Operations Research Letters 37 (2009) 387–391

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Optimal control of terrorism and global reputation: A case study with novel threshold behavior

J.P. Caulkins [a], G. Feichtinger [b], D. Grass [b,*], G. Tragler [b]

[a] H. John Heinz III College and the Qatar Campus, Carnegie Mellon University, Qatar

[b] Institute for Mathematical Methods in Economics, Vienna University of Technology, Argentinierstrasse 8/105-4, A-1040 Vienna, Austria

ARTICLE INFO

Article history:
Received 16 January 2007
Accepted 27 July 2009
Available online 12 August 2009

Keywords:
Optimal control
Threshold curve
DNSS curve
Multiple equilibria
Economic terror model

ABSTRACT

A control model is presented which studies optimal spending for the fight against terrorism. Under the assumptions that economic damages are larger the greater the number of terrorists and that the success of counter terror operations depends on public opinion, it is demonstrated that a so-called DNSS threshold may exist, separating the basin of attraction of optimal paths.

1. Introduction

Refs. [1–3] and [4] discuss points of indifference where the decision maker in an optimal dynamic control problem is indifferent between choosing any of multiple optimal strategies. These so-called DNSS points (referring to Dechert, Nishimura, Sethi and Skiba) have now been studied in many models [5], and for higher dimensional models whole DNSS sets can occur. Nevertheless, few theoretical arguments exist concerning the existence of DNSS points [6], apart from some analytic arguments about the topology of DNSS sets/manifolds in higher dimensions (see, e.g., [7]). This paper likewise does not offer any unifying theory of DNSS sets, but it provides a numerical example that reveals some interesting properties of DNSS sets. Following the results of [7] we can speak of a DNSS curve, not only of a DNSS set.

2. Model

A growing number of papers seek to guide counter-terror policy by modeling explicitly the time evolution of a state variable representing the number of terrorists. There is also a conventional wisdom that winning the "hearts and minds" of non-combatants is an important strategy in non-conventional conflict. We merge both themes by introducing a two-state optimal control model that explicitly includes a state variable representing the level of public sympathy for the counter-terror forces.

The size of the terrorist organization, denoted by x, increases as new terrorists are recruited and diminishes because of both counter-terror operations and "natural outflow". [8] and [9] model recruitment as a constant, but others (e.g., [10]) assume that the more terrorists there are, the more new terrorists the organization can recruit because recruiting occurs through personal interaction, as in diffusion models. We include both a constant recruitment term, τ, and, following many drug and crime models (e.g., [11]), a term that is a power function in x.

A "natural" outflow is presumed to occur at a constant per capita rate μ. This reflects terrorists leaving the organization, dying in suicide attacks, or falling victim to on-going law enforcement efforts that are in addition to those the decision maker is actively managing in this model.

The second outflow is proportional to the level of counter-terror operations, u. It is presumed to be increasing and concave in x because the more terrorists there are, the more that can be eliminated by a given investment in counter-terror. The effect is concave, not linear, because counter-terror is typically driven by intelligence, not just random search.

The key innovation in this model is to presume that this outflow is also increasing (and concave) in the level of public sympathy for those operations, y. The reason for this is that public support encourages the civilian population, within which the terrorists are embedded, to provide information or otherwise assist the counter terror forces, or at least to refrain from actively helping the terrorists. The dependence on both x and y is modeled by a power function, as in Cobb–Douglas production functions, so the first state dynamic equation can be written as (omitting explicit denotation of the dependent variable t for time here and throughout):

$$\dot{x} = \tau + kx^{\alpha} - \mu x - \eta u x^{\beta} y^{\gamma}. \tag{1}$$

* Corresponding author.
E-mail address: dieter.grass@tuwien.ac.at (D. Grass).

0167-6377/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.orl.2009.07.003

Table 1
The parameter values for the base case.

r	γ	α	b	β	c	δ	η	k	κ	μ	ρ	τ
0.01	1	0.8	1	0.5	1	0.25	0.085	0.05	0.05	0.05	1	10^{-5}

In the absence of terror attacks and counter-terror operations, public opinion (y) is presumed to adjust to an underlying constant level of sympathy, $b = 1$. That level could be higher or lower depending on a range of factors, but those considerations are all exogenous to this model and so are not modeled as varying over time.

