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Functional Minimization Method Addressed to the Vacuum Finding for an Arbitrary Driven Quantum Oscillator

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The old problem exists for a driven (time-dependent) quantum oscillator: to differ a true vacuum state from a squeezed one. We suggest finding of a true vacuum state by minimization of the functional containing the difference of the potential and kinetic energies of oscillator. Analytical and numerical examples confirming this offer are considered.

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1. Introduction

A time-dependent (driven) oscillator arises naturally in a number of fields of the theoretical physics [1, 2]. In particular, it has an application in cosmology and astrophysics, where the scalar, fermion, gravitational, and other quantum fields evolve in an expanding Universe [2, 3]. Particle creation by the nonstationary gravitational field is long considered as one of the possible sources of the matter origin in the Universe, and, to talk about a "particle" one has to understand what is the vacuum. It should also be mentioned that according to the modern view the vacuum fluctuations were the seeds for the structure formation in Universe [4].

Nevertheless, the definition of the ground (vacuum) state remains to be obscure [2, 5–7]. This forces one to use the well-known adiabatic states in concrete calculations [8], whereas for systems which admit analytical consideration, for instance, quantum field in the De Sitter Universe, Bunch-Davis vacuum states [9] can be built. However it would be desirable to define vacuum state without appealing to the adiabatic series or analytical solution (it may be impossible). This issue is addressed in our paper. The suggested method allows finding numerically the true vacuum state (if it exists).

Let us remind the problem in more detail.

Hamiltonian of the time-dependent oscillator has the following form:

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2(t)x^2. \quad (1)$$

Its quantization in the Heisenberg picture consists in a replacement of the coordinate x by the time-dependent operator:

$$\hat{x}(t) = \hat{\mathrm{a}}\, u(t) + \hat{\mathrm{a}}^+ u^*(t), \quad (2)$$

where the operators $\hat{\mathrm{a}}$ and $\hat{\mathrm{a}}^+$ obey the commutator relation

$$[\hat{\mathrm{a}}, \hat{\mathrm{a}}^+] = 1, \quad (3)$$

whereas the function u satisfies the equation

$$\dot{u}^* u - \dot{u}\, u^* = \mathrm{i}. \quad (4)$$

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These rules provide the standard commutation relation for the momentum and coordinate operators

$$[\hat{p}, \hat{x}] = [\hat{\dot{x}}, \hat{x}] = -\mathrm{i}. \quad (5)$$

The vacuum state is defined as the state, which is the null-space of the annihilation operator: $\hat{a}|0>= 0$. However, there remains a problem in the concrete definition of u. This function satisfies the oscillator equation of motion

$$\ddot{u} + \omega^2 u = 0, \quad (6)$$

and one has to define (at some instant) the initial condition corresponding to the true vacuum state. It should be noted that there exists a family [2, 6] of the functions u, which satisfy Eq. (4) and are interrelated by the Bogolubov's transformation:

$$u(t) = \cosh r\, u_0(t) + \sinh r\, \mathrm{e}^{\mathrm{i}\delta} u_0^*(t)$$
$$u^*(t) = \cosh r\, u_0^*(t) + \sinh r\, \mathrm{e}^{-\mathrm{i}\delta} u_0(t). \quad (7)$$

In the terms of an oscillator with constant frequency $\omega = const$, this means that it is necessary to differ the true vacuum from the squeezed vacuum states. When $\omega = const$ the vacuum choice can be made by minimization of the mean value of $< 0|H|0 >$, but it is no so for time-dependent oscillator.

However let us remind that when $\omega = conat$ the mean value of an observable oscillates with time for a squeezed vacuum state, whereas it is zero or constant for a true vacuum state. One can suppose, that this is the clue to issue of the true vacuum state of time-dependent oscillator. Namely, an observable value oscillates with time for a squeezed vacuum state, but it is monotonic function for a true vacuum state. It is convenient to choose the difference of the oscillator kinetic and potential energies to be this observable. For the vacuum state of oscillator with constant frequency, this quantity equals to zero according to the virial theorem and we will see that this quantity is a monotonic function of time for a vacuum state of time-dependent oscillator (if such a state exists).

2. Time-dependent oscillator: examples of the vacuum states

We intend to concentrate the different examples, which grade with the asymptotic of the adiabatic parameter $\dot{\omega}/\omega^2$: i) $\omega \to const$, $\dot{\omega}/\omega^2 \to 0$; ii) ω does not tends to a constant but $\dot{\omega}/\omega^2 \to 0$; iii) then $\dot{\omega}/\omega^2 = const$, and, at last, iv) $\dot{\omega}/\omega^2 \to \infty$.

Let ω depends on t in the well-known form [2]:

$$\omega(t) = k\sqrt{1 + \tanh(H\,t)}. \quad (8)$$

The solution of Eqs. (4) and (6) is [2]:

$$u_0 = 2^{-3/4} k^{-1/2} \mathrm{e}^{-\frac{ikt}{\sqrt{2}}} \left(\mathrm{e}^{-Ht} + \mathrm{e}^{Ht}\right)^{-\frac{ik}{\sqrt{2}H}}_2 F_1\left(\frac{\mathrm{i}k}{\sqrt{2}H}, \frac{\mathrm{i}k}{\sqrt{2}H} + 1; \frac{\mathrm{i}\sqrt{2}k}{H} + 1; \frac{1}{1 + \mathrm{e}^{2Ht}}\right), \quad (9)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [10]. An arbitrary solution from a family of the squeezed vacuum states is given by Eqs. (7).

