tu \) = \( O(\epsilon^{1/2}Tu^2) \)

Neish & Smith (JFM, 1992)
Finite turbulence intensity level  


asymptotic analysis of Reynolds-averaged NS eqs

\[ \epsilon := Re^{-1/2} \to 0, \quad Tu := \sqrt{\langle u'^2 + v'^2 + w'^2 \rangle}/3 \ll 1 \]

hypothesis of \textit{locally isotropic} turbulence

\[ [\tau, \sigma_s, \sigma_n, \sigma_z] := -[\langle u'v' \rangle, \langle u'^2 \rangle, \langle v'^2 \rangle, \langle w'^2 \rangle] = O(Tu^2) \]

classical inner expansions (Prandtl-type BL)

\[ p - p_F \sim -u_e^2(\theta; k)/2 + O(\epsilon), \quad dp_e/d\theta = -u_e du_e/d\theta \]

\[ [\psi, r] \sim \epsilon [\psi_0, Tr_0](\theta, N; k, T) + O(\epsilon^2), \quad r := \tau, \sigma_s, \sigma_n, \sigma_z, \quad N := Re^{1/2}n \]

time-mean flow governed by three parameters

Brillouin–Villat parameter \( k \)
Reynolds number \( Re \)
turbulence level gauge factor \( T = O(Re^{1/2} Tu^2) \)

Neish & Smith (JFM, 1992)
Finite turbulence intensity level

asymptotic analysis of Reynolds-averaged NS eqs

\[ \epsilon := Re^{-1/2} \to 0 , \quad Tu := \sqrt{\langle u'^2 + v'^2 + w'^2 \rangle} / 3 \ll 1 \]

hypothesis of *locally isotropic* turbulence

\[ [\tau, \sigma_s, \sigma_n, \sigma_z] := -[\langle u'v' \rangle, \langle u'^2 \rangle, \langle v'^2 \rangle, \langle w'^2 \rangle] = O(Tu^2) \]

classical inner expansions (Prandtl-type BL)

\[ p - p_F \sim -u_e^2(\theta; k)/2 + O(\epsilon) , \quad dp_e/d\theta = -u_e du_e/d\theta \]

\[ [\psi, r] \sim \epsilon [\psi_0, T r_0](\theta, N; k, T) + O(\epsilon^2) , \quad r := \tau, \sigma_s, \sigma_n, \sigma_z , \quad N := Re^{1/2} n \]

time-mean flow governed by three parameters

Brillouin–Villat parameter \( k \)
Reynolds number \( Re \)
turbulence level gauge factor \( T = O(Re^{1/2} Tu^2) \)
Finite turbulence intensity level


asymptotic analysis of Reynolds-averaged NS eqs

\[ \epsilon := Re^{-1/2} \to 0, \quad Tu := \sqrt{\langle u'^2 + v'^2 + w'^2 \rangle}/3 \ll 1 \]

hypothesis of \textit{locally isotropic} turbulence

\[ [\tau, \sigma_s, \sigma_n, \sigma_z] := -[\langle u'v' \rangle, \langle u'^2 \rangle, \langle v'^2 \rangle, \langle w'^2 \rangle] = O(Tu^2) \]

classical inner expansions (Prandtl-type BL)

\[ \rho - p_F \sim -u_e^2(\theta; k)/2 + O(\epsilon), \quad \frac{dp_e}{d\theta} = -u_e \frac{du_e}{d\theta} \]

\[ [\psi, r] \sim \epsilon [\psi_0, Tr_0](\theta, N; k, T) + O(\epsilon^2), \quad r := \tau, \sigma_s, \sigma_n, \sigma_z, \quad N := Re^{1/2} n \]

time-mean flow governed by three parameters

- Brillouin–Villat parameter \( k \)
- Reynolds number \( Re \)
- turbulence level gauge factor \( T = O(Re^{1/2} Tu^2) \)
Finite turbulence intensity level — cont’d

leading-order BIV problem

\[
\frac{\partial \psi_0}{\partial N} \frac{\partial^2 \psi_0}{\partial N \partial \theta} - \frac{\partial \psi_0}{\partial \theta} \frac{\partial^2 \psi_0}{\partial N^2} = u_e \frac{\partial u_e}{\partial \theta} + \frac{\partial}{\partial N} \left[ T \tau_0 + \frac{\partial^2 \psi_0}{\partial N^2} \right]
\]

\( N = 0 : \) \( \partial \psi_0 / \partial N = \psi_0 = \tau_0 = 0 \)

\( N \to \infty : \) \( \partial \psi_0 / \partial N \to u_e(\theta; k), \ \tau_0 \to 0 \)

