On Reproducing Linear Estimators within the GM-Model with Stochastic Constraints

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Introducing the GM-Model with Stochastic (Fiducial) Constraints

$$y = A\xi + e,\ z_0 = K\xi + e_0,\ \begin{bmatrix} e \\ e_0 \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \begin{bmatrix} P^{-1} & 0 \\ 0 & Q_0 \end{bmatrix} \right),$$
$\ell = rk\, K \leq m = rk\, [A^T, K^T] < n,\ Q_0$ possibly *singular (p.s.d.)*.

Note that $e_0 \in \mathcal{R}\,(Q_0)$ with probability 1, implying that a vector λ exists such that $e_0 = -Q_0\lambda$.

Least-Squares principle:
$e^T Pe + \lambda^T Q_0 \lambda_0 = \min$ s.t. the above model

LEast-Squares Solution (LESS)

$$\begin{bmatrix} N & K^T \\ K & -Q_0 \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ \hat{\lambda} \end{bmatrix} = \begin{bmatrix} c \\ z_0 \end{bmatrix} \text{ for } \begin{bmatrix} N & c \end{bmatrix} := A^T P \begin{bmatrix} A & y \end{bmatrix}$$

The solution is *unique* since $rk \begin{bmatrix} A^T & K^T \end{bmatrix} = m$, with:

$$\sigma_0^2 \begin{bmatrix} N & K^T \\ K & -Q_0 \end{bmatrix}^{-1} = \begin{bmatrix} D\left\{\hat{\xi}\right\} & \mathrm{X}^T \\ \mathrm{X} & -D\left\{\hat{\lambda}\right\} \end{bmatrix},\ C\left\{\hat{\xi}, \hat{\lambda}\right\} = 0, \text{ and}$$

$\hat{\sigma}_0^2(n - m + \ell) = y^T P y - c^T \hat{\xi} - z_0^T \hat{\lambda} = \tilde{e}^T P \tilde{e} + \hat{\lambda}^T Q_0 \hat{\lambda}$ where

$\tilde{e} = y - A\hat{\xi} \sim (0, D\{y\} - A \cdot D\left\{\hat{\xi}\right\} \cdot A^T)$ and

$\tilde{e}_0 = z_0 - K\hat{\xi} = -Q_0\hat{\lambda} \sim (0, Q_0 \cdot D\left\{\hat{\lambda}\right\} \cdot Q_0)$,

$C\{\tilde{e}, \tilde{e}_0\} = -C\left\{y, \hat{\lambda}\right\} \cdot Q_0 = -AX^T Q_0$

Decomposition of the SSR:

$\tilde{e}^T P \tilde{e} + \hat{\lambda}^T Q_0 \lambda = \Omega + R(Q_0)$ where:

$\Omega = y^T P y - c^T N^- c,\ R(Q_0) = c^T(N^- c - \hat{\xi}) - z_0^T \hat{\lambda}$

Simplification if $P_0 = Q_0^{-1}$ exists

$$e^T P e + e_0^T P_0 e_0 = \min . \text{ s.t. the above model}$$

$$\Longrightarrow \boxed{N_K \hat{\xi} = c + K^T P_0 z_0} \text{ for } N_K := N + K^T P_0 K$$

$$\Longrightarrow \hat{\xi} = N_K^{-1}(c + K^T P_0 z_0) \sim (\xi, \sigma_0^2 N_K^{-1})$$

<u>with</u>: $\hat{\sigma}_0^2(n - m + \ell) = y^T P y - c^T \hat{\xi} - z_0^T \hat{\lambda} = \tilde{e}^T P \tilde{e} + \tilde{e}_0^T P_0 \tilde{e}_0 = 0$, if

$\hat{\lambda} = -P_0 \tilde{e}_0 = -P_0(z_0 - K\hat{\xi}) \sim (0, \sigma_0^2[P_0 - P_0 K N_K^{-1} K^T P_0]), C\left\{\hat{\xi}, \hat{\lambda}\right\} = 0,$

$$\left.\begin{array}{l} \tilde{e}_0 = z_0 - K\hat{\xi} \sim (0, \sigma_0^2[Q_0 - K N_K^{-1} K^T]) \\ \tilde{e} = y - A\hat{\xi} \sim (0, \sigma_0^2[P^{-1} - A N_K^{-1} A^T]) \end{array}\right\} \Rightarrow A^T P \tilde{e} + K^T P_0 \tilde{e}_0 = 0,$$

$$C\{\tilde{e}, \tilde{e}_0\} = C\{y, \tilde{e}_0\} = -C\left\{y, K\hat{\xi}\right\} = -\sigma_0^2 A N_K^{-1} K^T$$

