Entry: Simplicial Complex

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gz 4361 svn 832 actual: 2415 Note: Figures are not yet in final format in this document.

Synonyms
KW complex, polyhedron, cell complex

(*these are not synonyms, but terms used to describe very similar concepts – enter them as synonyms?)*

Definition

A simplicial complex is a topological space constructed by gluing together dimensional simplices (points, line segments, triangle, tetrahedrons, etc.).

A simplicial complex *K* is a set of simplices, which satisfies the two conditions:

1. any face of a simplex in *k* is also in *K*
2. the intersection of any two simplices in *k* is a face of both simplices.

Historical Background

Raster (field) or vector (object) are the two dominant conceptualizations of space. Applications focusing on object with 2 or 3 dimensional geometry structure the storage of geometry as points, lines, surface, and volumes and the relations between them; a classical survey paper discussed the possible approaches mostly from the perspective of Computer Aided Design (CAD) where individual physical objects are constructed [10].

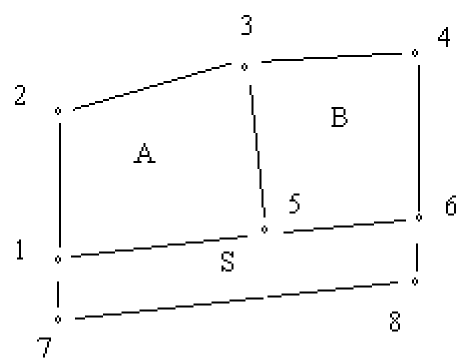

Figure 1: Cadastral parcels provide an example of a simplicial complex

The representation of geographic information, e.g., maps, introduces consistency constraints between the objects; consider the sketch of few cadastral parcels (lots) and the adjoining street (Figure 1). Land, in this case 2 dimensional space, is divided in lots, such that the lots do not overlap and there are no gaps between them; this is called a partition (definition next section). Corbett [2] proposed to check that a sequence of line segments around a face closes and that the neighbors of line segments around a point form pairs; these two conditions are dual to each other (Figure 7). This duality is the foundation of the DIME (dual independent map encoding) schema to store 2D line geometry for areas.

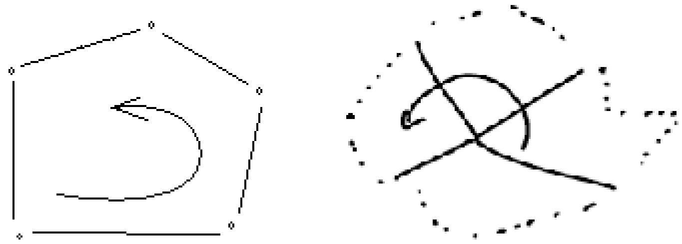

Figure 2: The two consistency checks: following the line segments around a face and following the line segments around a node.

Every line of a graph, which represents a partition is related to a start and an end point and to two adjacent faces. (Figure 3); such data structures were typical for the 1980s; implemented originally

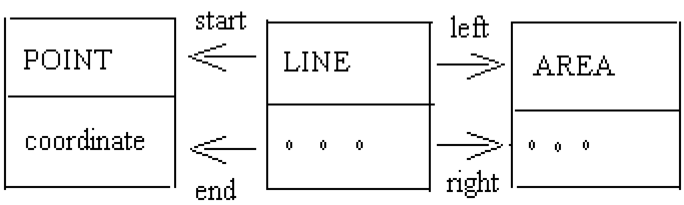

Figure 3: An UML object diagram for a database schema for partitions

with network and later relational DBMS. They did not perform acceptably fast with large Geographic Information System data, mostly because geometric operations do not translate to database operations directly (the so-called impedance mismatch of record oriented programming and tuple oriented database operations [7]), most obvious when checking geometric consistency. As late as 1985, all commercial programs to compute the overlay of two partitions, which is one of the most important operations in geographic information processing, failed.

In 1986 Frank observed that simplicial (and possibly cell) complexes enforced exactly the consistency constraints required by the large class of applications that manage geometry as 2D or became 3D partitions [5]. A commercial implementation available, designed concurrently by Dr. John Herring (then with Intergraph, now Oracle). Alternative approaches to manage the geometry of partitions without explicit representation of topology and to reconstruct topology when required were often used, but cause difficulties, because of the fundamental limitations of approximative numerical processing.