Mean value of the kinetic and potential energies difference is expressed as

$$< 0|\frac{1}{2}p^2 - \frac{1}{2}\omega x^2|0 >= \frac{1}{2}(\dot{u}\dot{u}^* - \omega^2 u u^*) = \dot{\sigma}(t). \quad (10)$$

Here

$$\sigma = \frac{1}{2} < 0|\hat{x}\hat{p} + \hat{p}\hat{x}|0 >= \frac{1}{2}(\dot{u}u^* + \dot{u}^* u) \quad (11)$$

has a sense of additional uncertainty arising in the Heisenberg uncertainty relation [11]:

$$<|(\hat{p}-p_0)^2|><|(\hat{x}-x_0)^2|>>1/4+\sigma^2, \quad (12)$$

where $p_0 =<\hat{p}>$, $x_0 =<\hat{x}>$ and $|>$ is the arbitrary state. For a family of the squeezed vacuum states, including the true vacuum, the inequality (12) becomes an equality.

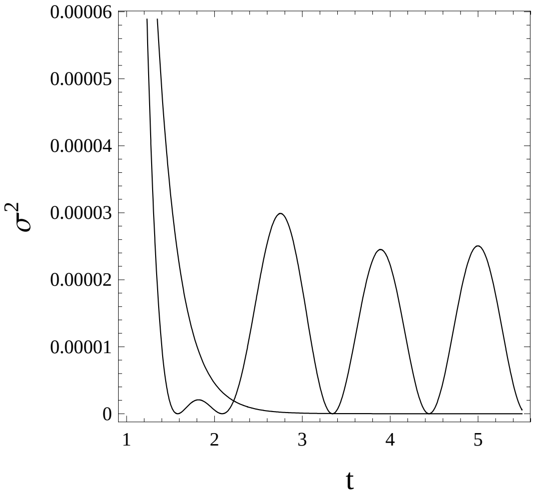

FIG. 1. The function σ^2 for the dependence of $\omega(t)$ given by (8) and $k = 1$, $H = 1$. Monotonic curve corresponds to the vacuum state ($r = 0$, $\delta = 0$) and oscillating curve corresponds to the squeezed state ($r = 0.005$, $\delta = 0$).

Fig. 1 shows the $\sigma(t)$-function for different values of the parameters r, δ. One can see that this function oscillates at some value of the parameters r, δ and the only parameter $r = 0$ results in the monotonic behavior of $\sigma(t)$.

It should be noted that the selection rule have been offered [6] for a vacuum state as a state having the minimal uncertainty at each moment of time. As one can see from the above example, this rule is not satisfied for the vacuum state. Indeed, there exists a region in Fig. 1, where the uncertainty for the slightly squeezed state is less than that for the vacuum state. Hence, one has to conclude that this selection rule is not valid in the general case.

Our suggestion is to correlate a vacuum state with the monotonic time-dependence of the functions σ or $\dot{\sigma}$. That is, for this example, the true vacuum corresponds to the function u_0 given by (9). As the criterium for choosing the function with monotonic behavior, one can use the minimization of the functional

$$Z(r,\delta)=\lim_{T\to\infty}\left(\frac{\int_{t_0}^{T}(\partial_t\sigma(t,r,\delta))^2\mathrm{d}t}{\int_{t_0}^{T}(\partial_t\sigma(t,r_0,\delta_0))^2\mathrm{d}t}\right), \quad (13)$$

where r_0, δ_0 are some fixed values used for normalization. The exact analytic calculation of the functional with the function u from (7), (9) and values $r_0 = \ln 2$, $\delta_0 = 0$ gives

$$Z=\frac{64}{225}\sinh^2 r. \quad (14)$$

Thus, the minimization of the functional leads to the value $r = 0$ for the vacuum state. This is because the function σ has the asymptotic

$$\sigma(t)\approx-\frac{1}{2}\sin\left(2\sqrt{2}k\,t+\delta\right)\sinh(2r) \quad (15)$$

at infinity.

Instead of the parameters r and δ, one can seek the initial conditions for Eq. (6) at some t_0. Really, the represention $u(t) = \mathrm{e}^{\mathrm{i}\varphi(t)}\theta(t)$ leads to $\dot{\varphi} = -\theta^{-2}/2$ from Eq. (4). That is, $\theta(t_0)$ and $\dot{\theta}(t_0)$ define $u(t_0)$ and $\dot{u}(t_0)$ completely because the phase $\varphi(t_0)$ can be chosen to be zero. Then one can solve Eq. (6) with some initial conditions and find the value of the functional (13). Initial conditions giving the minimum of the functional correspond to the vacuum state.

Moreover, one can write the differential equation directly for $\sigma(t)$ [12].

Straightforward computation shows that σ satisfies the equation

$$\dddot{\sigma}-\ddot{\sigma}\left(\frac{\dot{\omega}}{\omega}+\frac{\ddot{\omega}}{\dot{\omega}}\right)+4\dot{\sigma}\omega^2+\sigma\left(8\omega\dot{\omega}-\frac{4\omega^2\ddot{\omega}}{\dot{\omega}}\right)=0, \quad (16)$$

if $u(t)$ obeys (6).

