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Numerical calculations of the driving force on an Abrikosov vortex

D.-X. Chen [a,*], E. Pardo [b,c], A. Sanchez [c]

[a] ICREA and Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain
[b] Atominstitut, Vienna University of Technology, Stadionallee 2, 1020 Vienna, Austria
[c] Departament de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain

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ABSTRACT

The driving force on an Abrikosov vortex is calculated numerically from the London equation and involved energies for a vortex perpendicular to the screening current near the surface of a superconductor. Compared with previous analytical derivation of the total force, the partial magnetic, kinematic, and external forces are also obtained so that the nature of the driving force may be deeply discussed. It is shown that the force is neither a Lorentz force nor a Magnus force as often believed and that in order to get a correct result, the image effects and the work done by the applied field must be taken into account. A name of London force is suggested for the driving force. A deep understanding of the nature of the driving force on Abrikosov vortices may also be important in the study of vortex pinning and dynamics in type-II superconductors.

1. Introduction

Consider a superconductor with an infinite z dimension, along which there is an Abrikosov vortex (AV) carrying a flux quantum $\Phi_0 = \pi\hbar/e = 2.07 \times 10^{-15}$ Wb centered at $\mathbf{r} = \mathbf{r}_{01}$ on the xy plane. If there is an external supercurrent with a z-independent density $\mathbf{J}(\mathbf{r})$, the driving force $\mathbf{F}$ on the AV per unit length due to the supercurrent is known as

$$\mathbf{F} = \mathbf{J}(\mathbf{r}_{01}) \times \mathbf{\Phi}_0. \tag{1}$$

This force was first derived by de Gennes and Matricon for a system of two parallel AVs, where the force acting on the first AV was due to the external (with respect to this AV) supercurrent of the second AV centered at $\mathbf{r}_{02}$ [1,2]. The derivation was made based on the London equation through the derivative of the interaction energy between both AVs with respect to the AV distance $d = |\mathbf{r}_{02} - \mathbf{r}_{01}|$. After this derivation, several authors treated analytically a single AV driven by an arbitrary external supercurrent [3–5]. Based on the London equation and making a derivative of the interaction energy between the AV and the current with respect to the AV movement, they also obtained a driving force expressed by Eq. (1).

However, there is a contradiction found between the two AV and one AV cases. In the two AV case, writing the field and current density as $\mathbf{H}_1$ and $\mathbf{J}_1$ for the first AV and as $\mathbf{H}_2$ and $\mathbf{J}_2$ for the second AV, the interaction energy density is proportional to H_1H_2 for the field energy and $\mathbf{J}_1 \cdot \mathbf{J}_2$ for the kinetic energy. When both AVs are far away from each other, there will be two regions around each AV core where either H_1 and J_1 or H_2 and J_2 are high so that the interaction energy is concentrated there. Owing to the symmetry, each region carries a half of the change in energy after a small change in distance. Therefore, the force will be a half of that expressed by Eq. (1) if the system is considered to be one of the highly interacted regions centered at the first AV core. But this is actually an example of the one AV case mentioned above, for which the force has been derived as Eq. (1).

In order to resolve this contradiction, we deal with a typical example of the one AV case, which was already treated analytically by de Gennes [2]. After writing the formulas for the involved energies, we will make all the calculations numerically, so that possible approximations or errors in previous analytical derivations may be avoided. We will show that the force calculated from the interaction energy may be indeed a half of Eq. (1) under simplified, unphysical assumptions and that the practical validity of Eq. (1) is justified for the one AV case only when the boundary-value problem is correctly solved and all the necessary energies are considered. We will argue that without considering the image effects and the external work done during the AV movement, the derivations carried out in [3–5] are questionable. We will further discuss the nature of the driving force to distinguish it clearly from a Lorentz force and a Magnus force.

* Corresponding author.
E-mail address: duxing.chen@uab.cat (D.-X. Chen).

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2. Calculations for one AV case

2.1. The studied one AV case

The studied superconductor occupies the half-space $x > 0$ with its limiting surface on the yz plane. An AV parallel to the z axis is centered at $\mathbf{r}_{01} = (x_{01}, 0)$. Applying a uniform field $\mathbf{H}_a$ in the z direction, $\mathbf{H}(\mathbf{r})$ is calculated by the London equation with a two-dimensional delta function as [2]

$$\mathbf{H} + \lambda^2 \nabla \times \nabla \times \mathbf{H} = \mathbf{\Phi}_0 \delta_2(\mathbf{r} - \mathbf{r}_{01})/\mu_0, \tag{2}$$

where λ is the London penetration depth.

The resultant field is the sum of the screening and AV fields. Screening field and current density, $\mathbf{H_{sc}}$ and $\mathbf{J}_{sc}$, are induced by $\mathbf{H}_a$ in the z and y directions, respectively, which are calculated from the London equation in the half space as

$$H_{sc} = H_a \exp(-x/\lambda), \tag{3}$$

$$J_{sc} = H_a \exp(-x/\lambda)/\lambda. \tag{4}$$

The field for the AV is calculated from the London equation in an infinite superconductor by

$$H_1 = \frac{\Phi_0}{2\pi\mu_0\lambda^2} K_0\left(\frac{r_1}{\lambda}\right), \tag{5}$$

where K_i is the ith-order second-kind modified Bessel function and

$$r_1 = |\mathbf{r} - \mathbf{r}_{01}| \geqslant \xi, \tag{6}$$

ξ being the AV core radius equal to the coherence length.

