Quantum noise thermometry for bosonic Josephson junctions in the mean-field regime

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Bosonic Josephson junctions can be realized by confining ultracold gases of bosons in multiwell traps and studied theoretically with the M-site Bose-Hubbard model. We show that canonical equilibrium states of the M-site Bose-Hubbard model may be approximated by mixtures of coherent states, provided the number of atoms is large and the total energy is comparable to k_BT. Using this approximation, we study thermal fluctuations in bosonic Josephson junctions in the mean-field regime. Statistical estimates of the fluctuations of relative phase and number, obtained by averaging over many replicates of an experiment, can be used to estimate the temperature and the tunneling parameter or to test whether the experimental procedure is effectively sampling from a canonical thermal equilibrium ensemble.

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I. INTRODUCTION

Quantum degenerate Bose gases in double-well potentials exhibit coherent macroscopic tunneling dynamics analogous to those in superconducting Josephson junctions [1–3]. The observer can detect individual well populations by direct optical absorption imaging and can infer the relative phase of the wave packets from interference experiments [4–6]. Furthermore, atomic interactions can be tuned over a wide range by adjusting particle number and double-well parameters or by means of Feshbach resonances (cf., [7] for a review).

Double-well systems are often modeled within a two-mode approximation by the Bose-Hubbard Hamiltonian. The parameters of the model are the number $N \gg 1$ of atoms, the interaction energy E_C for a pair of particles in the same potential well, and the tunneling coupling energy E_J (cf., [8–10], which use the same notation). One distinguishes the "Rabi," "Josephson," and "Fock" regimes [11] according to the relations

$$E_C/E_J \ll N^{-2} \quad \text{(Rabi)},$$

$$N^{-2} \ll E_C/E_J \ll 1 \quad \text{(Josephson)},$$

$$1 \ll E_C/E_J \quad \text{(Fock)}.$$

Recent experiments making use of strong interactions deep in the Josephson regime have accomplished squeezing and macroscopic entanglement [4]. On the other hand, coherent tunneling dynamics and Bloch oscillations have been realized in completely noninteracting Bose gases [12,13]. The intermediate regime of moderate interactions (the Josephson-Rabi boundary regime) is virtually unexplored in experiment. This regime, where $N^2E_C \sim E_J$, is of particular interest, as it contains most of the stationary Josephson modes, such as 0 and π phase modes, and the onset of macroscopic quantum self-trapping [14]. Here, number and phase fluctuations are sensitive to the ratio of N^2E_C to E_J, both in the ground state [15] and, as we shall see, in thermal equilibrium at higher temperatures.

When a gas of ultracold bosons is released from a double-well potential trap and recombined in free expansion, interference fringes, analogous to those of Young's double-slit experiment, are observed in the atomic density. This does not necessarily mean that the double-well system was prepared in a coherent state, as individual images or "shots" will feature interference fringes *even if the gases are initially independent* [16]. To ascertain that the experimental procedure prepares the system in a coherent state, one needs to repeat the experiment many times and compare the results. If the fringes always lie in the same position, one may infer that the state is coherent and ascribe a definite value to the relative phase between the condensates in the two wells.

The decohering effect of temperature is seen in the fluctuations, from one shot to another, of the location of the interference fringes. These fluctuations reduce the visibility of the interference fringes when the density profiles are averaged. The fringe contrast in the average density profile is called the "coherence factor" and, for double-well systems in the Josephson regime, is found to be a certain function of k_BT/E_J [9]. This function is used to calibrate the "thermometer" of noise thermometry [10,17].

In this paper we study canonical thermal equilibrium states of $N \gg 1$ bosons in double-well and multiwell potentials, focusing on regimes where both $E_J/k_BT \ll N$ and $N^2E_C/k_BT \ll N$. This includes the Rabi-Josephson boundary regime, provided the temperature is high enough that $E_J/k_BT \ll N$. We find that the coherence factor is sensitive to the ratios E_J/k_BT and N^2E_C/k_BT. Our results are rigorous inasmuch as they are derived from a general theorem about canonical statistics of M-mode boson models [18].

The regimes we consider are not normally attained in atom interferometry experiments. For example, the noise thermometry experiments reported in [10,17] involved trapping a few thousand ^{87}Rb atoms at 15–80 nK. The parameter E_J/k_BT ranged between about 0.02 and 20, but the parameter N^2E_C/k_BT was large because the experiments were performed deep within the Josephson regime. However, it should be possible to engineer bosonic Josephson junctions in the Rabi-Josephson boundary regime by taking advantage of Feshbach resonances to reduce the interaction parameter E_C [12,13].