\( \theta \to 0_+ : \) \( u_e \sim b(k) \theta + O(\theta^2), \ \psi_0 \sim b^{1/2} \theta H(b^{1/2} N), \ \tau_0 = O(\theta^2) \)

flow laminar close to \( F \)

Hiemenz (1911)

\( H''^2 - HH'' = 1 + H''' , \quad H(0) = H'(0) = 0 , \quad H'(\infty) = 1 \)

present study: BL in the ‘turbulent limit’ \( T \to \infty \)

\( T \tau_0 \gg \frac{\partial^2 \psi_0}{\partial N^2} \) \( \Rightarrow \) problem singularly perturbed

\[
\begin{cases}
\theta \to 0, & N = O(1) \\
N \to 0, & \theta = O(1)
\end{cases}
\]
Finite turbulence intensity level — cont’d

leading-order BIV problem

\[
\frac{\partial \psi_0}{\partial N} \frac{\partial^2 \psi_0}{\partial N \partial \theta} - \frac{\partial \psi_0}{\partial \theta} \frac{\partial^2 \psi_0}{\partial N^2} = u_e \frac{du_e}{d\theta} + \frac{\partial}{\partial N} \left[ T \tau_0 + \frac{\partial^2 \psi_0}{\partial N^2} \right]
\]

\( N = 0 : \quad \frac{\partial \psi_0}{\partial N} = \psi_0 = \tau_0 = 0 \)

\( N \to \infty : \quad \frac{\partial \psi_0}{\partial N} \to u_e(\theta; k), \quad \tau_0 \to 0 \)

\( \theta \to 0^+ : \quad u_e \sim b(k)\theta + O(\theta^2), \quad \psi_0 \sim b^{1/2} \theta H(b^{1/2} N), \quad r_0 = O(\theta^2) \)

flow laminar close to \( F \)

Hiemenz (1911)

\[ H''^2 - HH''' = 1 + H'''' \]

\( H(0) = H'(0) = 0 \), \( H'(\infty) = 1 \)

present study: BL in the ‘turbulent limit’ \( T \to \infty \)

\( T \tau_0 \gg \frac{\partial^2 \psi_0}{\partial N^2} \) \Rightarrow \text{problem singularly perturbed} \begin{align*}
\theta &\to 0, & N &= O(1) \\
N &\to 0, & \theta &= O(1)
\end{align*}
Finite turbulence intensity level — cont’d

leading-order BIV problem

\[
\frac{\partial \psi_0}{\partial N} \frac{\partial^2 \psi_0}{\partial N \partial \theta} - \frac{\partial \psi_0}{\partial \theta} \frac{\partial^2 \psi_0}{\partial N^2} = u_e \frac{d u_e}{d \theta} + \frac{\partial}{\partial N} \left[ T \tau_0 + \frac{\partial^2 \psi_0}{\partial N^2} \right]
\]

\(N = 0:\) \quad \frac{\partial \psi_0}{\partial N} = \psi_0 = \tau_0 = 0

\(N \to \infty:\) \quad \frac{\partial \psi_0}{\partial N} \to u_e(\theta; k), \quad \tau_0 \to 0

\(\theta \to 0+:\) \quad u_e \sim b(k) \theta + O(\theta^2), \quad \psi_0 \sim b^{1/2} \theta H(b^{1/2} N), \quad r_0 = O(\theta^2)

flow laminar close to \(F\) \hspace{1cm} \text{Hiemenz (1911)}

\(H''^2 - H H''' = 1 + H''', \quad H(0) = H'(0) = 0, \quad H'(\infty) = 1\)

present study: BL in the ‘turbulent limit’ \(T \to \infty\)

\[T \tau_0 \gg \frac{\partial^2 \psi_0}{\partial N^2} \Rightarrow \text{problem singularly perturbed}\]

\[
\begin{cases}
\theta \to 0, & N = O(1) \\
N \to 0, & \theta = O(1)
\end{cases}
\]
Finite turbulence intensity level — cont’d

leading-order BIV problem

\[
\frac{\partial \psi_0}{\partial N} \frac{\partial^2 \psi_0}{\partial N \partial \theta} - \frac{\partial \psi_0}{\partial \theta} \frac{\partial^2 \psi_0}{\partial N^2} = u_e \frac{du_e}{d\theta} + \frac{\partial}{\partial N} \left[ T \tau_0 + \frac{\partial^2 \psi_0}{\partial N^2} \right]
\]

\(N = 0:\quad \frac{\partial \psi_0}{\partial N} = \psi_0 = \tau_0 = 0\)