Correspondingly, the modulus of the current density for the AV is calculated using the Ampère law from Eq. (5) as

$$\mathbf{J}_1 = \frac{\Phi_0}{2\pi\mu_0\lambda^3} K_1\left(\frac{r_1}{\lambda}\right), \tag{7}$$

the direction of $\mathbf{J}_1$ being defined by the angle ϕ_1 it makes with the x axis,

$$\phi_1 = \frac{\pi}{2} + \arccos\frac{x - x_{01}}{r_1}. \tag{8}$$

The field and current density in the AV core of $r_1 < \xi$ are approximated by $H_1(r_1) = H_1(\xi)$ and $J_1(r_1) = J_1(\xi)r_1/\xi$.

Our purpose is to calculate the force on the AV in comparison with Eq. (1).

2.2. Force calculation from internal energy under simplified consideration

In order to demonstrate the involved physics and the various origins of the driving force, we make numerical calculations. This is different from the previous works mentioned earlier, where only analytical derivations for the total force are carried out. Instead of the interaction energy used in most analytical derivations, we use the energy itself for our numerical calculations, since both are equivalent when only their derivative with respect to the AV movement is concerned and since the expression of the latter is sometimes simpler and physically clearer than the former. Of course, in the actual force calculation, terms not changing with the AV movement may be removed to save computation time.

We write the internal energy of the studied system per unit z length as the sum of the magnetic field energy and the kinetic energy by

$$E = E_m + E_k = \int_{x>0} \left[\frac{\mu_0}{2}(H_{sc} + H_1)^2 + \frac{\mu_0\lambda^2}{2}(\mathbf{J}_{sc} + \mathbf{J}_1)^2\right] dv. \tag{9}$$

For numerical calculations, the xy cross-section of superconductor is divided into a large number (up to 10^8) of square cells of size equal to $\xi/5$, so that the core diameter equals 10 cell sizes. The

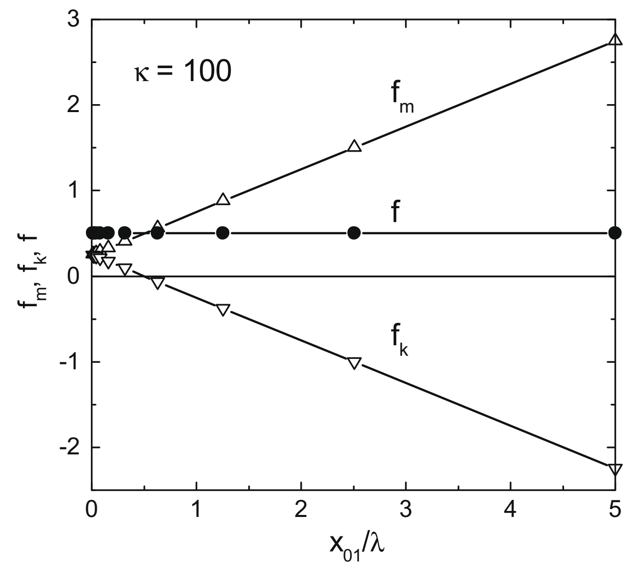

Fig. 1. For the studied one AV case of $\kappa = 100$, the magnetic, kinematic, and total force, F_m, F_k, and F, acting on the AV normalized to $J(\mathbf{r}_{01})\Phi_0$, as functions of the normalized distance x_{01}/λ of the AV center to the superconductor surface. The forces are calculated from the internal energy alone without considering the boundary effects (Section 2.2).

force in the x direction acting on the AV was calculated by $F = -\partial E/\partial x_{01}$ with one cell movement of the AV core. F_m, F_k, and F normalized to $J(\mathbf{r}_{01})\Phi_0$, where

$$J(\mathbf{r}_{01}) = J_{sc}(x_{01}), \tag{10}$$

f_m, f_k, and f, are obtained as functions of x_{01}/λ. The results are very weak functions of $\kappa \equiv \lambda/\xi$ when $\kappa \gg 1$, and plotted in Fig. 1 for $\kappa = 100$. We see that with increasing x_{01} from ξ to 5λ, f_m increases from 0.25 to 2.75 whereas f_k decreases from 0.25 to -2.25 so that $f = 0.50$ always.