<u>Decomposition of the SSR:</u>

$$\tilde{e}^T P \tilde{e} + \tilde{e}_0^T P_0 \tilde{e}_0 = y^T P \tilde{e} + z_0^T P_0 \tilde{e}_0 + \hat{\xi}^T (A^T P \tilde{e} + K^T P_0 \tilde{e}_0) =$$

$$= (y^T P y - c^T N^- c) + c^T (N^- c - \hat{\xi}) - z_0^T \hat{\lambda} = \Omega + R(Q_0)$$

Definition of Reproducing Estimates

$\bar{\xi} = L_1 \cdot y + L_2 \cdot z_0$ such that $K\bar{\xi} = z_0 \sim (K\xi, \sigma_0^2 Q_0)$
Note: $K\hat{\xi} = KN_K^{-1}(c + K^T P_0 z_0) = z_0 - \tilde{e}_0 \neq z_0$ (in general)
with $D\left\{K\hat{\xi}\right\} = \sigma_0^2 \cdot KN_K^{-1}K^T \neq \sigma_0^2 Q_0$ (in general)

Ad-hoc attempt:

$$\begin{bmatrix} N & K^T \\ K & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}_c \\ \hat{\lambda}_c \end{bmatrix} = \begin{bmatrix} c \\ z_0 \end{bmatrix} \text{ in spite of } Q_0 \neq 0$$

$$\Longrightarrow \begin{bmatrix} N_K & K^T \\ K & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}_c \\ \hat{\lambda}_c \end{bmatrix} = \begin{bmatrix} c + K^T P_0 z_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} N_K \hat{\xi} \\ z_0 \end{bmatrix}$$

$$\Longrightarrow \hat{\xi}_c = \hat{\xi} - N_K^{-1}K^T\hat{\lambda}_c \text{ and } z_0 = K\hat{\xi}_c = K\hat{\xi} - KN_K^{-1}K^T \cdot \hat{\lambda}_c$$

$$\Longrightarrow \hat{\lambda}_c = -(KN_K^{-1}K^T)^{-1}(z_0 - K\hat{\xi})$$

$$\Longrightarrow \boxed{\hat{\xi}_c = \hat{\xi} + N_K^{-1}K^T(KN_K^{-1}K^T)^{-1}(z_0 - K\hat{\xi})} \Longrightarrow K\hat{\xi}_c = z_0$$

Characterizing the ad-hoc Reproducing Estimate

$$\hat{\xi}_c = \hat{\xi} + N_K^{-1}K^T(KN_K^{-1}K^T)^{-1}\cdot \tilde{e}_o \quad \text{for } \tilde{e}_o = z_0 - K\hat{\xi}$$

$$\Rightarrow \hat{\xi}_c \sim (\xi,$$

$$\sigma_0^2\left[N_K^{-1} + N_K^{-1}K^T(KN_K^{-1}K^T)^{-1}\cdot D\left\{\tilde{e}_0\right\}\cdot(KN_K^{-1}K^T)^{-1}KN_K^{-1}\right])$$

$$\Rightarrow D\left\{\hat{\xi}_c\right\} = \sigma_0^2\left[N_K^{-1} - N_K^{-1}K^T(KN_K^{-1}K^T)^{-1}KN_K^{-1} + \right.$$

$$\left. + N_K^{-1}K^T(KN_K^{-1}K^TP_0KN_K^{-1}K^T)^{-1}KN_K^{-1}\right]$$

$$\Rightarrow D\left\{\hat{\xi}_c\right\} - D\left\{\hat{\xi}\right\} = \sigma_0^2\cdot N_K^{-1}K^T(KN_K^{-1}K^T)^{-1}\cdot$$

$$\cdot\left[Q_0 - K(N + K^TP_0K)^{-1}K^T\right](KN_K^{-1}K^T)^{-1}KN_K^{-1} =$$

$$= \sigma_0^2\cdot N_K^{-1}K^T(KN_K^{-1}K^T)^{-1}Q_0\cdot$$

$$\cdot\left[P_0 - P_0KN_K^{-1}K^TP_0\right]Q_0(KN_K^{-1}K^T)^{-1}KN_K^{-1}$$

is n.n.d. and describes the "*loss in efficiency*".