This paper is organized as follows. In Sec. II we state our main results and proposals concerning noise thermometry of two-site bosonic Josephson junctions. In Sec. III we state the central result of this paper, theorem 1, a general result concerning canonical statistics of M-site Bose-Hubbard models. We return to the particular case $M=2$ in Sec. IV and discuss the fluctuations of density observables such as interference fringes in time-of-flight (TOF) matter wave interferometry. We outline a proof of theorem 1 in the Appendix.

II. NOISE THERMOMETRY WITH BOSONIC JOSEPHSON JUNCTIONS

Double-well systems may be modeled using a two-mode approximation [19]. In a symmetric two-well potential, the ground state is degenerate when the wells are separated by an infinitely high barrier: the *gerade* and *ungerade* modes have the same energy. If the barrier between the wells is finite, tunneling lifts the ground state degeneracy. Provided the tunneling barrier is not *too* low, the energy splitting of these low energy modes is small compared to the energy difference between them and the higher excited states, and, at low enough temperatures, there are effectively only two modes in play.

Two-mode approximations have been derived using either a "semiclassical" or "second-quantized" approach. The former approach attempts to constrain the semiclassical Gross-Pitaevskii dynamics to a two-dimensional subspace [1,14]. The latter approach begins with the second-quantized Hamiltonian and attempts to restrict it to one involving only two modes [20], which might be the lowest energy solutions of the Gross-Pitaevskii equation [15] or which might be found by self-consistent variational minimization of the energy over all suitable two-mode approximations [21]. The relationship between the second-quantized and the semiclassical theories is discussed in [8].

Two-mode models become quite sophisticated [22]. The simplest one is the two-site Bose-Hubbard model, whose Hamiltonian is

$$H_N = -E_J\hat{x} + (N^2E_C/4)\hat{z}^2. \tag{1}$$

In this formula, the operators $\hat{x}=\frac{1}{N}(a_1^\dagger a_2 + a_2^\dagger a_1)$ and $\hat{z}=\frac{1}{N}(a_1^\dagger a_1 - a_2^\dagger a_2)$ are understood to operate on the N-particle component of the boson Fock space. The observable $\hat{z}$ is the relative number imbalance between the wells. The observable $\hat{x}$ is the relative occupation difference of the gerade and ungerade modes, which is related to relative phase.

Canonical thermal equilibrium states of the two-site Bose-Hubbard Hamiltonian can be approximated by certain mixtures of coherent states. We shall show that this approximation is rigorously justified for regimes where $E_J/k_BT \ll N$ and $N^2E_C/k_BT \ll N$ (and N is large). Define the dimensionless parameters,

$$\delta = E_J/k_BT,$$

$$\varepsilon = N^2E_C/4k_BT.$$

For regimes where $\delta,\varepsilon \ll N$, we will derive the following formulas for the coherence factor [Eq. (2)] and the second moment of $\hat{z}$ [Eq. (5)].

The coherence factor α is defined to be the fringe contrast in the ensemble averaged density profile of a double-well interference experiment [9,17,23]. We will show that

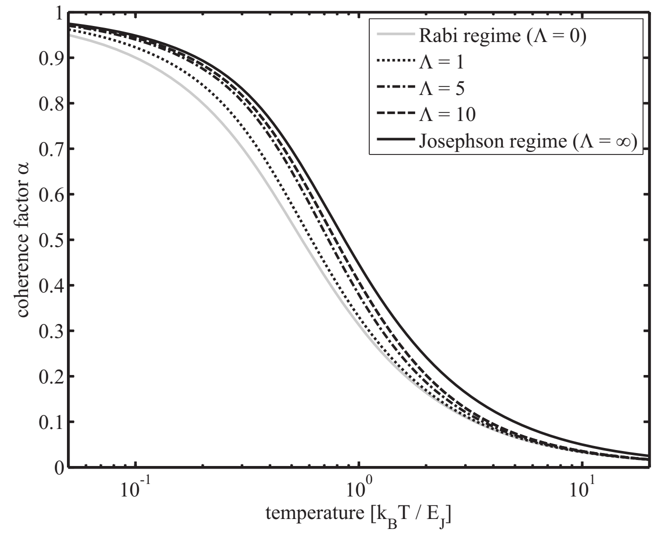

FIG. 1. Coherence factor α vs temperature for several values of $\Lambda = N^2E_C/4E_J$ (logarithmic scale).