\(N \to \infty:\quad \frac{\partial \psi_0}{\partial N} \to u_e(\theta; k), \quad \tau_0 \to 0\)

\(\theta \to 0^+: \quad u_e \sim b(k)\theta + O(\theta^2), \quad \psi_0 \sim b^{1/2} \theta H(b^{1/2} N), \quad r_0 = O(\theta^2)\)

flow laminar close to \(F\)

\(H''^2 - HH''' = 1 + H'''', \quad H(0) = H'(0) = 0, \quad H'(\infty) = 1\)

present study: BL in the ‘turbulent limit’ \(T \to \infty\)

\(T \tau_0 \gg \frac{\partial^2 \psi_0}{\partial N^2} \Rightarrow \text{problem singularly perturbed}\)

\[
\begin{cases}
\theta \to 0, & N = O(1) \\
N \to 0, & \theta = O(1)
\end{cases}
\]
7 Separation for high turbulence intensities
   Boundary layer splitting
   Asymptotically ‘underdeveloped’ turbulence
   Numerical results
Large-$T$ asymptotics

region I: two BL types possible

velocity scale $u_1$: \[ r_0 = O(u_1^2) \] \[ \{ \text{velocity defect: } u_e - u = O(u_1) \} \text{ Re} \to \infty \]

\[ [\psi_0, \tau_0] = [u_e \Delta f, Tu_e^2 s] \quad \zeta := N/\Delta(\theta; k, \text{Re, } T) \]

(1) \[ [f, s] \sim [f_0, s_0](\theta, \zeta; k) + \cdots, \quad s_0(\theta, 0; k) = 0, \quad u_1 \text{ finite} \]

(2) \[ [f, s] \sim [\zeta, 0] + [-\gamma f_1, \gamma^2 s_1](\theta, \zeta; k) + \cdots, \quad \gamma := u_1/u_e \to 0 \]

region II: transition

\[ X := Tb(k)^{1/2} \theta = O(1) \quad \eta := b^{1/2} N \quad \rho := \zeta/\eta \]

\[ [u_e, \psi_0] \sim T^{-1} X[b^{1/2}, F(X, \eta)], \quad \tau_0 \sim T^{-2} X^2 b S(X, \eta) \]

(1) \[ [F, S] \sim [F_0, S_0](\zeta), \quad \rho \propto 1/X, \quad S_0(0) > 0 \]

(2) \[ [F, S] \sim [\gamma_F(X) F_1(\zeta)/\rho, \gamma_S(X) S_1(\zeta)], \quad \gamma_F \to 0 \]

$X \to \infty$
Large-$T$ asymptotics

region I: two BL types possible

velocity scale \( u_1 \): \[ r_0 = O(u_1^2) \]
velocity defect: \[ u_e - u = O(u_1) \]

\[
\begin{align*}
[\psi_0, \tau_0] &= [u_e \Delta f, T u_e^2 s] \\
\zeta &= N/\Delta(\theta; k, Re, T)
\end{align*}
\]

(1) \[ [f, s] \sim [f_0, s_0](\theta, \zeta; k) + \cdots, \quad s_0(\theta, 0; k) = 0, \quad u_1 \text{ finite} \]

(2) \[ [f, s] \sim [\zeta, 0] + [-\gamma f_1, \gamma^2 s_1](\theta, \zeta; k) + \cdots, \quad \gamma := u_1/u_e \to 0 \]

region II: transition \[ X := T b(k)^{1/2} \theta = O(1) \quad \eta := b^{1/2} N \quad \rho := \zeta/\eta \]

\[
[u_e, \psi_0] \sim T^{-1} X[b^{1/2}, F(X, \eta)] , \quad \tau_0 \sim T^{-2} X^2 b S(X, \eta)
\]

(1) \[ [F, S] \sim [F_0, S_0](\zeta), \quad \rho \propto 1/X, \quad S_0(0) > 0 \]

(2) \[ [F, S] \sim [\gamma_F(X) F_1(\zeta)/\rho, \gamma_S(X) S_1(\zeta)], \quad \gamma_F \to 0 \]
Large-$T$ asymptotics

region I: two BL types possible

velocity scale $u_1$: $r_0 = O(u_1^2)$
velocity defect: $u_e - u = O(u_1)$ \( \{ \)
\[ \psi_0, \tau_0 \] = \[ u_e \Delta f, Tu_e^2 s \] \hspace{1cm} \zeta := N/\Delta(\theta; k, Re, T) \]