The relation (4) leads to

$$\frac{1}{\omega\dot{\omega}^2}\left(4\sigma\omega^2+\ddot{\sigma}\right)\left(4\sigma\,\omega^3+\ddot{\sigma}\,\omega-2\dot{\sigma}\dot{\omega}\right)-4\sigma^2=1, \quad (17)$$

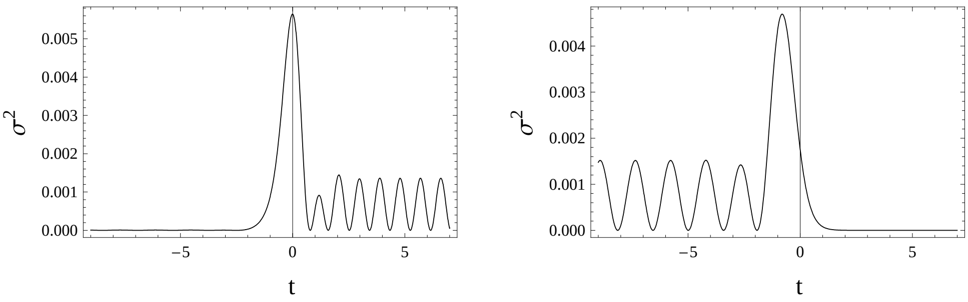

FIG. 2. The function σ^2 for in- and out-vacuum states. The dependence $\omega(t)$ is given by (20) and $k = 1$, $H = 1$.

for the states belonging to a family of the squeezed vacuum states including the true vacuum. Left hand side of Eq. (17) is the integral of motion of (16).

One can connect the initial condition for Eq. (16) with that for Eq. (6):

$$\sigma = \theta\,\dot{\theta}, \tag{18}$$

$$\dot{\sigma} = \dot{\theta}^2 + \theta^2(\dot{\varphi}^2 - \omega^2). \tag{19}$$

Second derivative of σ can be expressed through Eq. (17). Thus, the determination of $\sigma(t_0)$ and $\dot{\sigma}(t_0)$ allows solving Eq. (16) instead of defining $\theta(t_0)$ and $\dot{\theta}(t_0)$ and solving Eq. (6).

In the above example, the function σ has the monotonic behavior in the vacuum state within all range of t. This means that the single global vacuum exists. The more complicated case [2] is

$$\omega(t) = k\sqrt{2 + \tanh(H\,t)}, \tag{20}$$

where there are two different non zero values of ω at $t \to \infty$ and $t \to -\infty$.

One can see from Fig. 2, that two vacuum solutions exist. One of them has the monotonic behavior at $t \to +\infty$ (out-vacuum state) and the second one has such a behavior at $t \to -\infty$ (in-vacuum state). In this paper we do not discuss an important issue concerning the dependencies $\omega(t)$ providing the unique global vacuum [a], but if in- or out- vacuums exist, the out-vacuum can be found by the minimization of the functional (13), whereas the in-vacuum state can be found by setting $T \to -\infty$ in (13).

The considered cases are simple in the sense that ω tends to a constant and a notion of particle is asymptotically defined. For example, the out-vacuum means an absence of the particles at $t \to \infty$ and, simultaneously, the function σ has the monotonic behavior.

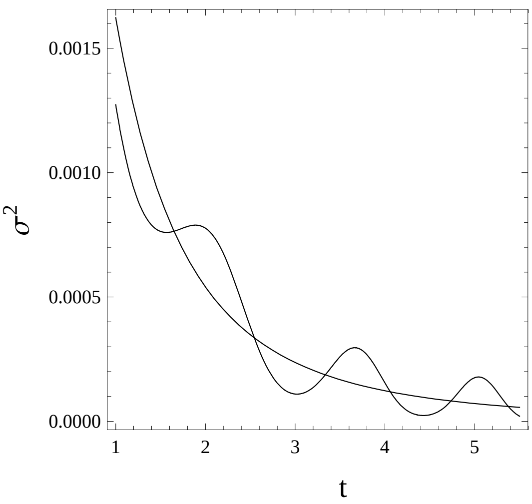

FIG. 3. The function σ^2 for the dependence $\omega(t)$ given by (21) and $k = 1$, $H = 1$. Monotonic curve corresponds to the vacuum state ($r = 0$, $\delta = 0$) and oscillating curve corresponds to the squeezed state ($r = 0.005$, $\delta = 0$).

Now let us consider the example

$$\omega(t) = \sqrt{1 + Ht}, \tag{21}$$

where $\omega(t)$ does not tend to a constant at infinity, but the adiabatic condition $\left|\frac{\dot{\omega}}{\omega^2}\right| \to 0$ is still

[a] Under existence of global vacuum we understand an existence of a state for which σ has monotonic behavior at all range of t, whereas σ for the non-vacuum states oscillates.

satisfied at $t \to \infty$. Eqs. (6), (4) are solvable in the closed form

$$u_0 = H^{-1/6} k^{-1/3} \sqrt{2\pi}\, \mathrm{Ai}\left(\frac{2k^{2/3}(Ht+1)}{H^{2/3}\left(1-i\sqrt{3}\right)}\right), \tag{22}$$

where $\mathrm{Ai}(z)$ denotes the Airi function [10]. Again, the calculation of the functional (13) gives Eq. (14). It should be noted that the asymptotic of σ is (see also Fig. 3)

$$\sigma \approx -\frac{1}{4}\left(\cos\left(\frac{4k(Ht+1)^{3/2}}{3H}+\delta\right) + \sqrt{3}\sin\left(\frac{4k(Ht+1)^{3/2}}{3H}+\delta\right)\right)\sinh(2r). \tag{23}$$

Let us come to the example, where the adiabatic condition is not fulfilled:

$$\omega(t) = \frac{k}{1+2Ht}. \tag{24}$$

The adiabatic parameter $\left|\frac{\dot\omega}{\omega^2}\right| = \frac{2H}{k}$ is constant. However, as it will be shown, the vacuum state exists in this case, as well. The solution of Eqs. (6), (4) is

$$u_0 = \frac{(1+2Ht)^{-\mathrm{i}\sqrt{k^2-H^2}/(2H)+1/2}}{\sqrt{2}(k^2-H^2)^{1/4}}, \tag{25}$$

and results in the following asymptotic

$$\sigma \approx \frac{1}{2\sqrt{k^2-H^2}}\Bigg(H\cosh^2(r) + H\sinh^2(r) - \sqrt{k^2-H^2}\sin\left(\delta + \frac{1}{H}\sqrt{k^2-H^2}\ln(2Ht+1)\right)\sinh(2r) + H\cos\left(\delta+\frac{1}{H}\sqrt{k^2-H^2}\ln(2Ht+1)\right)\sinh(2r)\Bigg). \tag{26}$$

The comparison with the previous cases demonstrates that the constant component appears in the asymptotic [b]. However, again the functional (13) has the form (14) for $k > H$. In the opposite case the function σ has non-oscillating behavior at infinity under arbitrary initial conditions.