2.3. Force calculations from internal energy

The result of $f = 0.50$ is consistent with the above deduction from the two AV result and is in discrepancy with $f = 1$ in [3–5] derived from the interaction energy for the one AV case. In fact, because it does not consider the influence of the boundary, the above treatment for the one AV case is over-simplified. With the boundary, a part of the complete AV at $x < 0$ has been removed so that H_1 and $\mathbf{J}_1$ are not the field and current density owing to the existence of the AV. It is well known that the correct solution at $x > 0$ is the sum of those for the AV at $(x_{01}, 0)$ and its antiparallel image at $(x_{02} = -x_{01}, 0)$ [6]. After having the correct solution to the London equation, the correct internal energy becomes

$$E = E_m + E_k = \int_{x>0} \left[\frac{\mu_0}{2}(H_{sc} + H_1 + H_2)^2 + \frac{\mu_0\lambda^2}{2}(\mathbf{J}_{sc} + \mathbf{J}_1 + \mathbf{J}_2)^2\right] dv, \tag{11}$$

where H_1 and $\mathbf{J}_1$ are the field and current density for the AV, as expressed by Eqs. (5)–(8), and H_2 and $\mathbf{J}_2$ are the field and current density produced by its image. H_2 is expressed by

$$H_2 = -\frac{\Phi_0}{2\pi\mu_0\lambda^2} K_0\left(\frac{r_2}{\lambda}\right), \tag{12}$$

where

$$r_2 = |\mathbf{r} - \mathbf{r}_{02}| \geqslant \xi, \tag{13}$$

and the modulus of $\mathbf{J}_2$ is expressed by

$$J_2 = \frac{\Phi_0}{2\pi\mu_0\lambda^3} K_1\left(\frac{r_2}{\lambda}\right), \tag{14}$$

with its direction defined by the angle ϕ_2 it makes with the x axis,

$$\phi_2 = -\frac{\pi}{2} + \arccos\frac{x - x_{02}}{r_2}. \quad (15)$$

2.3.1. Results for $H_a = 0$

When $H_a = J_{sc} = H_{sc} = 0$, the force in the x direction acting on the AV is calculated by $F = -\partial E/\partial x_{01}$. For $\kappa = 100$, F_m, F_k, and F normalized to $J(\mathbf{r}_{01})\Phi_0$, where

$$J(\mathbf{r}_{01}) = -J_2(\mathbf{r}_{01}), \quad (16)$$

f_m, f_k, and f, as functions of x_{01}/λ are plotted in Fig. 2. We see that with increasing x_{01} from ξ to 5λ, f_m increases from zero to 4.8, f_k decreases from 1 to −3.8, so that $f = 1$ remains. These results are the same as those of the analytically calculated two AV case, if x_{01} is replaced by $d/2$.

2.3.2. Results for nonzero H_a

With nonzero applied field, calculations have to be made at a fixed H_a by $F = -\partial E/\partial x_{01}$. We assume $\kappa = 100$ and $H_a = 3H_{c1}$, where H_{c1} is the lower critical field calculated by

$$H_{c1} = \frac{E_{AV}}{\Phi_0} = \frac{\Phi_0}{4\pi\mu_0\lambda^2} K_0\left(\frac{1}{\kappa}\right), \quad (17)$$

E_{AV} being the energy of an AV per unit length. The calculated F_m, F_k, and F normalized to $J(\mathbf{r}_{01})\Phi_0$, where

$$J(\mathbf{r}_{01}) = \frac{\Phi_0}{4\pi\mu_0\lambda^3}\left[3K_0\left(\frac{1}{\kappa}\right)\exp\left(\frac{-x_{01}}{\lambda}\right) - 2K_1\left(\frac{2x_{01}}{\lambda}\right)\right], \quad (18)$$

f_m, f_k, and f, as functions of x_{01}/λ are plotted in Fig. 3. We see that when $x_{01} > \lambda$, f_m and f_k increases and decreases, respectively, from zero linearly so that f remains about zero. All the three forces show an oscillation at $x_{01} < \lambda$, which will be explained later on.

2.4. Force calculation from internal energy and external work

The studied one AV system is not an isolated system since an external work will be done with the AV movement. When displacing the AV inward, i.e., increasing x_{01}, the separation between the AV and its antiparallel image increases, so that the flux in the superconductor increases. The flux change will induce an emf,

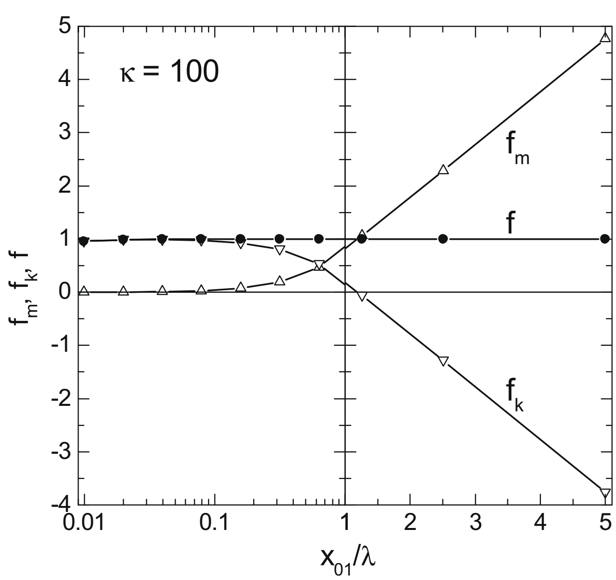

Fig. 2. Same as Fig. 1, but the forces are calculated from the internal energy alone at zero applied field (Section 2.3). Note that the scale of x_{01}/λ is logarithmic and linear in the left and right halves of the figure, respectively.