Characterizing the ad-hoc Reproducing Estimate (cont'd)

Furthermore:

$$\hat{\lambda}_c = -(KN_K^{-1}K^T)^{-1} \cdot \tilde{e}_0 \sim \left(0, (KN_K^{-1}K^T)^{-1} \cdot D\left\{\tilde{e}_0\right\} \cdot (KN_K^{-1}K^T)^{-1}\right)$$

$$\begin{aligned}
\Rightarrow D\left\{\hat{\lambda}_c\right\} &= \sigma_0^2\left[(KN_K^{-1}K^T)^{-1}\left[Q_0 - (KN_K^{-1}K^T)\right](KN_K^{-1}K^T)^{-1}\right] = \\
&= \sigma_0^2\left[(KN_K^{-1}K^TP_0KN_K^{-1}K^T)^{-1} - (KN_K^{-1}K^T)^{-1}\right] = \\
&= \sigma_0^2(KN_K^{-1}K^T)^{-1}K^TN_K^{-1}A^T \cdot \\
&\quad \cdot\left[P^{-1} - AN_K^{-1}K^T(KN_K^{-1}K^T)^{-1}KN_K^{-1}A^T\right]^{-1} \cdot \\
&\quad \cdot AN_K^{-1}K^T(KN_K^{-1}K^T)^{-1}
\end{aligned}$$

and:

$$C\left\{\hat{\xi}_c, \hat{\lambda}_c\right\} = C\left\{\hat{\xi}, \hat{\lambda}_c\right\} - N_K^{-1}K^T(KN_K^{-1}K^T)^{-1} \cdot D\left\{\tilde{e}_0\right\} \cdot (KN_K^{-1}K^T)^{-1} =$$

$$= 0 - N_K^{-1}K^T \cdot D\left\{\hat{\lambda}_c\right\} \neq 0 \quad \text{(in general)}$$

More Characteristics of the ad-hoc Reproducing Estimate

$$\tilde{e}_{0c} = z_0 - K\hat{\hat{\xi}}_c = 0\checkmark \Rightarrow D\left\{K\hat{\hat{\xi}}_c\right\} = K \cdot D\left\{\hat{\hat{\xi}}_c\right\}K^T = \sigma_0^2 Q_0 = D\{z_0\}\checkmark$$

$$\tilde{e}_c = y - A\hat{\hat{\xi}}_c = \tilde{e} - AN_K^{-1}K^T(KN_K^{-1}K^T)^{-1} \cdot \tilde{e}_0 \sim (0, D\{\tilde{e}_c\}) \text{ with}$$

$$\begin{aligned}
D\{\tilde{e}_c\} = \; & D\{\tilde{e}\} - AN_K^{-1}K^T(KN_K^{-1}K^T)^{-1} \cdot C\{\tilde{e}_0, \tilde{e}\} - \\
& -C\{\tilde{e}, \tilde{e}_0\} \cdot (KN_K^{-1}K^T)^{-1}KN_K^{-1}A^T + \\
& +AN_K^{-1}K^T(KN_K^{-1}K^T)^{-1} \cdot D\{\tilde{e}_0\} \cdot (KN_K^{-1}K^T)^{-1}KN_K^{-1}A^T = \\
= \; & \sigma_0^2\left[P^{-1} + AN_K^{-1}K^T(KN_K^{-1}K^T)^{-1}KN_K^{-1}A^T - AN_K^{-1}A^T + \right. \\
& \left. +AN_K^{-1}K^T(KN_K^{-1}K^T)^{-1}Q_0(KN_K^{-1}K^T)^{-1}KN_K^{-1}A^T\right] = \\
= \; & D\{y\} - D\left\{A\hat{\hat{\xi}}_c\right\} + \\
& 2 \cdot AN_K^{-1}K^T(KN_K^{-1}K^T P_0 KN_K^{-1}K^T)^{-1}KN_K^{-1}A^T, \;\; C\{\tilde{e}_c, \tilde{e}_{0c}\} = 0;
\end{aligned}$$

