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Is hilltop buckling imperfection sensitive or insensitive ?

Gerhard Hoefinger[∗1], **Xin Jia**[1], and **Herbert A. Mang**[1]

[1] Vienna University of Technology, Karlsplatz 13/202, 1040 Vienna, Austria

Hilltop buckling is the coincidence of a snap-through point with a bifurcation point on the primary equilibrium path. An originally imperfection sensitive structure cannot be modified such that it shows both hilltop buckling and imperfection insensitivity. This is a rule for designing structures for imperfection insensitivity and guarantees validity of Koiter's postbuckling analysis, which would fail for imperfection insensitive hilltop buckling.

1 Introduction

In geometrically nonlinear elastic structures two modes of loss of stability exist. These are snap-through, where the primary (equilibrium) path has a maximum in terms of the load, and bifurcation buckling, where two equilibrium paths intersect. Hilltop buckling is characterized by the coincidence of a snap-through point and a bifurcation point on the primary path. Thus, the primary path has a local maximum (with respect to the load level at the stability limit) where the tangent to the equilibrium path is not unique because of the intersection of the primary and the secondary path. In context of the FEM, this is reflected by at least two eigenvalues of the tangent stiffness matrix being zero. At least one of the respective eigenvectors is orthogonal to the reference load vector.

In order to decide on imperfection sensitivity, Koiter's postbuckling analysis is used. Therefore, the parametrization of the secondary path is expanded into a Taylor series at the stability limit. The series for the load level is written as $\lambda(\eta) = \sum \lambda_i \eta^i$. A necessary condition for imperfection insensitivity is that the first non-vanishing term has an even exponent and the coefficient of this term in the series is positive. In an analysis aimed at improving the postbuckling behavior and consequently achieving imperfection insensitivity, a design parameter κ is varied. Hence, the investigated terms are functions of the (in this work scalar) parameter κ.

2 Limitations for Koiter's postbuckling analysis

An important motivation to prove the imperfection sensitivity of hilltop buckling is to ensure the validity of Koiter's postbuckling analysis also for this special case. In the course of an analysis with this method, the basis for calculation of points on the secondary path is a point on the a priori known primary path at the same load level. An ansatz $(\,\tilde{\mathbf{u}}(\lambda(\eta)) + \mathbf{v}(\eta), \lambda(\eta)\,)$ is made for a parametrisation of the secondary path and then the series

$$\begin{aligned} \mathbf{v}(\eta) &= \mathbf{v}_1\,\eta + \mathbf{v}_2\,\eta^2 + \mathbf{v}_3\,\eta^3 + \dots \\ \lambda(\eta) &= \lambda_C + \lambda_1\,\eta + \lambda_2\,\eta^2 + \lambda_3\,\eta^3 + \dots \end{aligned} \tag{1}$$

are calculated [1]. It is self-evident that $((\tilde{\mathbf{u}}(\lambda(\eta))), \lambda(\eta))$ has to exist as equlibrium point on the primary path for all values of η in a neighborhood of 0. For simple bifurcation points, this is always the case, both for imperfection sensitivity and insensitvity, because a neighborhood of the buckling load can be chosen for which the primary path increases monotonically. The situation is different at a hilltop buckling point. There, the load level of an imperfection insensitive postbuckling path would increase, while the primary path has its maximum there. This is shown in Figure (1). As this case does not exist - the proof follows below - the limitation of Koiter's method is only of theoretical nature.

3 The coefficient λ_2

In general, to be able to convert a system from imperfection sensitivity to insensitivity, $\lambda_1 = 0$ has to be ensured. This will very often be done by using symmetries in the structure. Then, up to special cases, λ_2 decides on the qualitative postbuckling behavior. Let $\mathbf{K}_T$ be the tangent stiffness matrix of the system and $(\tilde{\mathbf{u}}(\lambda), \lambda)$ be a parametrization of the primary equilbrium path. With $\widetilde{\mathbf{K}}_T(\lambda) := \mathbf{K}_T(\tilde{\mathbf{u}}(\lambda))$ and $\widetilde{\mathbf{K}}_T(\eta) := \widetilde{\mathbf{K}}_T(\tilde{\mathbf{u}}(\lambda(\eta)))$ we obtain [2]

$$\lambda_2 = \frac{1}{2} \frac{\mathbf{v}_1^T \cdot \widetilde{\mathbf{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \mathbf{v}_1}{\mathbf{v}_1^T \cdot \widetilde{\mathbf{K}}_{T,\lambda}\Big|_{\lambda=\lambda_C} \cdot \mathbf{v}_1} \tag{2}$$