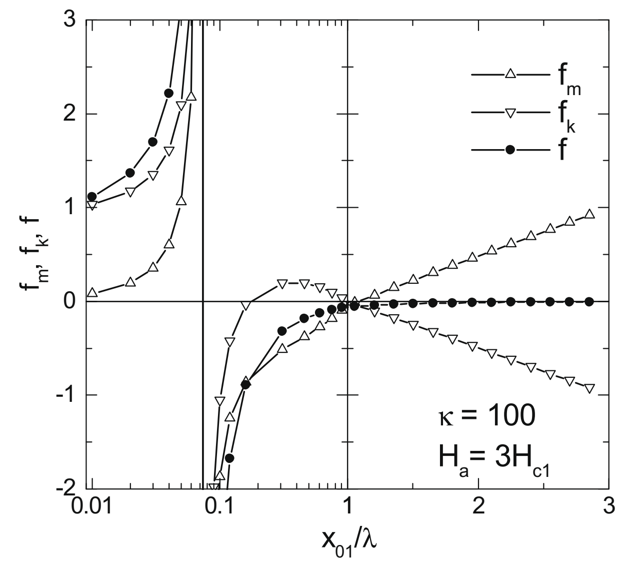

Fig. 3. Same as Fig. 1, but the forces are calculated from the internal energy alone at applied field $H_a = 3H_{c1}$ (Sec. II C 2). Note that the scale of x_{01}/λ is logarithmic and linear in the left and right halves of the figure, respectively.

which corresponds to an electrical field $\mathbf{E}(0, y)$ on the surface in the y direction, so that energy will enter the superconductor in the x direction according to the Poynting vector $\mathbf{E} \times \mathbf{H}_a$. Therefore, when displacing the AV inward, the work done on the AV is not only coming from the internal energy reduction but also from the work done by the applied field. As a result, for calculating F in terms of energies, one has to replace the internal energy E by the microscopic Gibbs potential G and to use $F = -\partial G/\partial x_{01}$ [2],

$$G = E_m + E_k + E_e = \int_{x>0}\left[\frac{\mu_0}{2}(H_{sc} + H_1 + H_2)^2 + \frac{\mu_0\lambda^2}{2}(\mathbf{J}_{sc} + \mathbf{J}_1 + \mathbf{J}_2)^2 - \mu_0 H_a(H_{sc} + H_1 + H_2)\right] dv, \quad (19)$$

where E_e is the external energy whose change equals the external work.

For this one AV system with external work being considered, the numerically calculated results at $\kappa = 100$ and $H_a = 3H_{c1}$ are plotted in Fig. 4, where f_m, f_k, f_e, and f are E_m, E_k, E_e, and G normalized to $J(\mathbf{r}_{01})\Phi_0$, where $J(\mathbf{r}_{01})$ is expressed by Eq. (18).

In fact, de Gennes has treated this problem analytically by surface integrations, obtaining [2]

$$G = \Phi_0\left[H_a \exp\left(\frac{-x_{01}}{\lambda}\right) - \frac{\Phi_0}{4\pi\mu_0\lambda^2}K_0\left(\frac{2x_{01}}{\lambda}\right) + H_{c1} - H_a\right], \quad (20)$$

from which the vortex entry field is calculated as $H_s = \Phi_0/(4\pi\mu_0\lambda\xi) \approx \kappa H_{c1}/(\ln\kappa + 0.116) \approx 21H_{c1}$ for $\kappa = 100$. In our case of $H_a = 3H_{c1} < H_s$, $F = -\partial G/\partial x_{01}$ changes from negative to positive with increasing x_{01}/λ to 0.0739, where $F = J_{AVcent} = 0$ according to Eq. (20). We see from Fig. 4 that $f_k \approx 1$ and $f_e < 0$ occur at $x_{01}/\lambda = 0.01$, which means that the attractive kinematic interaction between the actual and image AVs dominates when the AV is near the surface, whereas the external work pushes the AV inward. With increasing x_{01}, an oscillation between $\pm\infty$ occurs for all fs, owing to $J_{AVcent} = 0$ at $x_{01}/\lambda \approx 0.0739$. Since F_m and F_k are negative and F_e is positive with significant magnitudes when $x_{01}/\lambda < 0.1$, the oscillation for f_m, f_k, and f_e extends to a wide x_{01} interval. In contrast, f undergoes a sharp oscillation because although $F < 0$ at