$$\alpha = \frac{\int_{-1}^{1} xI_0[\varepsilon(1-x^2)/4]e^{\delta x+\varepsilon x^2/4}dx}{\int_{-1}^{1} I_0[\varepsilon(1-x^2)/4]e^{\delta x+\varepsilon x^2/4}dx}, \tag{2}$$

where I_0 denotes the modified Bessel function of the first kind (of order zero). In the noninteracting case, when $\varepsilon=0$, formula (2) reduces to

$$\alpha_{\delta,0} = \coth(\delta) - 1/\delta. \tag{3}$$

In the strongly repulsive case, when $\varepsilon \gg 1$, the term $I_0[\varepsilon(1-x^2)/4]e^{\varepsilon x^2/4}$ is nearly proportional to $1/\sqrt{1-x^2}$ over much of the domain of integration, and formula (2) tells us that

$$\alpha_{\delta,\infty} \approx \frac{\int_{-1}^{1} x\frac{e^{\delta x}}{\sqrt{1-x^2}}dx}{\int_{-1}^{1} \frac{e^{\delta x}}{\sqrt{1-x^2}}dx} = \frac{I_1(\delta)}{I_0(\delta)}. \tag{4}$$

This agrees with the semiclassical formula for the coherence factor in the Josephson regime [9,10,17].

In a symmetric double-well potential, the expected value of the population imbalance $\hat{z}$ is zero, i.e., $\langle\hat{z}\rangle=0$. We will show that the variance of $\hat{z}$ is

$$\langle\hat{z}^2\rangle = \frac{\int_0^1 z^2 I_0(\delta\sqrt{1-z^2})e^{-\varepsilon z^2/2}dz}{\int_0^1 I_0(\delta\sqrt{1-z^2})e^{-\varepsilon z^2/2}dz}. \tag{5}$$

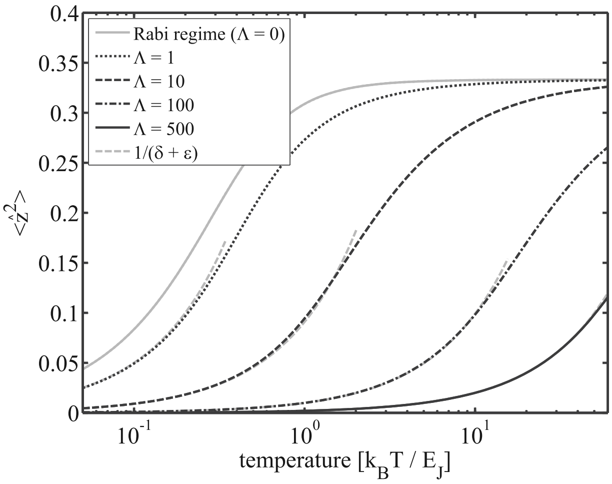

FIG. 2. Variance of the normalized population imbalance $\langle \hat{z}^2 \rangle$ vs temperature for various values of $\Lambda = N^2 E_C / 4E_J$. The gray dashed lines indicate the "low temperature" approximation (15).

In Figs. 1 and 2, α and $\langle \hat{z}^2 \rangle$ are plotted against $k_B T / E_J$ on a logarithmic scale for various values of

$$\Lambda = \varepsilon / \delta = N^2 E_C / 4E_J.$$

The method of "noise thermometry" developed in [10,17] uses statistical estimates of α, obtained by replicating a double-well experiment under identical conditions, to determine $k_B T / E_J$. If E_J is known, or estimable, the temperature T can be deduced even when this temperature is so low that it cannot be found by the usual technique (fitting a Gaussian to the "wings" of the density profile after some time of flight). This method is suitable for the Josephson regime $E_J \ll N^2 E_C$, where the parameter Λ is formally equal to ∞.

To perform noise thermometry in the Rabi-Josephson boundary regime, where $0 \ll \Lambda \ll \infty$, one needs to know both $k_B T / E_J$ and Λ. Estimates of $k_B T / E_J$ and Λ can be deduced from statistical estimates of the coherence factor α and the variance $\langle \hat{z}^2 \rangle$ of the number fluctuations, thanks to formulas (2) and (5).

There are a couple of benefits of doing noise thermometry in the Rabi-Josephson boundary regime:

(1) When a bosonic Josephson junction is fashioned in the laboratory, one usually has more accurate knowledge of the parameter E_C than the parameter E_J (the tunneling energy E_J is quite difficult to estimate due to its exponential sensitivity to the precise geometry of the double-well potential). By performing noise thermometry in the Rabi-Josephson boundary regime, one can take advantage of one's knowledge of E_C to estimate E_J as well as T.