Now let us consider the following example:

$$\omega = \frac{k}{1+H^2t^2}, \tag{27}$$

where the adiabatic parameter $\left|\frac{\dot\omega}{\omega^2}\right| = \frac{H^2 t}{k}$ becomes greater than unity at large t. The expression for the function u_0 has the form

$$u_0 = \frac{\sqrt{H^2t^2+1}}{\sqrt{2}\sqrt[4]{H^2+k^2}} \times \exp\left(-\frac{\mathrm{i}}{H}\sqrt{H^2+k^2}\,\arctan(Ht)\right) \tag{28}$$

as well as the expression for σ is

$$\sigma = \frac{t\cosh(2r)H^2}{2\sqrt{H^2+k^2}} + \left(\frac{H^2 t}{2\sqrt{H^2+k^2}}\cos\left(\delta + \frac{2\sqrt{H^2+k^2}\arctan(Ht)}{H}\right) - \frac{1}{2}\sin\left(\delta + \frac{2\sqrt{H^2+k^2}\tan^{-1}(Ht)}{H}\right)\right)\sinh(2r). \tag{29}$$

In this example, the function $\sigma(t)$ for an arbitrary state has non-oscillating behavior at

[b] To be sure that $\sigma = const$ minimizes the functional (13) for some family of the dependencies $\omega(t)$ including (24), it is sufficiently to write the extremum condition $\frac{\partial Z(r,t)}{\partial r} = \frac{\partial}{\partial r}\int_{t_0}^{T}\dot\sigma^2 dt = 2\int_{t_0}^{T}\dot\sigma\frac{\partial\dot\sigma}{\partial r}dt$. To satisfy this condition, one can suppose $\dot\sigma = 0$. Substitution of the last equality into Eq. (16) results in the particular family of frequencies so that $2\dot\omega - \frac{\omega\ddot\omega}{\dot\omega} = 0$.

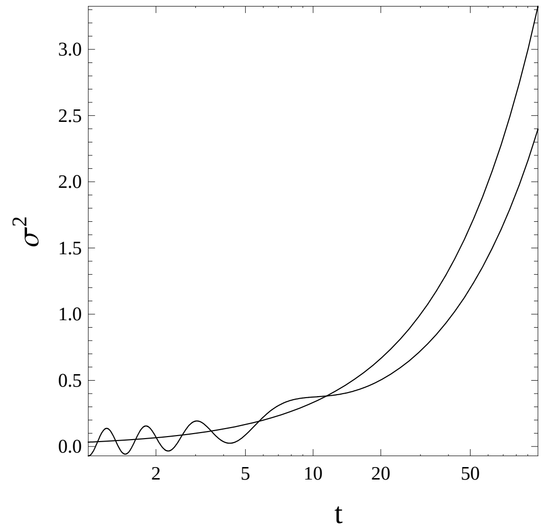

FIG. 4. The function σ^2 for the dependence $\omega(t)$ given by (27) and $k = 1$, $H = 1$. Monotonic curve corresponds to the vacuum state ($r = 0$, $\delta = 0$) and oscillating curve corresponds to the squeezed state ($r = 0.005$, $\delta = 0$).

infinity (Fig. 4). It occurs because the function u itself ceases to oscillate at $t \to \infty$. In the literature [3], such a phenomenon is interpreted as the transition from the quantum field to the classical one. The behavior of σ confirms this interpretation because the absence of oscillations means the absence of interference (i.e., in fact, absence of the main constituent of quantum mechanics). In any case we cannot talk about an existence of the out- vacuum state here. However, one can introduce a concept of the approximate vacuum state corresponding to some range of t. One can see from Fig. 4, that there exists a range, where the typical non-vacuum σ oscillates and thus, an approximate vacuum state corresponding to the non-oscillating σ can be defined.

3. Vacuums of the scalar field oscillator

In principle a number of the approximate vacuums corresponding to the different time regions can exist. Let us take an example, which does not admit an analytical consideration.

Lagrangian corresponding to the modes of the scalar field in an expanding Universe has the form [2]

$$\mathcal{L}_{scal} = \frac{1}{2}\sum_{\boldsymbol{k}} a^2\phi'_{\boldsymbol{k}}\phi'_{-\boldsymbol{k}} - a^2k^2\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}} - a^4m^2\phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}, \tag{30}$$

where $\phi_{\boldsymbol{k}}$ is the Fourier-transform of the scalar field $\phi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \phi_{\boldsymbol{k}} e^{i\boldsymbol{kr}}$ and $a(\tau)$ is the scale factor of Universe, τ is the conformal time $dt = a(\tau)d\tau$.