(1) $[f, s] \sim [f_0, s_0](\theta, \zeta; k) + \cdots, \quad s_0(\theta, 0; k) = 0, \quad u_1 \text{ finite}$
(2) $[f, s] \sim [\zeta, 0] + [-\gamma f_1, \gamma^2 s_1](\theta, \zeta; k) + \cdots, \quad \gamma := u_1/u_e \to 0$

region II: transition

$X := Tb(k)^{1/2} \theta = O(1) \quad \eta := b^{1/2} N \quad \rho := \zeta/\eta$

$[u_e, \psi_0] \sim T^{-1}X[b^{1/2}, F(X, \eta)], \quad \tau_0 \sim T^{-2}X^2 b S(X, \eta)$

(1) $[F, S] \sim [F_0, S_0](\zeta), \quad \rho \propto 1/X, \quad S_0(0) > 0$
(2) $[F, S] \sim [\gamma_F(X)F_1(\zeta)/\rho, \gamma_S(X)S_1(\zeta)], \quad \gamma_F \to 0 \{ \}

$X \to \infty$
Outline

7 Separation for high turbulence intensities
   Boundary layer splitting
   Asymptotically ‘underdeveloped’ turbulence
   Numerical results
‘Underdeveloped’ turbulence

matching: regions I, II

(1) \( s_0(\theta, 0, k) = 0 \), \( S_0(0) > 0 \) \( \Rightarrow \) no match

(2) \( s_1(\theta \rightarrow 0, 0, k) > 0 \) \( \Rightarrow \) \( S_1(0) > 0 \) \( \Rightarrow \) \( [f_1, s_1] \sim [F_1, S_1] \)

matching: outer region I, viscous wall layer III

main layer scale \( u_I \):
\[
\begin{align*}
    r_0 &= O(u_I^2) \\
    u_e - u &= O(u_I)
\end{align*}
\]

wall layer scale \( u_{III} \):
\[
\begin{align*}
    r_0 &= O(u_{III}^2) \\
    u &= O(u_{III})
\end{align*}
\]

(2) \( s_1(\theta, 0, k) := 1 \) \( u_t := u_I = u_{III} \) \( \partial f_1 / \partial \zeta \sim -\kappa^{-1} \ln \zeta + B_1(\theta) \), \( \zeta \rightarrow 0 \)

small-defect scaling \( \gamma = u_t / u_e \sim \kappa / (2 \ln T) \)

classical scaling \( \Rightarrow T \sim Re^{1/2} \)

flow structure near \( F \) and \( S \) \( \Rightarrow T \ll Re^{1/2} \) Scheichl & Kluwick (AIAA P, 2008)
‘Underdeveloped’ turbulence

matching: regions I, II

\[(1) \quad s_0(\theta, 0, k) = 0, \quad S_0(0) > 0 \Rightarrow s_0 \not\sim S_0 \quad \text{no match}\]

\[(2) \quad s_1(\theta \to 0, 0, k) > 0 \Rightarrow S_1(0) > 0 \Rightarrow [f_1, s_1] \sim [F_1, S_1]\]

matching: outer region I, viscous wall layer III

main layer scale \( u_I \):
\[
\begin{align*}
  r_0 &= O(u_I^2) \quad u_e - u = O(u_I)
\end{align*}
\]

wall layer scale \( u_{III} \):
\[
\begin{align*}
  r_0 &= O(u_{III}^2) \quad u = O(u_{III})
\end{align*}
\]

\[(2) \quad s_1(\theta, 0, k) := 1 \quad u_t := u_I = u_{III} \quad \partial f_1/\partial \zeta \sim -\kappa^{-1} \ln \zeta + B_1(\theta), \quad \zeta \to 0\]

small-defect scaling \( \gamma = u_t/u_e \sim \kappa/(2 \ln T) \)

classical scaling \( \Rightarrow T \sim Re^{1/2} \)

flow structure near \( F \) and \( S \) \( \Rightarrow T \ll Re^{1/2} \)

Scheichl & Kluwick (AIAA P, 2008)
‘Underdeveloped’ turbulence

matching: regions I, II

(1) \( s_0(\theta, 0, k) = 0 \), \( S_0(0) > 0 \) \( \Rightarrow \) \( s_0 \not\sim S_0 \) no match

(2) \( s_1(\theta \to 0, 0, k) > 0 \) \( \Rightarrow \) \( S_1(0) > 0 \) \( \Rightarrow \) \([f_1, s_1] \sim [F_1, S_1]\)

matching: outer region I, viscous wall layer III

main layer scale \( u_I \):
\[
\begin{align*}
    r_0 &= O(u_I^2) \\
    u_e - u &= O(u_I)
\end{align*}
\]

wall layer scale \( u_{III} \):
\[
\begin{align*}
    r_0 &= O(u_{III}^2) \\
    u &= O(u_{III})
\end{align*}
\]