Scientific Fundamentals

Topology, specifically the theory of homotopy, provides the mathematical theory to program geometric operations. Homotopy captures the notion that multiple metric (coordinative) descriptions of a single geometry may be different but represent “essentially” the same geometry. Figure 1 can be transformed continuously to Figure 4 but not to Figure 5.

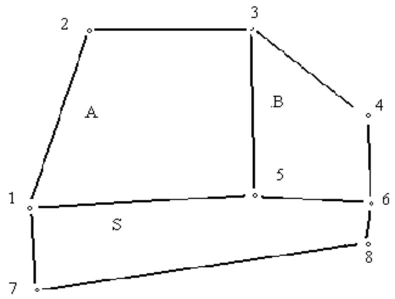

Figure 4: A deformed, but homotopic, copy of Figure 1

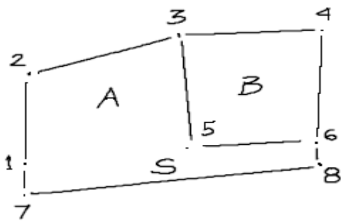

Figure 5: Metric is preserved, but the figure is not homotopic to figure 1, because elements are missing

Homotopy creates equivalence classes for geometric figures. Many applications are interested in exactly these equivalence classes and benefit from the achieved abstraction that leaves out imprecisions caused, e.g., by measurements or approximative numerical processing.

Topology studies the invariants of space under continuous (homeomorphic) transformations, which preserve neighborhoods. Algebraic topology, also called combinatorial topology [1], studies invariants of spaces under homotopy with algebraic methods. The perspective of point set topology, which sees geometric figures as (infinite) sets of points is not practical for programming and the discretization of geometry achieved through algebraic topology is crucial: the unmanageable infinite sets are converted in countable objects, namely points, lines between points and faces bounded by the boundary lines. Algebraic topology studies different 'spaces' like Figure 4 and Figure 5 (both are embedded in ordinary 2D space, but the embedding is not in focus in algebraic topology).

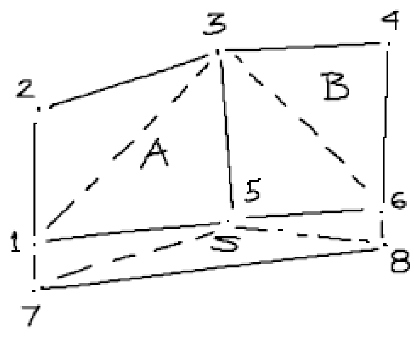

Figure 6: The geometry of figure 1 triangulated

The complexity of operations on arbitrary cells of a partition can be reduced by forcing a triangulation; all elements are then convex! Figure 1 is a cell complex and the corresponding simplicial complex is Figure 6.

Algebraic topology studies simplices and their relations: A simplex is the simplest geometric figure in each dimension. A zero dimensional simplex (0-simplex) is a point, a one dimensional simplex (1-simplex) is a straight line segment, a two dimensional simplex (2-simplex) is a triangle, a three dimensional simplex (3-simp) a tetrahedron, etc. $n + 1$ points in general position define an n-simplex. Each simplex consists of (is bounded) by $(n + 1)$ n-simplexes: a line (1-simplex) is bounded by 2 0-simplices (points).

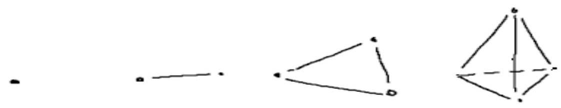

Figure 7: The simplices of 0, 1, 2, and 3 dimensions

A k-simplicial complex K is a complex in which at least one simplex has dimension k and none a higher dimension. A homogeneous (or pure) k-complex K is a complex in which every simplex with dimension less than k is the face of some higher dimension simplex in K. For example, a triangulation is a homogeneous 2-simplicial complex, a graph is a homogeneous 1-simplicial complex. Homogeneous simplicial complexes are models of partitions of space and used therefore to model geographic spatial data. Whitehead gave for CW-complexes a slightly more general, more categorical definition mostly used in homotopy theory.