The equation of motion can be deduced

$$\phi''_{\boldsymbol{k}} + (k^2 + a^2m^2)\phi_{\boldsymbol{k}} + 2\frac{a'}{a}\phi'_{\boldsymbol{k}} = 0. \tag{31}$$

Quantization of the scalar field [2]

$$\hat{\phi}_{\boldsymbol{k}} = \hat{a}^+_{-\boldsymbol{k}}\chi^*_k(\tau) + \hat{a}_{\boldsymbol{k}}\chi_k(\tau) \tag{32}$$

leads to the operators of creation and annihilation with the commutation rules $[\hat{a}_{\boldsymbol{k}}, \hat{a}^+_{\boldsymbol{k}}] = 1$. The complex functions $\chi_k(\tau)$ satisfy the relations [2]:

$$\chi''_k + (k^2 + m^2a^2)\chi_k + 2\frac{a'}{a}\chi'_k = 0,$$
$$a^2(\tau)(\chi_k\,\chi'^*_k - \chi^*_k\,\chi'_k) = \mathrm{i}. \tag{33}$$

Substitution of $\chi_k = u_k/a$ results in the time dependent oscillator:

$$u''_k + (k^2 + a^2m^2 + \frac{a''}{a})u_k = 0. \tag{34}$$

Now we consider some illustrative time-dependence $a(\tau)$

$$a(\tau) = \tau\,(1 + \exp(3-\tau))^{-1} + 4\left(1 - \frac{\tau + 15}{1 + \exp(\tau + 25)}\right)^{-1}, \tag{35}$$

which is shown in Fig. 5. There exist three ranges, where one can search for the vacuum state. Namely, one can try to find the in-, out- vacuum states and, besides, the approximate vacuum state for the central range (from ≈ -20 to 0 shown in Fig. 5). The numerical minimization of the functional

$$Z(\alpha,\beta) = \int_{\tau_1}^{\tau_2} (\partial_\tau\sigma(\tau,\alpha,\beta))^2 \mathrm{d}\tau, \tag{36}$$

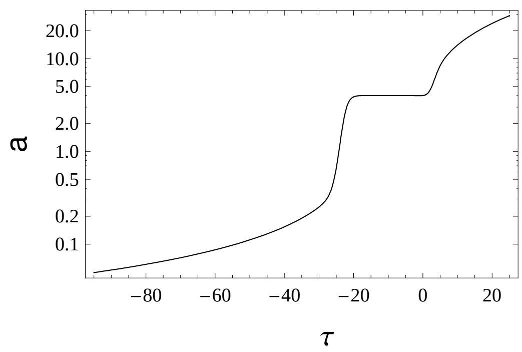

FIG. 5. The dependence (35) of the Universe scale factor $a(\tau)$ on the conformal time τ.

where $\sigma(\tau)$ obeys (16), (17) and $\alpha = \sigma(\tau_1)$, $\beta = \sigma'(\tau_1)$, allows finding the initial conditions corresponding to the vacuum. The solutions are shown in Fig. 6.

4. Vacuums of the fermionic oscillator

Let us come to the fermionic oscillator. After decomposition of the bispinor $\psi(\boldsymbol{r})$ in the complete set of modes $\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \psi_{\boldsymbol{k}} e^{i\boldsymbol{kr}}$, Lagrangian of the fermion field in the expanding Universe (see [13–16] and reference therein) takes the form

$$L = \sum_{\boldsymbol{k}} \frac{\mathrm{i}\, a^3}{2} \psi^+_{\boldsymbol{k}} \partial_\tau \psi_{\boldsymbol{k}} - \frac{\mathrm{i}\, a^3}{2} \partial_\tau \psi^+_{\boldsymbol{k}} \psi_{\boldsymbol{k}} - a^3 \psi^+_{\boldsymbol{k}} (\boldsymbol{\alpha k})\, \psi_{\boldsymbol{k}} - a^4\, m \psi^+_{\boldsymbol{k}} \beta \psi_{\boldsymbol{k}}. \tag{37}$$

The equation of motion is

$$i\psi'_k - (\boldsymbol{\alpha k})\psi_{\boldsymbol{k}} + \mathrm{i}\frac{3a'}{2a}\psi_{\boldsymbol{k}} - m\, a\beta\psi_{\boldsymbol{k}} = 0, \tag{38}$$

Fermion field is quantized as

$$\hat{\psi}_{\boldsymbol{k}} = \hat{b}^+_{-\boldsymbol{k},s} v_{-\boldsymbol{k},s} + \hat{a}_{\boldsymbol{k},s} u_{\boldsymbol{k},s}, \tag{39}$$

where the bispinor is [c]:

$$u_{\boldsymbol{k},s}(\eta) = \frac{\mathrm{i}\chi'_k + ma\chi_k}{a^{3/2}} \begin{pmatrix} \varphi_s \\ \frac{\chi_k(\boldsymbol{\sigma k})}{i\chi'_k + m\chi_k a}\varphi_s \end{pmatrix},$$

and spinors φ_s are $\varphi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\varphi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The bispinor $v_{\boldsymbol{k},s}$ is expressed as $v_{\boldsymbol{k},s} = \mathrm{i}\gamma^0\gamma^2(\bar{u}_{\boldsymbol{k},s})^T$, where the symbol T denotes the transpose vector and $\bar{u} = u^+\gamma^0$. The functions $\chi_k(\eta)$ satisfy [16]

$$\chi''_k + \left(k^2 + m^2a^2 - \mathrm{i}ma'\right)\chi_k = 0, \tag{40}$$
$$k^2\chi_k\chi^*_k + \left(am\chi^*_k - \mathrm{i}\chi'^*_k\right)\left(am\chi_k + \mathrm{i}\chi'_k\right) = 1, \tag{41}$$

and again there appears time-dependent oscillator (with the complex frequency), where the functions χ_k plays a role of the above mentioned u_k. The true vacuum state can be defined as that providing a non-oscillating behavior of the function

$$\sigma_k(t) = \frac{1}{2}\left(\chi^*_k\chi'_k + \chi'^*_k\chi_k\right). \tag{42}$$

One can deduce that if $\chi_k(t)$ obeys (40) then σ_k satisfy

$$\sigma'''_k - \frac{\sigma''_k M''}{M'} + 4(k^2 + M^2)\sigma'_k + \left(12MM' - \frac{4M''k^2}{M'} - \frac{4M^2M''}{M'}\right)\sigma_k = 0, \tag{43}$$

[c] Representation of the Dirac matrices is the same as in Refs. [17, 18].