(2) \( s_1(\theta, 0, k) := 1 \) \( u_t := u_I = u_{III} \) \( \partial f_1 / \partial \zeta \sim -\kappa^{-1} \ln \zeta + B_1(\theta), \zeta \to 0 \)

small-defect scaling \( \gamma = u_t / u_e \sim \kappa / (2 \ln T) \)

classical scaling \( \Rightarrow \) \( T \sim Re^{1/2} \)

flow structure near \( F \) and \( S \) \( \Rightarrow \) \( T \ll Re^{1/2} \) Scheichl & Kluwick (AIAA P, 2008)
Separation for high turbulence intensities

Boundary layer splitting
Asymptotically ‘underdeveloped’ turbulence
Numerical results
Solution of BIV problem for transitional BL

\[ [F, S](X, \eta) \quad \eta := b^{1/2} N \quad \rho(X) := \zeta/\eta \quad X \to \infty : [F', S] \sim [F'_1, S_1](\zeta) \]

\[ F'^2 - FF'' + X(F' \partial F'/\partial X - F'' \partial F/\partial X) = 1 + XS' + F'''' \]

\[ F(X, 0) = F'(X, 0) = 0 \quad F'(X, \infty) = 1 \quad F(0, \eta) = H(\eta) \]

- algebraic closure for \( \ell \), Klebanoff’s intermittency factor
  Michel, Quémard & Durant (1969), Grifoll & Giralt (J Heat Mass Tran, 2000), Klebanoff (1955)

- Keller–Box scheme / method of lines, adaptive step size control
Solution of BIV problem for transitional BL

\[ [F, S](X, \eta) \quad \eta := b^{1/2}N \quad \rho(X) := \zeta/\eta \quad X \to \infty : [F', S] \sim [F'_1, S_1](\zeta) \]

\[ F'' - FF'' + X(F' \partial F'/\partial X - F'' \partial F/\partial X) = 1 + X S' + F''' \quad S := [\ell(X, \eta)F'']^2 \]

\[ F(X, 0) = F'(X, 0) = 0 \quad F'(X, \infty) = 1 \quad F(0, \eta) = H(\eta) \]

- algebraic closure for \( \ell \), Klebanoff’s intermittency factor
  Michel, Quémard & Durant (1969), Grifoll & Giralt (J Heat Mass Tran, 2000), Klebanoff (1955)

- Keller–Box scheme / method of lines, adaptive step size control

\[ X_i = X_* e^{[i/10] \ln(X_{10}/X_*)} \]

\[ [X_*, X_{10}] = [100, 4.6] \]

\[ F'(\eta) \quad F'_1(\zeta) \]

\[ S(\eta) \quad S_1(\zeta) \]
Conclusions and outlook

turbulent BLs on flat plates and related bodies

- small velocity defect
  ⇒ unable to predict separation
- moderately large velocity defect
  ⇒ loss of uniqueness (self-preserving flows)
- large velocity defect
  ⇒ theory of marginal separation

turbulent BLs on bluff bodies

- stretch from front stagnation to separation point
  ⇒ apparently never attain a large velocity defect
  ⇒ apparently never reach a fully turbulent state
- modified small-defect theory
  ⇒ interaction theory, global separation
Conclusions and outlook

turbulent BLs on flat plates and related bodies

- small velocity defect
  ⇒ unable to predict separation
- moderately large velocity defect
  ⇒ loss of uniqueness (self-preserving flows)
- large velocity defect
  ⇒ theory of marginal separation

B. Scheichl (TU Vienna)

Asymptotic theory of turbulent boundary layers
Conclusions and outlook

Turbulent BLs on flat plates and related bodies

- small velocity defect
  ⇒ unable to predict separation
- moderately large velocity defect
  ⇒ loss of uniqueness (self-preserving flows)
- large velocity defect
  ⇒ theory of marginal separation

Turbulent BLs on bluff bodies

- stretch from front stagnation to separation point
  ⇒ apparently never attain a large velocity defect
  ⇒ apparently never reach a fully turbulent state
- modified small-defect theory
  ⇒ interaction theory, global separation
Conclusions and outlook

turbulent BLs on flat plates and related bodies

- small velocity defect
  ⇒ unable to predict separation
- moderately large velocity defect
  ⇒ loss of uniqueness (self-preserving flows)
- large velocity defect
  ⇒ theory of marginal separation

B. Scheichl (TU Vienna)

Asymptotic theory of turbulent boundary layers
Thank you for your attention!