It is presumed that when the terrorists are numerous (large x), they commit terrorist acts that build sympathy for the decision maker. Hence, public sympathy increases proportionally to x.

Counter-terror operations are assumed to erode public sympathy because they generate collateral damage to innocent parties, ranging from mere inconvenience (searching all cars at a check point) up to injury and death from errant bombs. Immediately after September 11th, there was considerable sympathy for the US around the world, including among many moderate Muslims. However, such sympathy can be undermined when warplanes accidentally bomb a wedding party (as in Afghanistan) or inhumane interrogation methods are employed.

We presume that this erosion is convex in the intensity of counter-terror operations. That is, some level of counter-terror activity is seen as more or less acceptable, but aggressive counter-terror actions can erode public sympathy disproportionately. In the absence of specific evidence, we opt for simplicity and make the outflow proportional to the square of the intensity of counter-terror operations.

Hence, the state equation for public sympathy y can be written:

$$\dot{y} = \delta x - \rho u^2 + \kappa\,(b - y)\,. \tag{2}$$

Our focus is on these state dynamics, so we keep the objective function very simple. The decision maker seeks to minimize a discounted (at rate r) sum of the number of terrorists plus the control costs, where the latter are squared for the usual diminishing returns arguments. Hence, the problem is formally written as

$$\begin{aligned} &\min_{u(\cdot)\geq 0} \int_0^\infty \mathrm{e}^{-rt}\left(cx + \frac{u^2}{2}\right)\mathrm{d}t \\ \text{s.t.}\quad &\dot{x} = \tau + kx^\alpha - \mu x - \eta u x^\beta y^\gamma \qquad\qquad \text{(TPS)} \\ &\dot{y} = \delta x - \rho u^2 + \kappa\,(b-y) \\ \text{and}\quad &x(0) = x_0 > 0, \quad y(0) = y_0 > 0. \end{aligned}$$

3. Analytical arguments

To apply Pontryagin's minimum principle (see, e.g., [12,13]) we consider the current value Hamiltonian

$$\begin{aligned} \mathcal{H} &= \lambda_0\left(cx + \frac{u^2}{2}\right) + \lambda\left(\tau + kx^\alpha - \mu x - \eta u x^\beta y^\gamma\right) \\ &\quad + \nu\left(\delta x - \rho u^2 + \kappa\,(b-y)\right), \end{aligned} \tag{3}$$

where $\lambda_0 \geq$ is a constant and λ and ν denote the co-state variables in current value terms. In a more detailed analysis it can be proved that λ_0 is positive and can therefore be assumed w.l.o.g. to be $\lambda_0 = 1$.

Following the standard methods, we derive the necessary optimality condition

$$u^* = \arg\min_{u\geq 0} \mathcal{H}. \tag{4}$$

Setting the derivative of $\mathcal{H}$ with respect to u equal to zero

$$\mathcal{H}_u = u - \lambda\eta x^\beta y^\gamma - 2\nu\rho u = 0,$$

implies

$$u = \frac{\lambda\eta x^\beta y^\gamma}{1 - 2\rho\nu}. \tag{5}$$

For $\nu < \frac{1}{2\rho}$ the second derivative

$$\mathcal{H}_{uu} = 1 - 2\rho\nu \tag{6}$$

is positive, so if $u^* > 0$ it has to satisfy (4). Using the results of [14] it can be shown that $\lambda(t) > 0$ and $\nu(t) < 0$ for all t, so clearly $\nu(t) < \frac{1}{2\rho}$, implying the strict convexity of the Hamiltonian along an optimal path.

Since the co-states can be interpreted as the shadow prices for the corresponding states, this makes sense. An increase in y means more support for anti-terror actions. That benefit can be written as a negative cost, and hence has a negative shadow-price.

The positivity of the states, which will be proved below, implies, together with (5) and (6), that $u^*(t) > 0$ for all t.