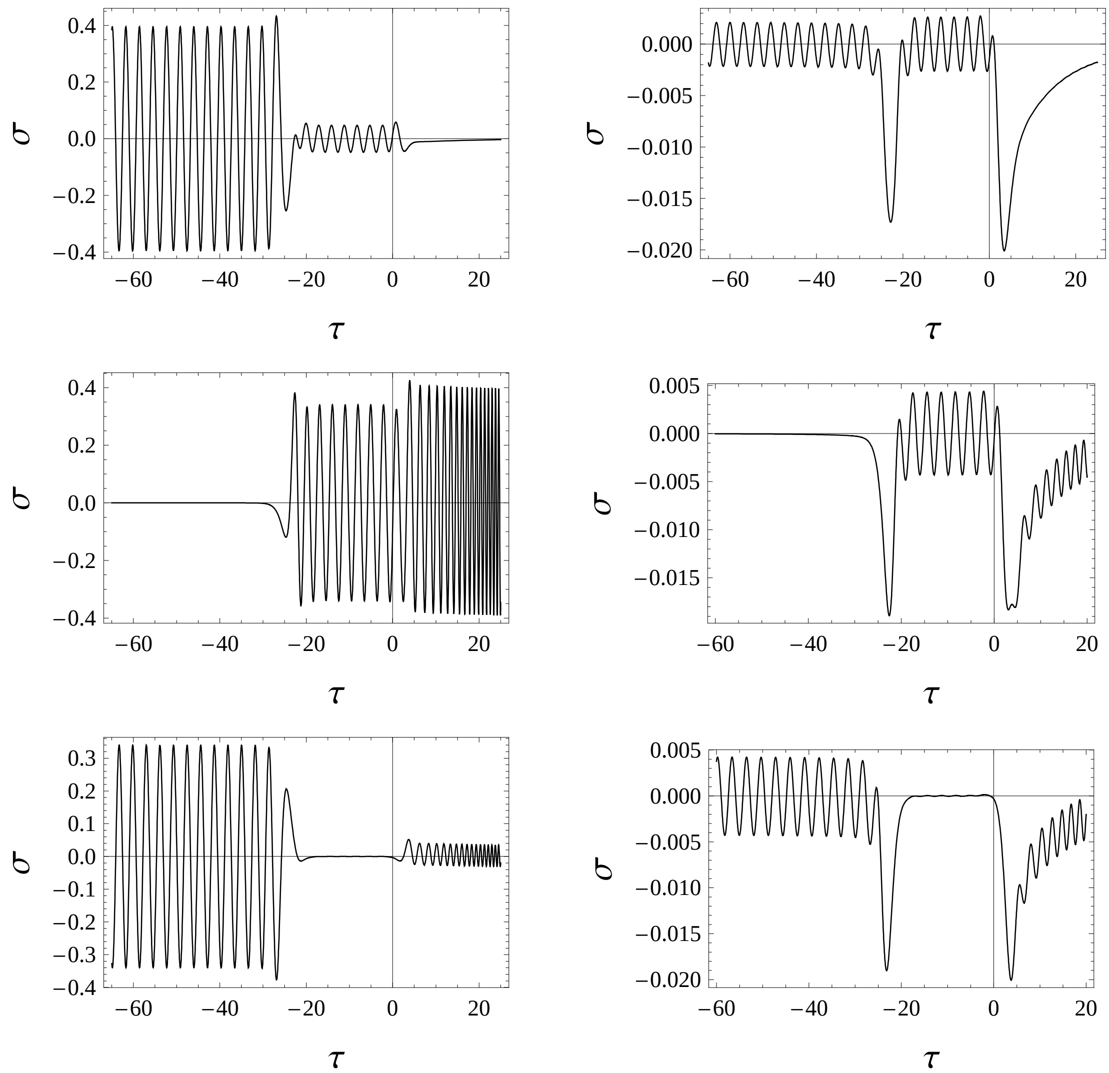

FIG. 6. The function σ for the vacuum states inspired by the dependence (35) and $k = 1$, $m = 1/16$. Left column corresponds to the scalar field oscillator, whereas right column corresponds to the fermionic one. Upper, middle and bottom rows correspond to the solutions representing the vacuums for the out-, in-, and central-time-ranges, respectively.

where $M(\tau) = ma(\tau)$. The relation (41) gives

$$\frac{1}{M'^2}\left(k^2+M^2\right)\left(4\sigma_k k^2+4\sigma_k M^2+\sigma_k''\right)^2 - 2\frac{M}{M'}\sigma_k'\left(4\sigma_k k^2+4\sigma_k M^2+\sigma_k''\right)+4k^2\sigma_k^2+\sigma_k'^2=1. \tag{44}$$

The vacuum solutions obtained by minimization of the functional (36) in the three different ranges are shown in Fig. 6 (right column).

5. Vacuums of the two coupled oscillators with constraint

Now we address ourself to a little more complicated system: namely, the two time-dependent coupled oscillators with constraint. This system appears in the theory of anisotropy of the Cosmic Microwave Background [19, 20]. One can expect, that some difficulties will arise with the vacuum definition for this system, because the quantization of constrained systems can reveals some nontrivial features. However, we will see that there are no pathologies in this particular case.