* Corresponding author: e-mail: gerhard.hoefinger@tuwien.ac.at, Phone: +43 1 58801 20254, Fax: +43 1 58801 20299

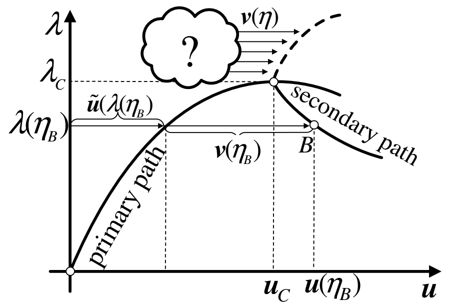

Fig. 1 Koiter's postbuckling analysis for hilltop buckling: The dashed line depicts an imperfection insensitive post-buckling path, the cloud symbolizes the missing reference $\widetilde{\mathbf{u}}(\lambda(\eta))$ for the vertical offset $\mathbf{v}(\eta)$.

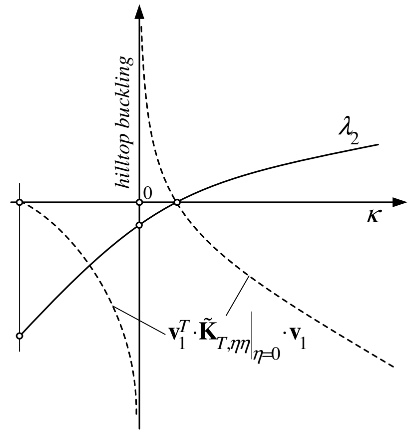

Fig. 2 A typical configuration where modification towards hilltop buckling leads to negative λ_2 and $\mathbf{v}_1^T \cdot \widetilde{\mathbf{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \mathbf{v}_1 \to \infty$

At the hilltop buckling point,

$$\left|\mathbf{v}_1^T \cdot \widetilde{\mathbf{K}}_{T,\lambda} \cdot \mathbf{v}_1\right| \to \infty \quad \text{because of} \quad \widetilde{\mathbf{K}}_{T,\lambda} = \mathbf{K}_{T,u} \cdot \tilde{\mathbf{u}}_{,\lambda} \;\; \text{and} \;\; ||\tilde{\mathbf{u}}_{,\lambda}|| \to \infty \tag{3}$$

Accordingly, in order to achieve a finite value for λ_2, $\mathbf{v}_1^T \cdot \widetilde{\mathbf{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \mathbf{v}_1$ has to be infinite in a hilltop buckling configuration.

4 Hilltop buckling is imperfection sensitive

What we actually show is that at the hilltop, $\lambda_2 \neq 0$. The consequence is that $\lambda_2 = 0$ is an (upper) bound for all possible values of λ_2 at a hilltop buckling point.
Two modifications of a system towards hilltop buckling with $\lambda_2 = 0$ are thinkable. The first is to have hilltop buckling right from the beginning and increase λ_2. In this way, $\mathbf{v}_1^T \cdot \widetilde{\mathbf{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \mathbf{v}_1$ would jump from ∞ to a finite value, which is not possible. The second way is to modify the system such that the snap through point and the bifurcation point with $\lambda_2 = 0$ converge. Then, the value of $\mathbf{v}_1^T \cdot \widetilde{\mathbf{K}}_{T,\eta\eta}\Big|_{\eta=0} \cdot \mathbf{v}_1$ at the hilltop buckling point would jump from ∞ to 0. Thus, also the second case is not possible. If $\lambda_2 = 0$ is not possible for hilltop buckling, a continuous transformation of the system from imperfection sensitivity to insensitivity is excluded. Thus, hilltop buckling is imperfection sensitive.

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References

[1] H.A. Mang, C. Schranz and P. Mackenzie-Helnwein, Conversion from imperfection-sensitive into imperfection-insensitive elastic structures I: Theory, Comput. Methods Appl. Mech. Engrg. **195**, 1422-1457 (2006).
[2] H.A. Mang, X. Jia, G. Hoefinger, Hilltop buckling as the Alpha and Omega in sensitivity analysis of the inital postbuckling behavior of elastic structures, J. Civ. Eng. Manag., **15**, 35-46 (2009).