To prove the positivity of the states along an optimal solution, we have to consider the co-state equations

$$\dot{\lambda} = \lambda\left(r - k\alpha x^{\alpha-1} + \mu + \frac{\eta^2\lambda x^{2\beta-1} y^{2\gamma}\beta}{(1-2\rho\nu)}\right) - \nu\delta - c \tag{7}$$

$$\dot{\nu} = \nu(r+\kappa) + \frac{\lambda^2\eta^2 x^{2\beta} y^{2\gamma-1}\gamma}{(1-2\rho\nu)}. \tag{8}$$

We can now verify the positivity of the state variables x and y. First consider (1). We get $\dot{x} = \tau > 0$ for $x = 0$ and hence $x(t) > 0$ if $x(0) > 0$. Positivity of y follows because if $y = 0$ (2) reduces to $\dot{y} = \delta x - \rho u^2 + \kappa b$. From (5) we see that $u = 0$ for $y = 0$ implying $\dot{y} > 0$, so $y(t) > 0$ if $y(0) > 0$.

We do not have empirical estimates for the parameters, so we simply select values that are round numbers and seem to have the right order of magnitude. See Table 1. Absence of a more precise parameterization is not a severe problem because the insights we draw pertain to the qualitative structure of the solutions and are of the existence variety. We claim only that certain things can happen, not that they do necessarily happen in practice.

4. Numerical analysis

To numerically compute the solution paths, we use a boundary value problem (BVP) approach, together with a continuation technique described in [15]. Thus we truncate the infinite time interval to a finite time horizon $[0, T]$, where the value of the truncation time T depends on the parameter values and the eigenvalues of the Jacobian. Assuming that the long run optimal solution is a steady state, one can state asymptotic transversality conditions (see, e.g., [7,16,17]) ensuring that the solution ends up at the linearized stable manifold of the limit set.

If the stable manifold is two-dimensional, the asymptotic transversality conditions, together with the initial conditions, yield a well-posed BVP. A trivial solution for this BVP is the steady state solution itself. Starting with this trivial solution, a numerical continuation technique can be applied (see, e.g., [18,19]) allowing one to follow the (optimal) solution for arbitrary initial states $(x(0), y(0)) \in \mathbb{R}^+ \times \mathbb{R}^+$. (Analogous considerations hold in case of a limit cycle solution.)

To compute and continue a DNSS point (curve), with the two optimal solution paths $(x_1(\cdot), y_1(\cdot), u_1^*(\cdot))$ and $(x_2(\cdot), y_2(\cdot), u_2^*(\cdot))$ approaching two distinct steady states $\hat{x}_i,\ i = 1, 2$, the BVP is given by:

Fig. 1. Some optimal paths approaching the steady states together with the DNSS curve Γ and the one dimensional stable manifold m_s are depicted in the projection into the state space for (a) small and (b) large x. The "weak" part of the DNSS curve coincides with the stable manifold m_s for all $0 \le x \le x_T$. For $x > x_T$ the curve Γ denotes a typical DNSS curve. The connecting point (x_T, y_T) is a "weak" DNSS point, which indicates the boundary between the "weak" and the "strong" part of the DNSS curve. For one specific DNSS point (x_D, y_D) the two corresponding optimal paths $p_1(\cdot)$ and $p_2(\cdot)$, satisfying $p_1(0) = p_2(0) = (x_D, y_D)$, are depicted.

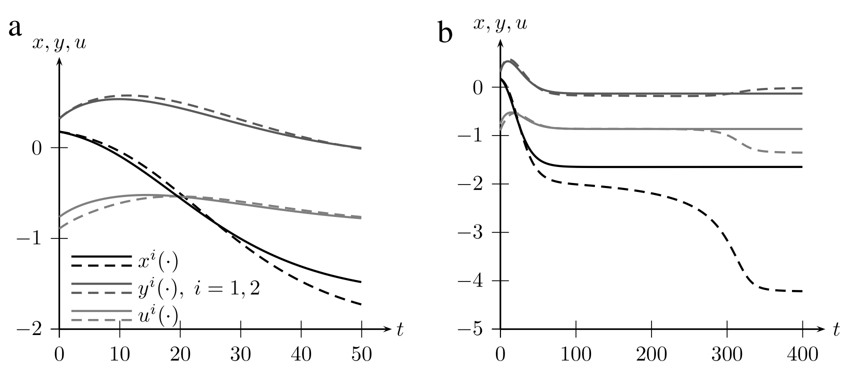

Fig. 2. Two solution paths $(x^i(\cdot),\ y^i(\cdot),\ u^i(\cdot)),\ i = 1, 2$ are depicted in logarithmic scale on the time intervals (a) $0 \le t \le 50$ and (b) $0 \le t \le 400$. The paths correspond to the two solutions, discriminated by solid and dashed lines, starting at the DNSS point $(x_D, y_D) = (1.5, 2.085718)$. (See Section 4).