(2) If one does happen to know E_J with some accuracy, one obtains *two* estimates of the temperature. If these estimates differ significantly, it may be evidence that the experimental procedure has failed to prepare the double-well system in a canonical thermal equilibrium state. The assumption that replication of the experiment yields samples from the canonical ensemble ought to be tested because some ways of preparing the system can fix it in a noncanonical equilibrium state, for example, if the double well is formed by ramping up a potential barrier too quickly [23].

III. CANONICAL STATISTICS OF THE M-SITE BOSE-HUBBARD MODEL

The M-site Bose-Hubbard Hamiltonian for bosons with nearest-neighbor hopping is

$$-J \sum_{i=1}^{M-1} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) + \frac{U}{2} \sum_{i=1}^{M} a_i^\dagger a_i^\dagger a_i a_i.$$

We are going to discuss systems of exactly N bosons and take a limit $N \to \infty$. Accordingly, we will consider the restriction of the above Hamiltonian to the N-particle subspaces of the boson Fock space and allow the parameters J and U to depend on N. Let $\mathcal{P}_N$ denote the orthogonal projector onto the N-particle component of the boson Fock space over $\mathbb{C}^M$ and let

$$\left(-J_N \sum_{i=1}^{M-1} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) + \frac{U_N}{2} \sum_{i=1}^{M} a_i^\dagger a_i^\dagger a_i a_i \right) \mathcal{P}_N \tag{6}$$

be the N-boson Hamiltonian H_N.

The density operator

$$e^{-H_N / k_B T} / \mathrm{Tr}(e^{-H_N / k_B T})$$

represents the canonical ensemble of N bosons in thermal equilibrium at temperature T in the sense that

$$\langle X \rangle_{N,T} = \mathrm{Tr}(X e^{-H_N / k_B T}) / \mathrm{Tr}(e^{-H_N / k_B T}) \tag{7}$$

is the expected value of any observable X when the system is in the canonical thermal equilibrium state. We are going to show that this state may be approximated by a mixture of coherent states, provided that $N \gg 1$ and both $N J_N / k_B T$, $N^2 U_N / k_B T \ll N$.

We parametrize the pure states of the M-site system by the product of the standard $(M-1)$-dimensional simplex and the M-dimensional torus. Let Δ_M denote the standard $(M-1)$-dimensional simplex

$$\{ p = (p_1, p_2, \ldots, p_M) \in \mathbb{R}^M \mid p_1 + \cdots + p_M = 1, \ \ p_i \geq 0 \}$$

and let $[0, 2\pi)^M$ denote

$$\{ \phi = (\phi_1, \phi_2, \ldots, \phi_M) \mid \phi_i \in [0, 2\pi) \text{for } \ i = 1, 2, \ldots, M \}.$$

To each point $(p, \phi) \in \Delta_M \times [0, 2\pi)^M$ we associate the unit vector

$$\psi_{p,\phi} = (\sqrt{p_1} e^{i\phi_1}, \sqrt{p_2} e^{i\phi_2}, \ldots, \sqrt{p_M} e^{i\phi_M}). \tag{8}$$

The parametrization $(p, \phi) \mapsto \psi_{p,\phi}$ is many–to–one because a global change of phase in ϕ does not change $\psi_{p,\phi}$.

Let $\mu_M(dp)$ denote the uniform probability measure on Δ_M; in particular, μ_2 is equivalent to the length measure dp on the unit interval $p \in [0, 1]$. Let $\nu_M(d\phi)$ denote the uniform probability measure $(2\pi)^{-M} d\phi_1 d\phi_2 \ldots d\phi_M$ on $[0, 2\pi)^M$.

Theorem 1. Let J_N and U_N be two sequences of parameter values. For each N, let H_N denote operator (6) and let $\langle X\rangle_{N,T}$ denote ensemble average (7) for the canonical ensemble at temperature T. If

$$\lim_{N\to\infty}\frac{NJ_N}{k_BT}=\delta \quad \text{and} \quad \lim_{N\to\infty}\frac{N^2U_N}{4k_BT}=\varepsilon, \tag{9}$$

then, for any vectors $\chi_1,\ldots\chi_k,\ \chi_1',\ldots,\chi_k'\in\mathbb{C}^M$,