Both scalar field and gravitation can be assumed to be specified by the action [2, 3]

$$S=-\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}R + \int \mathrm{d}^4x\sqrt{-g}[(\frac{1}{2}\partial_\mu\phi)^2-V(\phi)], \tag{45}$$

Representation of the metric tensor in the form [3, 19]

$$\mathrm{d}s^2=(1+2\Phi(\boldsymbol{r},t))\mathrm{d}t^2-a^2(t)(1-2\Phi(\boldsymbol{r},t))\mathrm{d}\boldsymbol{r}^2, \tag{46}$$

and considering the scalar field as that possessing a spatially uniform component with a small perturbation around it:

$$\phi(\boldsymbol{r},t)=\phi(t)+\theta(\boldsymbol{r},t) \tag{47}$$

allows obtaining the system of equations [3] of zero order in θ and Φ,

$$-\dot a^2 a+\dot\phi^2a^3+2a^3V(\phi)=0, \tag{48}$$

$$\ddot a=-\frac{3}{2}a\dot\phi^2-\frac{\dot a^2}{2a}+3aV(\phi), \tag{49}$$

$$\ddot\phi+3\frac{\dot a}{a}\dot\phi+\frac{\mathrm{d}V}{\mathrm{d}\phi}=0, \tag{50}$$

where we use system of units $4\pi G/3=1$. The first order equations for the Fourier-transformed perturbations of the scalar field $\theta(\boldsymbol{r},t)=\sum_{\boldsymbol{k}}\theta_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{kr}}$ and metric $\Phi(\boldsymbol{r},t)=\sum_{\boldsymbol{k}}\Phi_{\boldsymbol{k}}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{kr}}$ have the following form [3, 19]:

$$\frac{1}{3}\Phi_{\boldsymbol{k}}k^2+\frac{\mathrm{d}V}{\mathrm{d}\phi}\theta_{\boldsymbol{k}}a^2+\dot\Phi_{\boldsymbol{k}}\dot a a+\dot\theta_{\boldsymbol{k}}\dot\phi a^2+2\Phi_{\boldsymbol{k}}a^2V(\phi)=0, \tag{51}$$

$$-\frac{1}{3}\dot\Phi_{\boldsymbol{k}}-\frac{\Phi_{\boldsymbol{k}}\dot a}{3a}+\theta_{\boldsymbol{k}}\dot\phi=0, \tag{52}$$

$$-3\frac{\mathrm{d}V}{\mathrm{d}\phi}\theta_{\boldsymbol{k}}-\ddot\Phi_{\boldsymbol{k}}-4\dot\Phi_{\boldsymbol{k}}\frac{\dot a}{a}+3\dot\theta_{\boldsymbol{k}}\dot\phi-6V(\phi)\Phi_{\boldsymbol{k}}=0, \tag{53}$$

$$\ddot\theta_{\boldsymbol{k}}+3\frac{\dot a}{a}\dot\theta_{\boldsymbol{k}}+\frac{k^2}{a^2}\theta_{\boldsymbol{k}}+\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\theta_{\boldsymbol{k}}+2\frac{\mathrm{d}V}{\mathrm{d}\phi}\Phi_{\boldsymbol{k}}-4\dot\phi\dot\Phi_{\boldsymbol{k}}=0. \tag{54}$$

Eqs. (53), (54) are the equations of motion. They can also be obtained from Lagrangian

$$L=\sum_{\boldsymbol{k}}-\frac{1}{2}\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}\theta_{\boldsymbol{k}}\theta_{-\boldsymbol{k}}a^3+2\frac{\mathrm{d}V}{\mathrm{d}\phi}\theta_{\boldsymbol{k}}\Phi_{-\boldsymbol{k}}a^3-10V\Phi_{\boldsymbol{k}}\Phi_{-\boldsymbol{k}}a^3-\frac{1}{2}ak^2\theta_{\boldsymbol{k}}\theta_{-\boldsymbol{k}} -\frac{1}{6}ak^2\Phi_{\boldsymbol{k}}\Phi_{-\boldsymbol{k}}+\frac{1}{2}a^3\dot\theta_{\boldsymbol{k}}\dot\theta_{-\boldsymbol{k}}-\frac{1}{2}a^3\dot\Phi_{\boldsymbol{k}}\dot\Phi_{-\boldsymbol{k}}-4\Phi_{\boldsymbol{k}}\dot\Phi_{-\boldsymbol{k}}a^2\dot a-4a^3\Phi_{\boldsymbol{k}}\dot\theta_{-\boldsymbol{k}}\dot\phi. \tag{55}$$

Eqs. (51), (52) are the constraints. However, Eq. (51) can be derived from Eqs. (52), (53), (54) and, thus, it is not independent. That is there are two time dependent oscillators with one constraint of the first kind [21]. Using this constraint one can exclude the scalar field perturbation from (53) and obtain

$$\ddot{\Phi}_{\boldsymbol{k}} - \dot{\Phi}_{\boldsymbol{k}} \frac{\mathrm{d}}{\mathrm{d}t} \ln\left(\frac{\dot{a}^2}{a^3} - \frac{\ddot{a}}{a^2}\right) + \left(\frac{k^2}{a^2} + \frac{2\ddot{a}}{a} - \frac{2\dot{a}^2}{a^2} - \frac{\dot{a}}{a}\frac{\mathrm{d}}{\mathrm{d}t} \ln\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)\right)\Phi_{\boldsymbol{k}} = 0 \quad (56)$$