(i) The asymptotic transversality conditions for both steady states.

(ii) Coincidence of the initial states at time zero, i.e., $x_1(0) = x_2(0),\ y_1(0) = y_2(0)$.

(iii) The same objective value for both paths is represented by the Hamiltonian at the initial point, i.e.,

$$\begin{aligned}&\mathcal{H}(x_1(0), y_1(0), u_1^*(0), \lambda_1(0), \nu_1(0))\\ &\quad = \mathcal{H}(x_2(0), y_2(0), u_2^*(0), \lambda_2(0), \nu_2(0)).\end{aligned}$$

(iv) A phase condition identifying a unique solution in the case of a DNSS curve.

As an example, consider the computation of a DNSS point (x_D, y_D) with fixed $x_D = 1.5$ and y_D to be determined. One corresponding path of the canonical system $X_l(\cdot) = (x_l(\cdot), y_l(\cdot), \lambda_l(\cdot), \nu_l(\cdot))$ converges to the low steady state

$$\begin{aligned}\hat{X}_l &:= (\hat{x}_l, \hat{y}_l, \hat{\lambda}_l, \hat{\nu}_l)'\\ &= (0.000060, 0.961256, 74.731485, -0.036306)',\end{aligned}$$

and the other path $X_h(\cdot) = (x_h(\cdot), y_h(\cdot), \lambda_h(\cdot), \nu_h(\cdot))$ converges to the high steady state

$$\begin{aligned}\hat{X}_h &:= (\hat{x}_h, \hat{y}_h, \hat{\lambda}_h, \hat{\nu}_h)'\\ &= (0.022613, 0.737899, 95.1429846, -2.776039)',\end{aligned}$$

that is depicted in Figs. 1(b) and 2. Next the eigenvalues and eigenvectors of the Jacobian matrices $J_l = J(\hat{X}_l)$ and $J_h = J(\hat{X}_h)$ are computed. From the eigenvectors of J_l and J_h two matrices

$$F_l = \begin{pmatrix} 0.999952 & -0.000729 & -0.0000018 & 0.009699\\ 0.999999 & 0.000645 & 0.000005 & 0.001067\end{pmatrix}$$

and

$$F_h = \begin{pmatrix} -0.999936 & 0.011195 & -0.000269 & -0.001116\\ 0.999868 & 0.011371 & 0.000410 & 0.011516\end{pmatrix}$$

can be derived, which determine the (linearized) stable manifolds near the steady states. These matrices yield the (linear) asymptotic boundary conditions, which are stated as

$$\begin{aligned}&F_l(X_l(T) - \hat{X}_l) = (0, 0)'\\ &F_h(X_h(T) - \hat{X}_h) = (0, 0)',\end{aligned}$$

where T is the truncation time for the infinite time horizon which, for this specific case, can be set to 1000. Adding the condition at the initial states

$$\begin{aligned}&x_l(0) = x_h(0) = 1.5\\ &y_l(0) = y_h(0)\\ &\mathcal{H}(x_l(0), y_l(0), u_l^*(0), \lambda_l(0), \nu_l(0))\\ &\quad = \mathcal{H}(x_h(0), y_h(0), u_h^*(0), \lambda_h(0), \nu_h(0))\end{aligned}$$

completes the formulation of the BVP. Solving this BVP for approximate solutions, which can be derived from a previous continuation process, finally yields the initial pairs of states and costates

$$\begin{aligned}&(x_l(0), y_l(0), \lambda_l(0), \nu_l(0))\\ &\quad = (1.5, 2.085717, 4.968004, -3.717041)\\ &(x_h(0), y_h(0), \lambda_h(0), \nu_h(0))\\ &\quad = (1.5, 2.085717, 6.397329, -3.533227),\end{aligned}$$

and therefore $y_D = y_l(0) = y_h(0) = 2.085717$.