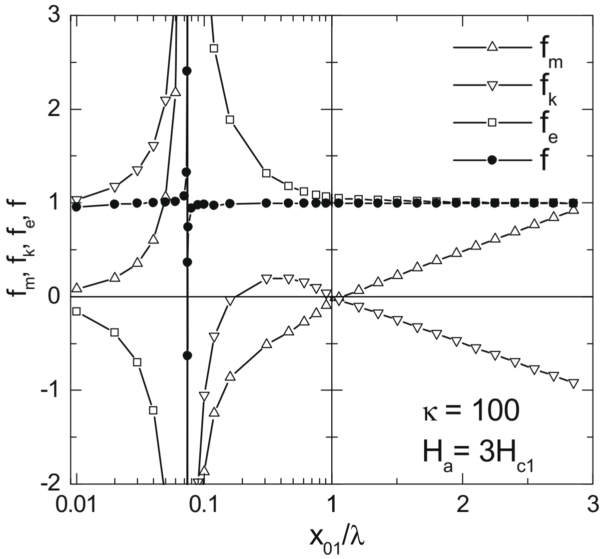

Fig. 4. For the studied one AV case of $\kappa = 100$ and $H_a = 3H_{c1}$, the magnetic, kinematic, external, and total force, F_m, F_k, F_e, and F, acting on the AV normalized to $J(\mathbf{r}_{01})\Phi_0$, as functions of x_{01}/λ. The forces are calculated correctly from the internal energy and external work considering boundary effects (Section 2.4). Note that the scale of x_{01}/λ is logarithmic and linear in the left and right halves of the figure, respectively.

$x_{01}/\lambda = 0.0739$, it becomes zero at x_{01} merely 0.5% greater. A significant feature occurs in the high-x_{01} region, where f is dominated by f_e with $f_m + f_k \approx 0$.

3. Discussion

3.1. Previous derivations from internal energy

After derived for the two AV case [1,2], Eq. (1) should be valid for a system consisting of $n(>1)$ AVs if $\mathbf{J}(\mathbf{r}_{01})$ is understood as the sum of the current densities of the 2nd to nth AVs with the force being acted on the first AV. This is expected since for the linear London equation a linear addition of its solutions is still a solution. After the validity of Eq. (1) was analytically confirmed for our one AV case [2], the meaning of $\mathbf{J}(\mathbf{r}_{01})$ could be extended to including a screening current in a homogeneous superconductor. Further steps should be to derive the force on an AV in more complicated systems even driven by an arbitrary external current.

Tachiki et al. have calculated the driving force on an AV near the interface between a strongly and a weakly superconducting layer due to an external current, which originates from the current of AVs other than the AV under consideration and from the surface current [4]. Castro and Lopez have considered a large superconductor subject to an applied field $\mathbf{H}_a$ in the z direction and an applied current in the x direction and calculated the driving force acting on an AV along the z direction [5]. Both calculations are similar. The authors of [5] first write the interaction energy between the AV and its surrounding as

$$W = W_m + W_k = \int_{SC} (\mu_0 \mathbf{H}_1 \cdot \mathbf{H}_{ex} + \mu_0 \lambda^2 \mathbf{J}_1 \cdot \mathbf{J}_{ex}) dv, \tag{21}$$

where the integration is performed over the volume of superconductor and $\mathbf{H}_{ex}$ and $\mathbf{J}_{ex}$ are the external field and supercurrent density of all the origins other than the AV under consideration. After $\mu_0 \mathbf{H}_1$ and $\mathbf{J}_{ex}$ in Eq. (21) are replaced by

$$\mu_0 \mathbf{H}_1 = \mathbf{\Phi}_0 \delta(\mathbf{r} - \mathbf{r}_{01}) - \mu_0 \lambda^2 \nabla \times \mathbf{J}_1, \tag{22}$$
$$\mathbf{J}_{ex} = \nabla \times \mathbf{H}_{ex}, \tag{23}$$

which are obtained from Eq. (2) with $\mathbf{H} = \mathbf{H}_1$ and from Ampère law for the external field and current density, respectively, one has

$$W = W_m + W_k = \int_{SC} [\mathbf{\Phi}_0 \cdot \mathbf{H}_{ex} \delta(\mathbf{r} - \mathbf{r}_{01}) + \mu_0 \lambda^2 (\mathbf{J}_1 \cdot \nabla \times \mathbf{H}_{ex} - \nabla \times \mathbf{J}_1 \cdot \mathbf{H}_{ex})] dv. \tag{24}$$

Then, they use a vector identity and Gauss theorem to transfer the second part of this volume integral into a surface integral that vanishes, except when the AV is closer to the surface than a distance of the order of λ. Thus, one has

$$W = \Phi_0 H_{ex}(\mathbf{r}_{01}) \tag{25}$$

and Eq. (1) is obtained by $\mathbf{F} = -\Phi_0 \nabla H_{ex}(\mathbf{r}_{01})$, where ∇ is performed with respect to $\mathbf{r}_{01}$. In [4], the cancellation of a term equivalent to the above second part is stated as by integrating by parts and neglecting the interface energy.

Although both cases seem to be more general than the one AV case we have treated, we may check their correctness using our one AV case by assuming $\mathbf{H}_{ex} = \mathbf{H}_{sc}$ as a simple example. Rewriting Eq. (9) into an interaction energy, we see that it is equivalent to Eq. (21) when $x_{01} \gg \lambda$. Since $f = 0.5$ is calculated from Eq. (9) for any value of x_{01}, as shown in Fig. 1, the result of $f = 1$ from Eq. (21) must be wrong, so that the cancellation of the second part in Eq. (24) is not justified even when the AV is far away from the surface. In fact, for an exact cancellation of that part, an inclusion of the image effect and $H_a = 0$ are required, in which case $f = 1$ for both cases is justified, as shown in Fig. 2. When the image effect is considered but $H_a \neq 0, f \neq 1$ occurs in general, as shown in Fig. 3, so that the cancellation of that part involves a huge error. In order to obtain a correct result shown in Fig. 4, the external work done during the AV movement must be taken into account. Therefore, the authors of [4,5] have not given a correct method to treat their AV problems.