$$\lim_{N\to\infty}\frac{1}{N^k}\langle a^\dagger_{\chi_1}a^\dagger_{\chi_2}\ldots a^\dagger_{\chi_k}a_{\chi_1'}a_{\chi_2'}\ldots a_{\chi_k'}\rangle_{N,T}$$
$$=\int_{\Delta_M}\int_{[0,2\pi)^M}\prod_{i=1}^{k}\langle\chi_i|\psi_{p,\phi}\rangle\langle\psi_{p,\phi}|\chi_i'\rangle\frac{\exp\left[2\delta\sum_{i=1}^{M-1}\sqrt{p_ip_{i+1}}\cos(\phi_{i+1}-\phi_i)-\varepsilon\sum_{i=1}^{M}p_i^2\right]}{\int\int\exp\left[2\delta\sum\sqrt{p_i'p_{i+1}'}\cos(\phi_{i+1}'-\phi_i')-\varepsilon\sum p_i'^2\right]\mu_M(dp')\nu_M(d\phi_i')}\nu_M(d\phi)\mu_M(dp). \tag{10}$$

This theorem can be deduced from propositions concerning Finetti representations for canonical states of M-mode bosons [18]. A proof is outlined in the Appendix.

IV. DERIVATION OF FORMULAS (2) and (5)

The two-site Bose-Hubbard Hamiltonian is [8]

$$-\frac{E_J}{N}(a_1^\dagger a_2+a_2^\dagger a_1)\mathcal{P}_N+\frac{E_C}{4}(a_1^\dagger a_1^\dagger a_1a_1+a_2^\dagger a_2^\dagger a_2a_2)\mathcal{P}_N$$

($\mathcal{P}_N$ restricts the operators to the N-particle component of the Fock space). When expressed in terms of the observables

$$\hat{x}=\tfrac{1}{N}(a_1^\dagger a_2+a_2^\dagger a_1)\mathcal{P}_N$$

$$\hat{z}=\tfrac{1}{N}(a_1^\dagger a_1-a_2^\dagger a_2)\mathcal{P}_N$$

on the N-boson space, this Hamiltonian just differs by the constant $E_CN(N-2)/4$ from Hamiltonian (1).

We are going to rewrite formula (10) for $M=2$, the double-well case. Changing variables

$$p_1=\tfrac{1}{2}+\tfrac{1}{2}z,\quad p_2=\tfrac{1}{2}-\tfrac{1}{2}z,\quad \phi=\phi_2-\phi_1,\quad \phi'=\phi_1+\phi_2$$

in formula (8), we write

$$\psi_{p,\phi}=e^{i\phi'/2}\left(\tfrac{\sqrt{1+z}}{\sqrt{2}}e^{-i\phi/2},\ \tfrac{\sqrt{1-z}}{\sqrt{2}}e^{i\phi/2}\right).$$

Define

$$\vec{u}_{z,\phi}=\left(\tfrac{\sqrt{1+z}}{\sqrt{2}}e^{-i\phi/2},\ \tfrac{\sqrt{1-z}}{\sqrt{2}}e^{i\phi/2}\right) \tag{11}$$

for $(z,\phi)\in[-1,1]\times[0,2\pi)$. The operators $a_1^\dagger$ and $a_2^\dagger$ in H_N are identified with the creation operators for the vectors $\vec{u}_{1,0}=(1,0)$ and $\vec{u}_{-1,0}=(0,1)$, respectively. Let us also define the probability density functions

$$P_{\delta,\varepsilon}(z,\phi)=\frac{\exp(\delta\sqrt{1-z^2}\cos\phi-\varepsilon z^2/2)}{\int\int e^{\delta\sqrt{1-z'^2}\cos\phi'-\varepsilon z'^2/2}d\phi' dz'} \tag{12}$$

on $[-1,1]\times[0,2\pi)$. Changing variables in formula (10) we find that

$$\lim_{N\to\infty}\frac{1}{N^k}\langle a^\dagger_{\chi_1}\ldots a^\dagger_{\chi_k}a_{\chi_1'}\ldots a_{\chi_k'}\rangle_{N,T}$$
$$=\int_0^{2\pi}\int_{-1}^{1}\prod_{i=1}^{k}\langle\chi_i|\vec{u}_{z,\phi}\rangle\langle\vec{u}_{z,\phi}|\chi_i'\rangle P_{\delta,\varepsilon}(z,\phi)dzd\phi \tag{13}$$

in the limit $N\to\infty$ with

$$E_J/k_BT\to\delta,\quad N^2E_C/4k_BT\to\varepsilon. \tag{14}$$

A. Population imbalance

Formula (13) may be applied directly to the observable $\hat{z}$. In a symmetric double well, $\langle\hat{z}\rangle=0$. Higher moments of $\hat{z}$ are those of the probability distribution