<u>such that</u>:

$$\begin{aligned}
\hat{\sigma}_{0c}^2[n - m + tr(Q_0(KN_K^{-1}K^T)^{-1})] &= \tilde{e}_c^T P \tilde{e}_c = \\
&= (\tilde{e}^T P \tilde{e} + \tilde{e}_0^T P_0 \tilde{e}_0) + \tilde{e}_0^T (KN_K^{-1}K^T)^{-1}\tilde{e}_0
\end{aligned}$$

More Characteristics of the ad-hoc Reproducing Estimate (cont'd)

Decomposition of the SSR:

$$\tilde{e}_c^T P \tilde{e}_c^T = (y^T P y - c^T N^- c) +$$
$$+ c^T (N^- c - \hat{\xi}) + z_0^T P_0 \tilde{e}_0 + \tilde{e}_0^T (K N_K^{-1} K^T)^{-1} \tilde{e}_0 = \Omega + R_0$$

$$\text{for } R_0 := c^T (N^- c - \hat{\xi}_c) - c^T N_K^{-1} K^T \hat{\lambda}_c - z_0^T P_0 (K N_K^{-1} K^T) \hat{\lambda}_c$$
$$- \left[z_0 - K N_K^{-1} (c + K^T P_0 z_0) \right]^T \hat{\lambda}_c = c^T (N^- c - \hat{\xi}_c) - z_0^T \hat{\lambda}_c$$

Obviously: $\boxed{\hat{\xi} \to \hat{\xi}_c \text{ if } Q_0 \to 0}$

thereby establishing some sort of *consistency* as $\hat{\xi}_c$ would then turn into the standard LESS with the *usual representations* for variances and covariances:

$$\left. \begin{array}{l} D\left\{\hat{\xi}\right\} \to \sigma_0^2 \left[N_K^{-1} - N_K^{-1} K^T (K N_K^{-1} K^T)^{-1} K N_K^{-1} \right] \\ \hat{\lambda} \to \hat{\lambda}_c \ \textit{and} \ \tilde{e}_0 \to \tilde{e}_{0c} \\ \hat{\sigma}_0^2 (n - m + \ell) \to \tilde{e}_c^T P \tilde{e}_c = \hat{\sigma}_{0c}^2 (n - m + \ell) \end{array} \right\} \textit{if } \ Q_0 \to 0$$

But: $\boxed{\text{Is } \hat{\xi}_c \text{ really the BEST reproducing estimate?}}$

Defining the repro-BLUUE

(i) "linear": $\bar{\xi} = L_1 y + L_2 z_0$, L_1 and L_2 to be determined;

(ii) "uniformly unbiased": $\xi = E\left\{\bar{\xi}\right\} = (A_1 A)\xi + (L_2 K)\xi$ for all $\xi \in \mathbb{R}^m \;\Leftrightarrow\; \boxed{L_1 A + L_2 K = I_m}$

(iii) "reproducing": $0 = K\bar{\xi} - z_0 = (K_1 L_1)y + (K_2 L_2 - I_\ell)z_0$ for arbitrary $y \in \mathbb{R}^n, z_0 \in \mathbb{R}^\ell \Leftrightarrow$
$\boxed{K \begin{bmatrix} L_1 & L_2 \end{bmatrix} = \begin{bmatrix} 0 & I_\ell \end{bmatrix}} \overset{(ii)}{\Rightarrow} A^T L_1^T K^T = 0$

(iv) "best" $\hat{=}$ "minimum MSE/dispersion":
$\sigma_0^{-2} \cdot tr\, D\left\{\bar{\xi}\right\} = tr(L_1 P^{-1} L_1^T) + tr(L_2 Q_0 L_2^T) = \min$
subject to (ii) and (iii)

Deriving the repro-BLUUE

$\ell_1 := vec\, L_1^T,\ \ell_2 := vec\, L_2^T \Rightarrow$

$$\Rightarrow \phi(\ell_1, \ell_2, \lambda_1, \lambda_2) :=$$
$$:= \ell_1^T (I_m \otimes P^{-1}) \ell_1 + \ell_2^T (I_m \otimes Q_0) \ell_2 + 2\lambda_1^T (K \otimes A^T) \ell_1 +$$
$$+ 2\lambda_2^T \left[(I_m \otimes A^T) \ell_1 + (I_m \otimes K^T) \ell_2 - vec\, I_m \right] =$$

stationary

where $\otimes$ denotes the Kronecker-Zehfuss product.