3.2. Previous derivation considering external work, Lorentz force

In contrast, Narayan has derived the driving force on an AV with both the internal energy and external work being considered [7]. In his interesting treatment under general circumstances, a transport current is injected into and extracted from the superconductor and the calculated force is due to the total current passing the AV. However, his derivation is too simple to be understood, especially concerning equations where the phase of superconducting order parameter is involved. Without a special gauge being chosen the derivation seems to be meaningless, if the problem involves the gauge covariant vector potential and phase. It would be helpful if the author could make more explanations for his derivation and could show how his derivation is applied to our one AV case. Since this work could be a significant step forward after [1,2], such an extension should be worth making.

Nevertheless, we would like to make some remarks on [7] before further understanding the involved derivation. It is stated that when the currents flow entirely in the superconductor, either due to other vortices or as closed current loops, there is no external work done and the driving force on the AV comes from the change of the internal energy. This is questionable, since our one AV case is actually such a case but external work must be taken into account.

It is also stated in [7] that when the driving currents are entirely outside the superconductor, there is no direct force on the AV. Thus, the same current flowing on or just off the surface of a superconductor gives rise to different forces on the AV, unlike the Lorentz force in classical electromagnetism.

A classical Lorentz force of current of density $\mathbf{J}$ acting on an AV is calculated by

$$\mathbf{F}_{Lor} = -\mu_0 \int \mathbf{J} \times \mathbf{H}_1 dv. \tag{26}$$

If $\mathbf{J}$ is assumed to be constant, this equation becomes

$$\mathbf{F}_{\text{Lor}} = -\mathbf{J} \times \Phi_0, \tag{27}$$

which was first used by Anderson for predicting the driving force on an AV [8,9]. In fact, he overlooked the opposite sign of this force to the later derived superconducting driving force in Eq. (1). Even so, the name of Lorentz force has been used for the superconducting driving force till now, which has caused much confusion on the force direction in the literature [10]. Therefore, some further discussion on this is necessary.

We think that if the superconductor is magnetically decoupled from external currents, the above second statement in [7] is correct. It will be interesting to mention our calculation of the movement of a current driven Josephson vortex (JV) along a resistively shunted long Josephson junction [11]. We have found that if an artificial uniform current is applied, both the driving force and damping force are the classical Lorentz force and that if the applied current is a supercurrent instead, the damping force is still a Lorentz force but the driving force becomes a force similar to Eq. (1) with an additional factor of 1.28. Since the artificial uniform current is unphysical, for a real system we may think of two overlapped films, of which one being superconducting with a perpendicular vortex and the other being normal conducting. Since both films are well magnetically coupled to each other, the situation should be similar to the above case of JV: applying a uniform external current through the latter, the vortex in the former will be acted by a Lorentz force from the external current but not by a driving force from internal supercurrent. Such a situation is worth being studied further.

3.3. Previous derivation from kinetic energy, Magnus force

Abrikosov has assumed the interaction energy for magnetic fields to be relatively small and calculated the components of the driving force from the kinematic interaction energy W_k as $F_x = -\partial W_k/\partial x_{01}$ and $F_y = -\partial W_k/\partial y_{01}$, when the AV is located at (x_{01}, y_{01}) [3]. Under the condition of

$$\nabla \times \mathbf{J}_1 = \frac{\Phi_0}{\mu_0 \lambda^2}\delta_2(\mathbf{r} - \mathbf{r}_{01}), \tag{28}$$

Eq. (1) is analytically derived.

Eq. (28) means that $\mathbf{J}_1$ is irrotational outside the vortex core, which is the characteristic of a vortex in uncharged superfluid. The $\mathbf{J}_1$ of an AV obeys Eq. (22), i.e., $\nabla \times \mathbf{J}_1 = -\mathbf{H}_1/\lambda^2$ outside the core. We know in superfluid that the vortex is exerted by a hydrodynamic Magnus force from the passing current, which also obeys Eq. (1). The Magnus force on a vortex was classically derived in a uniform driving current perpendicular to the vortex axis by means of the momentum flux through a cylindrical boundary of radius r_1 [12]. Abrikosov's derivation may serve as another derivation of the Magnus force by means of the interaction energy. We note that in this case no external work is done with vortex movement since the effect of magnetic field is neglected.

3.4. Energy analysis, London force

It is not easy to explain why the driving force Eq. (1) to be opposite to the Lorentz force Eq. (27). After the numerical calculations on planar Josephson junction arrays with a centered JV [13–15], it was found that Eq. (1) was also valid for the driving force on the JV by a transport current even if the magnetic effects were completely neglected. Thus, it was highlighted that the driving force in this case was not a magnetic Lorentz force as universally referred [16]. On the other hand, it was noticed that Eq. (1) may be written as

$$\mathbf{F} = -\pi\hbar n\mathbf{v} \times \mathbf{k}, \tag{29}$$

where n is the number density of superconducting electrons, $\mathbf{v}$, the velocity of the superconducting electrons at the AV core, and $\mathbf{k}$, the unit vector in the $\mathbf{\Phi}_0$ direction. Since no electromagnetic quantities but velocity of electron movements appear in Eq. (29), the driving force should be a hydrodynamic Magnus force but not an electromagnetic Lorentz force [10,17].