5. Results

Optimal solutions were computed numerically with MATLAB®, using the `OCMat` toolbox (cf., [15]), which can be downloaded for free at http://www.eos.tuwien.ac.at/OR/OCMat/index.html.

These numerical results, together with the analytical insights presented in [7], give a numerical proof for the existence of a DNSS curve. The `MATCONT` toolbox (cf., [20]) was used to perform bifurcation analysis.

Optimal strategy for small x

Fig. 1(a) summarizes the results for small x. There exist one high steady state at $(x_h, y_h) \approx (0.023, 0.738)$ and one low steady state $(x_l, y_l) \approx (0.0001, 0.961)$, both exhibiting a two dimensional stable manifold. At the low steady state terrorism has been all but eradicated, so only very modest efforts ($u = 0.044$) are needed to hold it in check. As a result, sympathy has returned to very nearly the constant level (of unity) that would pertain in the absence of terror related influences. At the high level steady state, recruitment is fifty times larger, so nontrivial levels of counter-terror efforts are needed to prevent the terrorist organization from growing. That, in turn, reduces the steady state level of sympathy for the counter-terror forces by about one quarter.

As one might expect, these two locally optimal steady states are separated by a DNSS curve. In particular, there is an unstable steady state $(x_u, y_u) \approx (0.016, 0.706)$ exhibiting a one dimensional stable manifold m_s. A numerically computed DNSS curve Γ divides the state space into two regions, where the DNSS curve is composed of parts of the one dimensional stable path of (x_u, y_u) and the set of DNSS points (see Fig. 1(b)). Both parts are connected by a "weak" DNSS point (x_T, y_T). Starting exactly on the DNSS curve left from x_T one is on the "weak" part of the DNSS curve, and one should remain on that curve and move towards the node (x_u, y_u). In contrast, when starting to the right of x_T, the decision maker has two optimal strategies, one leading to the lower equilibrium while the other leads to the higher equilibrium.

For any initial state to the lower right of the DNSS curve Γ in Fig. 1(a), the optimal trajectory approaches the higher steady state (x_h, y_h). For any initial state to the upper left, the optimal trajectory approaches the lower steady state (x_l, y_l).

The shape and direction of the DNSS curve Γ make intuitive sense for small x. The greater the initial level of sympathy, the more likely it is to be optimal to drive the number of terrorists down to the low steady state. With small x sympathy acts like a capital stock that is drawn down in order to drive the number of terrorists down to very low levels. Once the number of terrorists has been reduced, only modest levels of control are employed, and sympathy rebounds as the lower steady state is approached.

Likewise, for any given initial level of sympathy, $y(0)$, if the initial number of terrorists is smaller than (to the left of) the DNSS curve (see Fig. 1(a), (b)), then it is optimal to drive the number of terrorists down to the lower steady state. But if the initial number of terrorists is "too large", then the best one can do is to moderate growth in the number of terrorists toward the larger steady state. Such "eradicate or accommodate" choices have appeared before in optimal dynamic control models of "bad stocks" whose inflow is increasing in the level of the stock (cf., [11]).

That tidy interpretation does not, however, hold up if one moves further away from the origin. Fig. 1(b) parallels Fig. 1(a), but for x values up to almost 2. It shows that the DNSS curve that was upward sloping reaches a maximum, then bends back down all the way to the horizontal axis.

That the DNSS curve Γ might reach a maximum is not surprising. That says that if sympathy is large enough, it is not only possible but also optimal to use counter-terror operations to drive the number of terrorists down to the low steady state, no matter how many terrorists there are initially.

That the DNSS curve bends back down to the axis is more surprising. That says that, for initial numbers of terrorists around $x(0) = 1.5$ and initial sympathy levels that are not too high ($y < 3$ or so), the policy prescription is reversed. If the initial number of terrorists is very large (to the right of the DNSS curve), it is optimal to drive the number of terrorists down to very low levels. But if there are not so many terrorists initially, then the optimal strategy is to let them persist in the long run in greater numbers.