Four operations are important for simplicial complexes: the *closure* of a set of simplices S is the smallest complex containing all the simplices; it contains all the faces of every simplex in S. The *star* of a set of simplices S is the set of simplices in the complex that have simplices in S as faces. The *link* of a set of simplices S is a kind of boundary around S in the complex. The *skeleton* of simplicial complex K of dimension k is the subcomplex of faces of dimension k-1 in K.

Simplicial complexes can be represented as chains, which are lists of the ordered simplices included in the complex. Chains can be written as polynomials with integer factors for the simplices included in the complex, e.g., the 2-chain of the 2-complex in Figure 1 is

$$K = 1 \cdot A + 1 \cdot B + 1 \cdot S.$$

The boundary operator δ applied to a k-simplex gives the set of k-1-simplices, which form the boundary of the simplex; for example, the boundary of a 1-simplex gives the two 0-simplices, which are start and end point of the line. The boundary operator can be applied to a simplex written as a chain. The boundary of a closed simplicial complex is 0; in general, the boundary of the boundary is 0.

$$\delta A = l_{12} + l_{23} + l_{35} - l_{15}$$

$$\delta(\delta A) = \delta l_{12} + \delta l_{23} + \delta l_{35} - \delta l_{15}$$

$$= p_1 - p_2 + p_2 - p_3 + p_3 - p_s - p_1 + p_s$$

$$= 0$$

The boundary operator is important to deduce the topological 4- and 9-intersection (Egenhofer) relations between two subcomplexes, of the same complex [3, 4]. Chains and boundary operator are easy to implement with list operators and often times it is sufficient to generalize the code for operations on polynoms.

The theory of simplicial complexes can be generalized to cell complexes. Cells are homomorph to simplices, but can have arbitrary form; a 2-cell can have an arbitrary number of nodes in its boundary.

From an application point of view, it is often important, that objects do not overlap and all of space is accounted for. The concept of a *partition* captures this idea; a partition of a space *S* is a set of subsets of the space, such that

- all subsets cover all of space (jointly exhaustive) $\bigcup_i s_i = s$,
- no two subsets overlap (pairwise disjoint) $s_i \cap s_j = \emptyset \quad for\, i = j$.

These two properties are sometimes abbreviated as JEPD.

Partitions are changed by the Euler operations, *glue* and *split*, which maintain the Euler characteristic of the surface; the Euler characteristic is computed as $\chi = V - E + F$, where *V* is the number of nodes (vertices), *E* is the number of edges and *F* is the number of faces. From Figure 1 with $\chi = 8 - 9 + 3 = 2$ results Figure 8 when the two parcels are merged (glued together) with $\chi = 8 - 8 + 2 = 2$ and Figure 9 when parcel *A* is split into parcel *C* and *D* with $\chi = 11 - 13 + 4 = 2$.

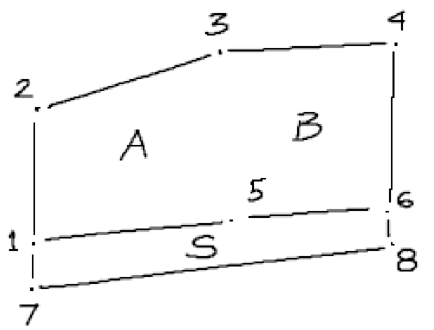

Figure 8: A and B merged

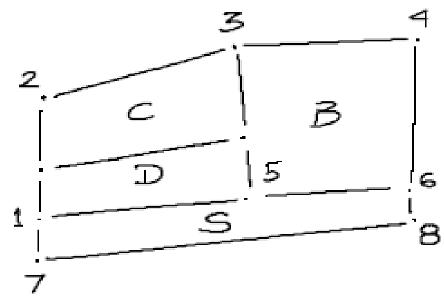

Figure 9: A subdivided in C and D

Consistency of these operations is difficult to check in cell complexes if "islands" occur as in Figure 10, which is realistic for many application areas. The problem is avoided by triangulation and therefore simplicial complexes are an effective representation for maintainable geometric data describing partitions.