However, the above arguments become illogical when the energy of the AV is concerned. The total energy E_{AV} of an AV and its magnetic part $E_{\text{AV},m}$ are approximated by [2]

$$E_{\text{AV}} = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2}\ln\kappa = 2E_{\text{AV},m}\ln\kappa. \tag{30}$$

From Eq. (30), we see that although $E_{\text{AV},m}/E_{\text{AV}} \to 0$ when $\kappa \to \infty$, it is still as large as 11% when $\kappa = 100$ for extreme type-II superconductors. Therefore the magnetic energy cannot be, as required, completely neglected for all the actual superconductors. After the present calculations, the above arguments even become wrong, since magnetic interaction can never be neglected even for the idealized case of $\kappa \to \infty$. For the actually two AV case shown in Fig. 2, we see that f is dominated by f_k when $x_{01} \ll \lambda$ but by f_m when $x_{01} > \lambda$. For the one AV case in Fig. 4, we see that both $|f_k|$ and $|f_m|$ may be much larger than f when $x_{01} < \lambda$, whereas when $x_{01} > \lambda, f_k + f_m \approx 0$ and $f \approx f_e$ hold. Since all these results are practically κ independent, magnetic energy plays an important and sometimes, decisive role in the resultant driving force.

Therefore, the driving force on AV is not a Lorentz force, which is not because of a kinetic energy dominance as argued before; the driving force is not a Magnus force even in the high κ limit, although both share the same expression, valid for the vortex and driving current being embedded in a single condensate [7]. With all the unique properties of the driving force derived from the London equation, a name of London force has been suggested for it for correctness and clarity [5,10,11]. Another name could be the de Gennes force, since the force was first derived by de Gennes for both the one and two AV cases.

We should mention that although the force on a JV is a Magnus force if magnetic effects are neglected as in [13–15], it will be a London force if both kinetic and magnetic field energies are considered, as calculated for a JV centered in a square-bar Josephson junction array [18].

3.5. The accuracy of Eq. (1)

When Eq. (1) was derived analytically by transforming volume integration to surface integration, a number of approximations had to be made [2]. It is unclear how big error will be introduced to the resultant Eq. (1) by these approximations. In our numerical calculations the original volume integration is directly used without such a transformation, so that the error of Eq. (1) may be estimated by comparing with our results. We conclude that Eq. (1) is very accurate except for the case of two AV cores to be very close to each other or equivalently, the one AV core to be very close to the surface, where the correct force may be lower than that calculated from Eq. (1) by a maximum 5%. However, since the property of AV core itself is over-simplified by the London model, such an error cannot be taken seriously, so that we may at least state that Eq. (1) is practically very accurate. The oscillation of f in Fig. 4 might be due to the errors in numerical calculation; an error of about 0.5% in the negative and positive forces from the image and the screening current leads to a huge relative error in the total force when it approaches zero. However, the oscillation may also be a true phenomenon, since, with a huge number of cells, the numerical error cannot be as large as 0.5%.

3.6. Internal and external forces

A remarkable feature in Fig. 4 is the high-x_{01} limit of the forces to be $f_k + f_m = 0$ and $f = f_e$. From Eq. (18), we calculate the second term to be less than 1% of the first term when $x_{01}/\lambda > 2.3$. Therefore, in the above limit the force is actually caused by screening current alone; this current drives the AV entirely by external work, whereas the internal energy has no effect on it. We may call this driving force as the external force. The origin of this force is the external electromagnetic work, but from the energy transport point of view, we may say that the applied field induces the screening current containing both magnetic and kinetic energies, and the screening current acts on the AV by both interactions with an magnetic domination similar to the high x_{01} case in Fig. 2.

In contrast to the external force exerted by the screening current, the force due to the image current is internal, since it is completely determined by the internal interaction between the AV and the surface after the AV creation [18,19].

It is interesting to point out that without considering external work as in [4,5], the result should be $f = 0$ but not 0.5 as mentioned above, if the field and current density distributions of the AV have been calculated by the boundary-value problem. The result of $f = 0.5$ in Fig. 1 is calculated under an over-simplified assumption where H_1 and $\mathbf{J}_1$ for a complete AV are used for the incomplete AV.

4. Importance of distinction between London and Lorentz forces

After Anderson's prediction of the driving force on an AV to be an electromagnetic Lorentz force [8,9], the same idea has been described in many books on superconductivity [20–27]. The identity between the supercurrent driving force and the electromagnetic Lorentz force is explicitly stated in some cases [24,27], and a sign change is made in some other cases [25]. Although the superconducting driving force is derived from London equation in several books, it is still regarded as the Lorentz force, a special case of the Lorentz force, or a so-called Lorentz force, without describing its essential differences in the meaning and direction from the electromagnetic Lorentz force [3,28,29].