where the uniform scalar field ϕ has been excluded by using (48), (49) as well. Quantization consists in

$$\hat{\Phi}_{\boldsymbol{k}} = \hat{\mathrm{a}}^{+}_{-\boldsymbol{k}} u^{*}_{k}(t) + \hat{\mathrm{a}}_{\boldsymbol{k}} u_k(t), \quad (57)$$

and the vacuum can be found by minimization of the quantity $Z(\alpha_k, \beta_k)$ (36), that allows finding α_k and β_k, which correspond to the vacuum state. In this we have solved Eq. (56) directly and have written the initial conditions at t_0 as $u_k(t_0) = \alpha_k$, $\dot{u}_k(t_0) = \beta_k - \mathrm{i}\frac{X(t_0)}{2\alpha_k}$, where $X(t_0) = \frac{\dot{a}^2}{a^3} - \frac{\ddot{a}}{a^2}\Big|_{t=t_0}$. These initial conditions are consistent with the relation

$$u_k \dot{u}^{*}_{k} - u^{*}_{k} \dot{u}_k = \mathrm{i}X(t), \quad (58)$$

which is analog of (33) and corresponds to the general case of quantization of oscillator with the time-dependent mass and frequency [22].

On the other hand, one can express $\Phi_{\boldsymbol{k}}$ through $\theta_{\boldsymbol{k}}$, and obtain the equation for $\theta_{\boldsymbol{k}}$ analogously to (56). This equation turns out to be more complicated and we do not write it here. The question arises: would be the vacuum state the same, if the scalar field perturbation is quantized as:

$$\hat{\theta}_{\boldsymbol{k}} = \hat{\mathrm{a}}^{+}_{-\boldsymbol{k}} \mathcal{U}^{*}_{k}(t) + \hat{\mathrm{a}}_{\boldsymbol{k}} \mathcal{U}_k(t)\,? \quad (59)$$

From Eq. (52), we have

$$\mathcal{U}_k = \frac{1}{3\dot{\phi}}\left(\dot{u}_k + \frac{\dot{a}}{a} u_k\right), \quad (60)$$

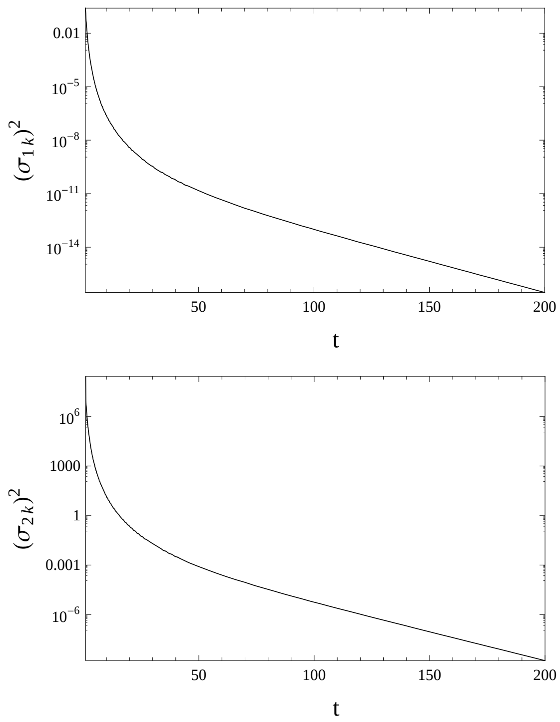

FIG. 7. The function σ^2 corresponding to the vacuum for $a(t) = \sinh(\gamma t)$, where $\gamma = \frac{1}{50}$ and $k = 1$. top panel shows result of the metric perturbation quantization, bottom panel shows the result of the scalar field perturbation quantization.

where Eqs. (48), (49) reduce $\dot{\phi}$ to the form $\dot{\phi} = \sqrt{\frac{1}{3}\frac{\dot{a}^2}{a^2} - \frac{1}{3}\frac{\ddot{a}}{a}}$.

Let us consider the particular case of $a(t) = \sinh(\gamma t)$. The numerical minimization of the functional allows finding the vacuum solution for which the function $\sigma_{1k} = \frac{1}{2}(\dot{u}_k u^{*}_{k} + \dot{u}^{*}_{k} u_k)$ has monotonic behavior and, thus, corresponds to the vacuum state. Now if one expresses $\sigma_{2k} = \frac{1}{2}(\dot{\mathcal{U}}_k \mathcal{U}^{*}_{k} + \dot{\mathcal{U}}^{*}_{k} \mathcal{U}_k)$ through the functions $\mathcal{U}_{\boldsymbol{k}}$ given by (60), it is seen from Fig.7, that $(\sigma_{2k})^2$ has also monotonic behavior. Thus, it is the vacuum state for the $\theta_{\boldsymbol{k}}$-oscillator too. Moreover, one can choose any convenient variable from a combination of $\Phi_{\boldsymbol{k}}$ and $\theta_{\boldsymbol{k}}$ as it usually done [19, 20, 23].

Procedure "NMinimize"of the Wolfram software "Mathematica"is used in all the numerical calculations.

6. Conclusion

We have considered a method to find the vacuum state of a driven quantum oscillator numerically by the means of minimization of the functional containing the square of derivative of the additional uncertainty σ arising in the Heisenberg uncertainty relation. For a time-dependent oscillator, the derivative of σ coincides with the difference of kinetic and potential energies. We show that this method can also be applied to both fermionic oscillator and pair of the coupled constrained oscillators. The last example is widely used in the theory of the microwave background anisotropy. We have verified that there is no problem with a selection of the vacuum state for the last system in spite of some discussions appearing in the literature [24, 25].

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