$$\frac{I_0(\delta\sqrt{1-z^2})e^{-\varepsilon z^2/2}}{\int_0^1 I_0(\delta\sqrt{1-(z')^2})e^{-\varepsilon(z')^2/2}dz'}dz,$$

that is, in limit (14) for each fixed k,

$$\lim\langle\hat{z}^k\rangle_{N,T}=\frac{\int_{-1}^{1}z^kI_0(\delta\sqrt{1-z^2})e^{-\varepsilon z^2/2}dz}{\int_{-1}^{1}I_0(\delta\sqrt{1-z^2})e^{-\varepsilon z^2/2}dz}.$$

We demonstrate this for $k=2$,

$$\langle \hat{z}^2\rangle_{\delta,\varepsilon} := \lim \langle \hat{z}^2\rangle_{N,T} = \lim \frac{1}{N^2}\langle (a_1^\dagger a_1 - a_2^\dagger a_2)^2\rangle_{N,T}$$

$$= \lim \frac{1}{N^2}\langle a_1^{\dagger 2}a_1^2 + a_2^{\dagger 2}a_2^2 - 2a_1^\dagger a_2^\dagger a_1 a_2\rangle_{N,T}$$

$$+ \lim \frac{1}{N^2}\langle a_1^\dagger a_1 + a_2^\dagger a_2\rangle_{N,T}$$

$$= \int_0^{2\pi}\int_{-1}^{1} (|\langle \vec{u}_{1,0}|\vec{u}_{z,\phi}\rangle|^2 - |\langle \vec{u}_{-1,0}|\vec{u}_{z,\phi}\rangle|^2)^2 \times P_{\delta,\varepsilon}(z,\phi) dz\, d\phi$$

$$= \int_0^{2\pi}\int_{-1}^{1} z^2 P_{\delta,\varepsilon}(z,\phi) dz\, d\phi$$

$$= \frac{\int_{-1}^{1} z^2 I_0(\delta\sqrt{1-z^2}) e^{-\varepsilon z^2/2} dz}{\int_{-1}^{1} I_0(\delta\sqrt{1-z^2}) e^{-\varepsilon z^2/2} dz}.$$

This proves formula (5) for $\langle \hat{z}^2\rangle_{\delta,\varepsilon}$. Figure 2 shows that

$$\langle \hat{z}^2\rangle_{\delta,\varepsilon} \approx \frac{1}{\delta+\varepsilon} \tag{15}$$

is a good approximation at lower temperatures.

B. Coherence factor

In a TOF experiment on double wells, the potential trap is suddenly shut off and the gas expands into free space for awhile before it is imaged. The images constitute a measurement of the "integrated density" observable $\int a^\dagger(\mathbf{r})a(\mathbf{r})dr_3$, where $a^\dagger(\mathbf{r})$ denotes the usual field operator at $\mathbf{r}=(r_1,r_2,r_3)$, and the integral is over the spatial coordinate r_3 parallel to the imaging light beam and perpendicular to the line that passes through the two wells, the r_1 axis. We turn our attention to the density observables $a^\dagger(\mathbf{r})a(\mathbf{r})$ and their linear combinations.

Moments of such density observables are easily computed if atom-atom interactions during the TOF are neglected. In a two-mode approximation, each vector $\vec{u}_{z,\phi}\in\mathbb{C}^2$ is identified with some wave function $\Psi_{z,\phi}(\mathbf{r})$. The vectors $\vec{u}_{-1,0}$ and $\vec{u}_{1,0}$ are identified with the wave functions of the "left" and "right" wells, respectively. The specific map $\vec{u}\mapsto\Psi$ depends on the two-mode approximation adopted, but the precise form of the initial left and right well wave functions hardly affects the interference pattern observed after a long TOF, and we may simply assume that $\Psi_{-1,0}$ and $\Psi_{+1,0}$ are Gaussian wave packets centered at $(-d/2,0,0)$ and $(d/2,0,0)$ [24]. After a long enough [25] time t of free expansion, the wave function $\Psi_{\pm 1,0}(\mathbf{r},t)$, which describes the state of an atom that was initially in right (+) or left (−) well, will be nearly proportional to

$$\exp\left(-i\frac{m}{2\hbar t}(\|\mathbf{r}\|^2 + d^2/4)\right)\exp\left(\mp i\frac{md}{2\hbar t}r_1\right) \tag{16}$$

over the region where the density is imaged. Let $\Psi_{z,\phi}(\mathbf{r},t)$ denote

$$\tfrac{\sqrt{1+z}}{\sqrt{2}}e^{-i\phi/2}\Psi_{-1,0}(\mathbf{r},t) + \tfrac{\sqrt{1-z}}{\sqrt{2}}e^{i\phi/2}\Psi_{+1,0}(\mathbf{r},t).$$