Solution:

$$\boxed{L_2 = K^T (KK^T)^{-1} + \left[I_m - K^T (KK^T)^{-1} K \right] N^{-1} K^T (Q_0 + KN^{-1} K^T)^{-1}}$$

$\boxed{L_1 = (I_m - L_2 K) N^{-1} A^T P}$ if N^{-1} exists

$$\Rightarrow \bar{\xi} = \hat{\xi} + K^T (KK^T)^{-1} (z_0 - K\hat{\xi})$$

where $\hat{\xi}$ *is LESS* (independent of N^{-1})

Characteristics of the repro-BLUUE

$$\bar{\xi} = \hat{\xi} + K^T(KK^T)^{-1}(z_0 - K\hat{\xi}),\ K\bar{\xi} = z_0\ \checkmark$$

$$\begin{aligned}\Rightarrow D\{\bar{\xi}\} &= \sigma_0^2 Q_{\bar{\xi}} = D\{\hat{\xi}\} + D\{K^T(KK^T)^{-1}(z_0 - K\hat{\xi})\} = \\ &= D\{\hat{\xi}\} + \sigma_0^2 \underbrace{K^T(KK^T)^{-1}Q_0(Q_0 + KN^{-1}K^T)^{-1}Q_0(KK^T)^{-1}K}_{\text{symmetric, positive-(semi)definit (if } N^{-1} exists)}\end{aligned}$$

$$\Rightarrow D\{\bar{\xi}\} - D\{\hat{\xi}\} = \sigma_0^2 K^T(KK^T)^{-1}(Q_0 - KN_K^{-1}K^T)(KK^T)^{-1}K$$

if Q_0^{-1} exists

<u>Furthermore</u>: $\bar{\sigma}_0^2 \cdot (n - m + \ell + tr(NQ_{\bar{\xi}})) = (y - A\bar{\xi})^T P(y - A\bar{\xi}) =$
$= \tilde{e}^T P\tilde{e} - 2(A^T P\tilde{e})^T K^T(KK^T)^{-1}\tilde{e}_0 + \tilde{e}_0^T(KK^T)^{-1}KNK^T(KK^T)^{-1}\tilde{e}_0 =$
$= (\tilde{e}^T P\tilde{e} + \tilde{e}_0^T P_0\tilde{e}_0) + \tilde{e}_0^T\left[P_0 + (KK^T)^{-1}KNK^T(KK^T)^{-1}\right]\tilde{e}_0 =$
$= \hat{\sigma}_0^2(n - m + \ell) + \hat{\lambda}^T Q_0\hat{\lambda} + \hat{\lambda}^T Q_0(KK^T)^{-1}(KNK^T)(KK^T)^{-1}Q_0\hat{\lambda}$

<u>Obviously</u>: $\boxed{D\{\hat{\xi}\} \leq_L D\{\bar{\xi}\}}$ <u>and</u> $\boxed{tr\, D\{\bar{\xi}\} \leq tr\, D\{\hat{\xi}_c\}}$

Traditional Approach by Means of HELMERT's Transformation

$\overline{\xi}_{HT} := (N + SE^T Q_K^{-1} ES)^{-1} c \in \{N^- c\}$ for
$Q_K := ESK^T (KK^T)^{-1} Q_0 (KK^T)^{-1} KSE^T, \; rk\, Q_K = m - q \leq k \leq$
$\leq rk\, Q_0 = \ell,$

where E is a $(m - q) \times m$ matrix such that
(i) $NE^T = 0$ and (ii) $rk\, N = q = m - rk\, E$;
and S is a chosen "selection matrix",
$S = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$, such that $rk(ES) = m - q = rk\, E \leq k = rk\, S$