Clearly, if one could instantly move to either steady state at no cost, one would prefer the lower steady state. So if it is optimal to approach the higher steady state, the trajectory to that steady state must be better in the short run. That is the case here. When $x(0)$ is to the left of the right-hand side of the DNSS curve, the best way to reach the higher steady state is to quickly drive down the number of terrorists. When the initial number of terrorists is larger, specifically to the right of the DNSS curve, the number of terrorists will fall relatively quickly of its own accord because they greatly exceed the uncontrolled steady state number of $x = 1$. The optimal policy in that case is not to deploy counter-terror operations too aggressively at first, but instead to build up the stock of public sympathy. Once that stock has grown, then it becomes optimal to deploy counter-terror operations aggressively. Furthermore, given the accumulation of public sympathy, those counter-terror operations can and should be used to drive the number of terrorists down to the lower steady state. Fig. 2 makes these strategies and dependencies clear by plotting x, y and u versus time t.

With the base case parameters, larger initial levels of public sympathy $y(0)$ always increase the range of initial numbers of terrorists for which it is optimal to essentially eradicate terrorism. Thus, the DNSS curve bends back down, but it does not turn back on itself. Is that result robust? It turns out that the answer is no. Modest changes in parameter values can lead to a backward bending right hand side of the DNSS curve. For example (figure available from authors), this occurs if κ is reduced from 0.05 to 0.02.

6. Conclusions

The analysis yielded interesting results, both mathematically and substantially. From an optimal control perspective, we showed that the long-run optimal outcome can depend on the initial conditions in ways that are not monotonic with respect to the initial value of either state. That is, we found a so-called DNSS curve separating different regions in state space, for which it is optimal to drive the system to steady states with either a lower or a higher number of terrorists. There are places in the state space where a slight increase in the initial number of terrorists can tip the optimal strategy, from approaching the lower-level to approaching the higher-level of terrorists. But there are other places where the same slight increase can tip one in the other direction. The same odd trait can hold for the initial level of public sympathy. There can be places where a bit more initial sympathy can tip one from approaching the high to approaching the lower level equilibrium, and other places where the same shift tips one in the opposite direction.

It has long been recognized that one state models can exhibit so called "weak" DNSS points, where one has to stay put if starting exactly at this point, but slight deviations from this initial position can place one on a trajectory leading to different long run steady states. In our model, we found an analogous one dimensional "weak" DNSS curve, where, if one starts exactly on this curve, the optimal path leads to an intermediate steady state, whereas a slight deviation leads to a small or large steady state. Indeed, we examine an intermediate case where one part of the curve contains "weak" DNSS points whereas the other part contains "strong" DNSS points.

From an applications perspective, we do not for a minute think that this model, as currently formulated, provides much practical guidance for fighting terrorists. It is far too stylized. However, even these preliminary results make one thing perfectly clear. If there is any merit to the conventional wisdom that public sympathy can catalyze the effectiveness of counter-terror operations, then it may be important for counter-terror models to incorporate that fact explicitly. Even in this most elementary model, doing so can have dramatic implications for the results and policy prescriptions.

A principal limitation of the present analysis is the lack of validation of functional forms relating to the public sympathy variable, y, with respect to both how terrorists' actions generate sympathy for counter-terror forces and how zealous counter-terrorism efforts might erode that sympathy. Hence, a fruitful next step would be empirical analysis of longitudinal data on terror attacks, counter-terror operations, and public opinion. There are databases on terror attacks (e.g., the RAND Worldwide Terrorism Incident Database). Content analysis of media coverage might be a sufficient proxy for public opinion. (e.g., all the major parties in Iraq have radio stations and publish newspapers). Data on the tempo of counter-terror operations is often classified, but might be made available to scholars working closely with the military of the nation(s) involved.

Acknowledgments

The work for this paper was financed, in part, by the Austrian National Bank (OeNB) under grant No. 12138 ("Optimal Design of Counter-Terror Operations"), and by Qatar Foundation. The helpful and constructive comments of two referees are highly appreciated.