Supposing that atom-atom interactions during the period of expansion may be neglected, the state of the many-boson system at time t is just the one freely induced by the one-particle map $\vec{u}_{z,\phi}\mapsto\Psi_{z,\phi}$, and theorem 1 implies that

$$\lim_{N\to\infty}\frac{1}{N^k}\langle a^\dagger(\mathbf{r}_1)\ldots a^\dagger(\mathbf{r}_k)a(\mathbf{r}_1')\ldots a(\mathbf{r}_k')\rangle_{N,T} = \int_0^{2\pi}\int_{-1}^{1}\prod_{i=1}^{k}\Psi_{z,\phi}(\mathbf{r}_i,t)\overline{\Psi_{z,\phi}}(\mathbf{r}_i',t)P_{\delta,\varepsilon}(z,\phi)dzd\phi \tag{17}$$

for all k and points $\mathbf{r}_1,\ldots,\mathbf{r}_k$ and $\mathbf{r}_1',\ldots,\mathbf{r}_k'$ in limit (14). Substituting Eq. (16) into Eq. (17) and proceeding formally, one finds that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\langle a^\dagger(\mathbf{r})a(\mathbf{r})\rangle_{N,T} &= \int_0^{2\pi}\int_{-1}^{1}|\Psi_{z,\phi}(\mathbf{r},t)|^2 P_{\delta,\varepsilon}(z,\phi)dzd\phi\\ &\propto 1 + \int_{-1}^{1}\int_0^{2\pi}\sqrt{1-z^2}\cos\left(\frac{md}{\hbar t}r_1 - \phi\right)P_{\delta,\varepsilon}(z,\phi)d\phi dz\\ &= 1 + \alpha_{\delta,\varepsilon}\cos\left(\frac{md}{\hbar t}r_1\right),\end{aligned} \tag{18}$$

where

$$\alpha_{\delta,\varepsilon} = \int_{-1}^{1}\int_0^{2\pi}\sqrt{1-z^2}\cos\phi P_{\delta,\varepsilon}(z,\phi)d\phi dz. \tag{19}$$

Finally, formula (2) for $\alpha_{\delta,\varepsilon}$ is obtained by changing variables $x=\sqrt{1-z^2}\cos\phi$, $y=\sqrt{1-z^2}\sin\phi$ and integrating over y.

Formula (18) shows that the interference pattern will feature fringes with spacing ht/md and contrast equal to the coherence factor $\alpha_{\delta,\varepsilon}$. In particular, formula (18) implies that

$$\lim_{N\to\infty}\frac{1}{N}\langle\int a^\dagger(\mathbf{r})a(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}\rangle_{N,T}$$

is proportional to $\alpha_{\delta,\varepsilon}$ when $\mathbf{k}=(md/\hbar t,0,0)$. Thus the coherence factor can be estimated by averaging, over many replicates of a TOF experiment, the Fourier coefficient of the imaged density profiles at wave vector $\mathbf{k}$.

V. CONCLUSION

We have studied the canonical statistics of phase and number in the M-site Bose-Hubbard model (6). Theorem 1 provides a convenient way to approximate the canonical thermal equilibrium states by mixtures of coherent states.

From theorem 1 we have derived formulas (2) and (5) for the coherence factor α and the variance of the relative population imbalance in symmetric double-well bosonic Josephson junctions. These formulas are valid in the Rabi-Josephson boundary regime, provided $N \gg 1$ and $E_J/k_B T \ll N$.

We have proposed a way to perform noise thermometry in the Rabi-Josephson boundary regime. In this regime, canonical statistics depend on two parameters, e.g., the dimensionless parameters $E_J/k_B T$ and $\Lambda = N^2 E_C/4E_J$. Statistical estimates of the coherence factor and the variance of the number fluctuations can be used to obtain empirical estimates of E_J and T and to test the assumption that the system is being prepared in a canonical equilibrium state.