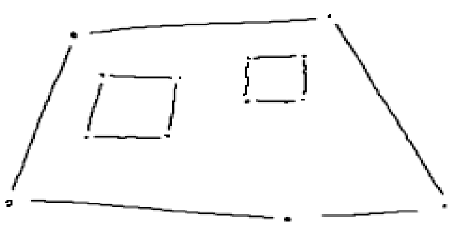

Figure 10: A parcel with "islands"

Simplicial complexes are of 2 dimensional space triangulation; they contain more objects than a partition represented as cells, but operations to maintain consistency in a triangulation are faster and simpler to program. The representation of a simplicial or cell complex requires the explicit representation of the boundary and converse co-boundary relation. The schema used initially (Figure 3) contains redundancy (which is used in Corbett's tests for consistency) and is therefore difficult to maintain. Popular today are schemes with half edges (Figure 11), where a half-edge points to the starting node and the corresponding other half edge or quad edges [6] (Figure 12), where each quad-edge points to the next quad-edge and either a boundary node or face; in a quad-edge structure, the boundary graph and its dual are maintained in a well-defined algebra with a single operation *splice*. For example, taking Figure 1 as a boundary graph (primal) the dual is Figure 13, which shows adjacency between faces.

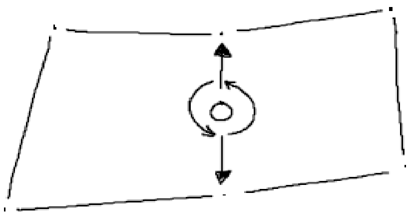

Figure 11: Two half edges, pointing to adjacent nodes

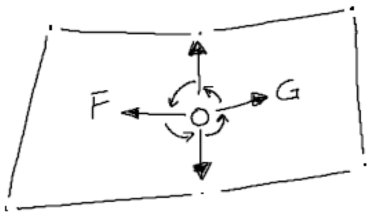

Figure 12: Four quad edges give one edge and point to adjacent nodes and faces

Quad edges represent efficiently without redundancy a much larger universe, namely partitions of orientable manifold. The Euler operations *glue* and *split* can be efficiently implemented and maintain a simplicial or cell complex. The geometry can be represented as generalized maps, for which efficient implementation using relational databases has been reported [9].

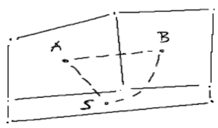

Figure 13: The dual graph of Figure 1 (dashed) shows the neighbor relations

Key Applications

Many applications include geometric descriptions of objects; Computer Aided Design for mechanical and civil engineering are important, but also Geographic Information systems, with many special applications like Utility Mapping for cities, Cadastral Maps to show ownership of land, but also car navigation systems, are popular examples.

Management of partitions is central for Geographic Information systems (GIS); 2D partitions are wisely used for land ownership parcels, soil types, etc. Increasingly 3D models of cities and buildings are built to produce visualizations for virtual trips. Town planning expect that changes in 3D models over time can be visualized, which requires 4 (3 spatial plus one temporal) dimensions.

Management of geometry of partitions of 3D space are important for CAD (Computer Aided Design), used for architecture, civil engineering but also mechanical engineering. Image processing intended to produce 3D representations of the environment is using hierarchically structured partitions and needs effective operations to subdivide these.

A generalizable approach to storing and maintaining geometry in a database integrates for many application areas the treatment of geometric data with other data. Approaches based on the theory of simplicial or cell complexes are now available as plug-ins to convert general purpose DBMS to spatial databases. They replace earlier systems where geometric data was managed in proprietary file structures and the connection between geometry and descriptive data established only in the application program.

Future Directions

Besides efforts to enhance performances of implementations three major research goals stand out:

1. efficient solutions for 3 dimensional data; required for example to build 3D city models and to construct operations for consistent updating these [12].
2. generalization to *n*-dimensions to include temporal data, especially 2 and 3 dimensional geometry and time required to include time related data, movement and, in general, processes in CAD and GIS applications [11].
3. hierarchical structures to have partitions at one level of resolution (e.g., countries of the world) and then allow subdivision (e.g., regions, departments, counties, towns) [13].

A fully general application independent, *n*-dimensional and hierarchical representation that supports Euler operations effectively within data stored in a database is the implied goal of research in the first decade of the 21st century.

Cross References

(Egenhofer) relations.

Recommended Reading

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