References

[1] S.P. Sethi, Nearest feasible paths in optimal control problems: Theory, examples, and counterexamples, Journal of Optimization Theory and Applications 23 (4) (1977) 563–579.
[2] S.P. Sethi, Optimal advertising policy with the contagion model, Journal of Optimization Theory and Applications 29 (4) (1979) 615–627.
[3] A.K. Skiba, Optimal growth with a convex-concave production function, Econometrica 46 (3) (1978) 527–539.
[4] W.D. Dechert, K. Nishimura, A complete characterization of optimal growth paths in an aggregated model with a non-concave production function, Journal of Economic Theory 31 (2) (1983) 332–354.
[5] C. Deissenberg, G. Feichtinger, W. Semmler, F. Wirl, Multiple equilibria, history dependence, and global dynamics in intertemporal optimization models, in: W.A. Barnett, C. Deissenberg, G. Feichtinger (Eds.), Economic Complexity: Non-linear Dynamics, Multi-agents Economies and Learning, Elsevier Science, Amsterdam, 2004, pp. 91–122.
[6] F.O.O. Wagener, Skiba points and heteroclinic bifurcations, with applications to the shallow lake system, Journal of Economic Dynamics and Control 27 (9) (2003) 1533–1561.
[7] W.J. Beyn, T. Pampel, W. Semmler, Dynamic optimization and Skiba sets in economic examples, Optimal Control Applications and Methods 22 (5–6) (2001) 251–280.
[8] N.O. Keohane, R.J. Zeckhauser, The ecology of terror defense, Journal of Risk and Uncertainty 26 (2–3) (2003) 201–229. Available at http://ideas.repec.org/a/kap/jrisku/v26y2003i2-3p201-29.html.
[9] E.H. Kaplan, A. Mintz, S. Mishal, C. Samban, What happened to suicide bombings in Israel? Insights from a terror stock model, Studies in Conflict and Terrorism 28 (3) (2005) 225–235.
[10] C. Castillo-Chavez, B. Song, Models of the transmission dynamics of fanatic behaviors, in: H.T. Banks, C. Castillo-Chavez (Eds.), Bioterrorism: Mathematical Modeling Applications in Homeland Security, in: Frontiers in Applied Mathematics, vol. 28, Society for Industrial and Applied Mathematics, Philadelphia, 2003, pp. 155–172.
[11] G. Tragler, J.P. Caulkins, G. Feichtinger, Optimal dynamic allocation of treatment and enforcement in illicit drug control, Operations Research 49 (3) (2001) 352–362.
[12] G. Feichtinger, R.F. Hartl, Optimale Kontrolle ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften, Walter de Gruyter, Berlin, 1986.
[13] S.M. Aseev, A.V. Kryazhimskiy, The pontryagin maximum principle and transversality conditions for a class of optimal control problems with infinite time horizons, SIAM Journal on Control and Optimization 43 (3) (2005) 1094–1119.
[14] D. Léonard, The signs of the costate variables and the sufficiency conditions in a class of optimal control problems, Economics Letters 8 (4) (1981) 321–325.
[15] D. Grass, J.P. Caulkins, G. Feichtinger, G. Tragler, D.A. Behrens, Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption, and Terror, Springer Verlag, Berlin, 2008.
[16] T. Pampel, Numerical approximation of generalized connecting orbits, Ph.D. Thesis, University of Bielefeld, 2000.
[17] A. Steindl, Numerische Behandlung von Randwertproblemen auf unendlichen Intervallen, Ph.D. Thesis, Technische Universität Wien, 1984.
[18] B. Krauskopf, H.M. Osinga, J. Galán-Vioque (Eds.), Numerical Continuation Methods for Dynamical Systems: Path Following and Boundary Value Problems, Springer, Dordrecht, 2007.
[19] E.L. Allgower, K. Georg, Introduction to Numerical Continuation Methods, SIAM Classics in Applied Mathematics, Philadelphia, PA, 2003.
[20] A. Dhooge, W. Govaerts, Yu.A. Kuznetsov, MATCONT: A Matlab package for numerical bifurcation analysis of ODEs, ACM Transactions on Mathematical Software (TOMS) 29 (2) (2003) 141–164.