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APPENDIX: PROOF OF THEOREM 1

Proposition 2 in [18] implies that

$$\lim_{N\to\infty} \frac{1}{N^k} \mathrm{Tr}[(a^\dagger_{\chi_1} \cdots a^\dagger_{\chi_k} a_{\chi'_1} \cdots a_{\chi'_k})\mathcal{P}_N]/\mathrm{Tr}(\mathcal{P}_N)$$
$$= \int_{\Delta_M} \int_{[0,2\pi)^M} \prod_{i=1}^{k} \langle \chi_i | \psi_{p,\phi}\rangle \langle \psi_{p,\phi} | \chi'_i\rangle \nu_M(d\phi)\mu_M(dp) \tag{A1}$$

for any vectors $\chi_1,\ldots,\chi_k,\chi'_1,\ldots,\chi'_k \in \mathbb{C}^M$. Indeed, formula (A1) holds even if the product of the $a^\dagger_{\chi_i}$ and $a_{\chi'}$ is not normally ordered.

Writing the operator defined in Eq. (6) as

$$H_N = [-J_N \hat{X}_1 + (U_N/2)\hat{X}_2]\mathcal{P}_N,$$

we may write

$$e^{-H_N/k_BT} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{(n_1+n_2)!} \left(\frac{NJ_N}{k_BT}\right)^{n_1} \left(\frac{N^2 U_N}{2k_BT}\right)^{n_2} \times \frac{1}{N^{n_1+2n_2}} (\hat{X}_1^{n_1}\hat{X}_2^{n_2} + \hat{X}_1^{n_1-1}\hat{X}_2\hat{X}_1\hat{X}_2^{n_2-1} + \cdots + \hat{X}_2^{n_2}\hat{X}_1^{n_1})\mathcal{P}_N.$$

We are going to take a limit of the trace of both sides of the preceding equation. Formula (A1) implies that a limit such as

$$\lim_{N\to\infty} \frac{1}{N^{n_1+2n_2}} \mathrm{Tr}(\hat{X}_1^{n_1-1}\hat{X}_2\hat{X}_1\hat{X}_2^{n_2-1}\mathcal{P}_N)/\mathrm{Tr}(\mathcal{P}_N)$$

is equal to the same limit for the normally ordered form of the operator, i.e., the limit here is identical to

$$\lim_{N\to\infty} \frac{1}{N^{n_1+2n_2}} \mathrm{Tr}(:\hat{X}_1^{n_1}\hat{X}_2^{n_2}:\mathcal{P}_N)/\mathrm{Tr}(\mathcal{P}_N). \tag{A2}$$

According to formula (A1), the limit in Eq. (A2) equals

$$\int_{\Delta_M} \int_{[0,2\pi)^M} f(p,\phi)^{n_1} g(p,\phi)^{n_2} \nu_M(d\phi)\mu_M(dp),$$

where $f(p,\phi) = 2\sum_{i=1}^{M-1}\sqrt{p_i p_{i+1}}\cos(\phi_{i+1}-\phi_i)$ and $g(p,\phi) = \sum_{i=1}^{M} p_i^2$. Therefore,

$$\lim_{N\to\infty} \frac{\mathrm{Tr}(e^{-H_N/k_BT})}{\mathrm{Tr}(\mathcal{P}_N)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{(n_1+n_2)!} \delta^{n_1}\varepsilon^{n_2} \binom{n_1+n_2}{n_1} \times \int_{\Delta_M}\int_{[0,2\pi)^M} f(p,\phi)^{n_1} g(p,\phi)^{n_2} \times \nu_M(d\phi)\mu_M(dp)$$
$$= \int_{\Delta_M}\int_{[0,2\pi)^M} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{1}{n_1!}\frac{1}{n_2!}[\delta f(p,\phi)]^{n_1} \times [-\varepsilon g(p,\phi)]^{n_2} \nu_M(d\phi)\mu_M(dp)$$
$$= \int_{\Delta_M}\int_{[0,2\pi)^M} e^{\delta f(p,\phi)-\varepsilon g(p,\phi)} \nu_M(d\phi)\mu_M(dp).$$

Similarly,

$$\lim_{N\to\infty} \frac{\mathrm{Tr}[(a^\dagger_{\chi_1}\cdots a^\dagger_{\chi_k} a_{\chi'_1}\cdots a_{\chi'_k})e^{-H_N/k_BT}]}{\mathrm{Tr}(\mathcal{P}_N)}$$
$$= \int_{\Delta_M}\int_{[0,2\pi)^M} \prod_{i=1}^{k} \langle \chi_i|\psi_{p,\phi}\rangle\langle\psi_{p,\phi}|\chi'_i\rangle \times e^{\delta f(p,\phi)-\varepsilon g(p,\phi)} \nu_M(d\phi)\mu_M(dp).$$

The last two equations imply formula (10).

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