$$\begin{aligned} \Rightarrow D\left\{\overline{\xi}_{HT}\right\} &= \sigma_0^2 (N + SE^T Q_K^{-1} ES)^{-1} N (N + SE^T Q_K^{-1} ES)^{-1} = \\ &= \sigma_0^2 (N + SE^T Q_K^{-1} ES)^{-1} - \sigma_0^2 E^T (ESE^T Q_K^{-1} ESE^T)^{-1} E \\ &= \sigma_0^2 Q_{\overline{\xi}_{HT}} \in \left\{\sigma_0^2 N_{rs}^-\right\} \end{aligned}$$

Note: $(ES)\overline{\xi}_{HT} = 0$, but: $K\overline{\xi}_{HT} \neq z_0$ (in general)

Traditional Approach by Means of HELMERT's Transformation (cont'd)

Also: $E\left\{\overline{\xi}_{HT}\right\} = (N + SE^T Q_K^{-1} SE)^{-1} N \cdot \xi = \xi - E^T(ESE^T)^{-1} ES \cdot \xi$
$\Rightarrow \overline{\xi}_{HT}$ is biased

and: $\tilde{e}_{HT} = y - A\overline{\xi}_{HT} \sim (0, \sigma_0^2 \left[P^{-1} - A(N + SE^T Q_K^{-1} ES)^{-1} A^T\right]),$

$$A^T P \tilde{e}_{HT} = 0,$$

$$\tilde{e}_{0HT} := z_0 - K\overline{\xi}_{HT} \sim (KE^T(ESE^T)^{-1} ES \cdot \xi, \sigma_0^2 Q_0 + \sigma_0^2 K Q_{\overline{\xi}_{HT}} K^T),$$

$$C\left\{\tilde{e}_{0HT}, \tilde{e}_{HT}\right\} = -K \cdot C\left\{\overline{\xi}_{HT}, \tilde{e}_{HT}\right\} = \\ = -\sigma_0^2 K(N + SE^T Q_K^{-1} ES)^{-1} A^T + K \cdot D\left\{\overline{\xi}_{HT}\right\} \cdot A^T = 0$$

Obviously:

$$\hat{\sigma}_{0HT}^2 \cdot (n - q) = y^T P y - c^T \overline{\xi}_{HT} = \tilde{e}_{HT}^T P \tilde{e}_{HT} = y^T P y - c^T N^- c = \Omega$$

Forgetful Helmert Transformation as Reproducing Estimate

$$\boxed{\bar{\xi}_{FHT} = \bar{\xi}_{HT} + K^T(KK^T)^{-1}(z_0 - K\bar{\xi}_{HT})}$$
$$\Rightarrow \; K\bar{\xi}_{FHT} = z_0\checkmark, \; D\left\{K\bar{\xi}_{FHT}\right\} = \sigma_0^2 Q_0\checkmark$$

Furthermore: $E\left\{\bar{\xi}_{FHT}\right\} = E\left\{\bar{\xi}_{HT}\right\} + K^T(KK^T)^{-1} \cdot E\left\{\tilde{e}_{0HT}\right\} =$

$$= \xi - \left[I_m - K^T(KK^T)^{-1}K\right] \cdot E^T(ESE^T)^{-1}ES \cdot \xi,$$

$$D\left\{\bar{\xi}_{FHT}\right\} = \sigma_0^2 \left[I_m - K^T(KK^T)^{-1}K\right] Q_{\bar{\xi}_{HT}} \cdot$$
$$\cdot \left[I_m - K^T(KK^T)^{-1}K\right] + \sigma_0^2 K^T(KK^T)^{-1}Q_0(KK^T)^{-1}K,$$

$$MSE\left\{\bar{\xi}_{FHT}\right\} =$$
$$= D\left\{\bar{\xi}_{FHT}\right\} + E\left\{\bar{\xi}_{FHT} - \xi\right\} \cdot E\left\{\bar{\xi}_{FHT} - \xi\right\}^T$$