The Lorentz force involved here is the magnetic part of the classical electromagnetic force derived by Lorentz in 1892 [30]. It was explicitly used for the driving force on vortices after 1964 [9]. For simplicity, the name of London force will be used below for the driving force of supercurrent acting on vortices expressed by Eq. (1), following [5,10,11]. Misnaming a London force as a Lorentz force may have important impacts on the study of superconductivity. A few examples are given below.

In many cases, London and Lorentz forces coexist in a system being studied so that they cannot be both called Lorentz force. As mentioned above, there are a driving London force and a damping Lorentz force acting on the moving JV along a resistively shunted long Josephson junction fed by a transport supercurrent [11], and both forces must be studied separately based on their own law. Another example is a Maglev system, where a bulk superconductor is placed above a permanent magnet. In this system, the force between the circulating supercurrents and the magnetic induction produced by the magnet is a Lorentz force, and the force between the same supercurrents and the AVs (or internal induction) is a London force. The opposite directions of both forces acting on the supercurrents are essential for the levitation force transmission from the magnet to supercurrents to the AV pinning centers and finally to the superconductor body.

Since supercurrents obey complicated laws, the London force has been frequently treated simply as a Lorentz force in the literature, but such treatments always contain an error in sign. For example, using a stream function $g(x, y)$ defined as the local magnetization by $\mathbf{J} = -\mathbf{k} \times \nabla g(x, y)$, where $\mathbf{k}$ is the unit vector in the direction of the applied field perpendicular to the film, to calculate electromagnetic properties of superconducting films, Brandt states that a vortex with flux $\mathbf{\Phi}_0$ in the $\mathbf{k}$ direction moves in the magnetic potential $V = -\mathbf{\Phi}_0 g(x, y)$, since the Lorentz force on a vortex is $-\mathbf{J} \times \mathbf{\Phi}_0 = \cdots = -\nabla V$ [31]. It is surprising that a correct direction of vortex movement is obtained by regarding the London force as an Lorentz force obeying Eq. (27). However, the sign of $g(x, y)$ should be negative in a positive applied field but not positive as shown in the paper.

Gurevich et al. have studied theoretically the flux flow of a linear array of Abrikosov–Josephson vortices (AJVs) along a grain boundary in high-temperature superconductors [32]. A normalized transport current density $\beta = J/J_b$, J and J_b being the current density and critical-current density, respectively, across the boundary, is included in their nonlocal equation for the guage-invariant phase difference on a grain boundary. Since β is not the normalized density of dc or ac Josephson current already included in the equation, J should not be the density of a supercurrent; it can be realized by checking the sign of each term in the equation that $-J$ should be the density of an artificially added normal current. Owing to such an error in sign, the resultant Lorentz force of J acting on AJVs turns out to have the same direction as a London force if a supercurrent of density $-J$ could be applied. Similar situations occur in many researches on the movements of JVs along a long Josepson junction, as explained in [11]. An important problem here is how a uniform net transport supercurrent can flow across a uniform long Josephson junction, for which the dc Josephson equation or the sine-Gordon equation can predict a net current near the edges only [18,33,34]. This difficult and essential problem has been bypassed in all previous works of such a kind, so that the London force has been incorrectly treated as a Lorentz force with a sign change, leading to important errors in the results.

5. Conclusion

After de Gennes and Matricon derived the driving force on an AV, $\mathbf{F} = f\mathbf{J}(\mathbf{r}_{01}) \times \mathbf{\Phi}_0$ with $f = 1$, from the interaction energy between two parallel AVs [1], Abrikosov and some other authors have treated systems with an AV driven by arbitrary external current [3–5]. They also used the interaction energy with the same result obtained. This is in contradiction to de Gennes' another work [2], where the same driving force was derived from Gibbs potential but not from internal energy for an AV parallel to the surface of a half-space superconductor. To seek for the reason for this, we have calculated numerically for this case with kinematic, magnetic, and external forces and boundary effects being studied separately. It is concluded that $f = 1$ occurs correctly only when a correct Gibbs potential is used, with the boundary effects and external work being taken into consideration. If the external work and boundary effects are incorrectly neglected, f will become 0.5; if the boundary effects on the field and current distribution of the AV is considered but the external work is incorrectly neglected, f can even become 0.

For a type-II superconductor with very large κ, the driving force is dominated by magnetic force when the distance between the AV core and the boundary is larger than λ and when the applied field is zero, although magnetic contribution to the AV energy is negligible. If the applied field is nonzero, the dominant driving force is kinematic when the AV core is distant from the boundary up to a few coherent lengths, but at a distance larger than λ, the force becomes purely external, with mutually cancellated positive magnetic and negative kinematic contributions.

The driving force consists of both magnetic and kinematic forces in general, so that it cannot be regarded as a magnetic Lorentz force or a kinematic Magnus force and we suggest calling it the London force to be distinguished from other forces. Since the London force and Magnus force share the same expression, $f = 1$ derived by Abrikosov [3], when neglecting magnetic effects, is actually for a Magnus force but not for a London force. After Narayan's interesting attempt [7], it remains to be a challenge to derive the London force in a general case including boundary effects and external work.

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