Forgetful Helmert Transformation as Reproducing Estimate (cont'd)

$$\tilde{e}_{0FHT} = z_0 - K\bar{\xi}_{FHT} = 0,\ \tilde{e}_{FHT} = y - A\bar{\xi}_{FHT} = \tilde{e}_{HT} - AK^T(KK^T)^{-1} \cdot \tilde{e}_{0HT}$$

$$\Rightarrow E\{\tilde{e}_{FHT}\} = -K^T(KK^T)^{-1} \cdot E\{\tilde{e}_{0HT}\} = = -K^T(KK^T)^{-1}K \cdot E^T(ESE^T)^{-1}ES \cdot \xi,$$

<u>and</u>: $D\{\tilde{e}_{FHT}\} = D\{\tilde{e}_{HT}\} + K^T(KK^T)^{-1} \cdot D\{\tilde{e}_{0HT}\} \cdot (KK^T)^{-1}K$

<u>Obviously</u> (if $ES \cdot \xi = 0$ is indeed fulfilled):

$$\hat{\sigma}^2_{0FHT} \cdot (n - q + tr\left[NK^T(KK^T)^{-1}(Q_0 + KQ_{\bar{\xi}_{HT}}K^T)(KK^T)^{-1}K\right]) =$$

$$= (y - A\bar{\xi}_{FHT})^T P(y - A\bar{\xi}_{FHT}) =$$

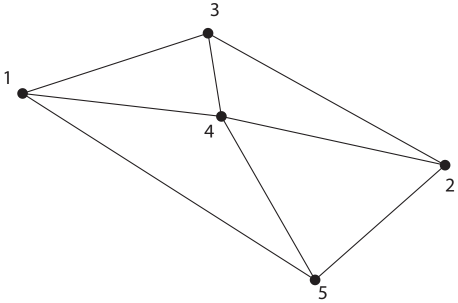
$$= \underbrace{\tilde{e}^T_{HT}P\tilde{e}_{HT}}_{=\Omega} - 2\underbrace{(A^TP\tilde{e}_{HT})^T}_{=0} \cdot K^T(KK^T)^{-1}\tilde{e}_{0HT} +$$

$$+ \underbrace{\tilde{e}^T_{0HT}(KK^T)^{-1}KNK^T(KK^T)^{-1}\tilde{e}_{0HT}}_{=R_{FHT}}$$

<u>and</u>: $\boxed{D\left\{\hat{\xi}\right\} \leq_L D\{\bar{\xi}_{FHT}\},\ tr\, D\{\bar{\xi}\} \leq D\{\bar{\xi}_{FHT}\}}$

Example: Level Network (Reissmann, 1976, p. 145)

- 5 points
- 7 level observations, known weights, no correlations
- 3 constraints
  - 2 previously estimated point heights (points 1 and 2)
  - a previously estimated height difference (between points 3 and 4)



Example: Incremental Parameter Estimates $[cm]$

Point	LESS	ad-hoc repEst.	repro-BLUUE	HT	repro-FHT
1	-0.69	0.00	0.00	-2.04	0.00
2	0.69	0.00	0.00	2.04	0.00
3	-2.03	-2.03	-2.23	-1.74	-2.58
4	0.58	0.97	0.77	-0.43	0.42
5	-8.82	-8.79	-8.82	-8.99	-8.99

Example: Cofactors $[cm^2]$, Sum of Squared Residuals, Variance Component Estimates

	LESS	ad-hoc repEst.	repro-BLUUE	HT	repro-FHT
Q_{11}	0.837	1.000	2.674	1.045	1.957
Q_{22}	0.837	1.000	2.674	1.045	1.957
Q_{33}	1.839	1.907	2.292	2.545	2.794
Q_{44}	1.752	1.811	2.205	1.975	2.794
Q_{55}	2.938	2.946	2.938	2.595	5.079
$tr\ Q$	8.204	8.664	12.783	9.205	14.581

	LESS	ad-hoc repEst.	repro-BLUUE	HT	repro-FHT
SSR	8.254	10.973	11.011	5.897	11.266
$\hat{\sigma}_0^2$	1.870	1.919	1.104	1.966	1.957

Example: Estimated Variances [cm^2]

Point	LESS	ad-hoc repEst.	repro-BLUUE	HT	repro-FHT
1	1.565	1.919	2.952	2.054	3.831
2	1.565	1.919	2.952	2.054	3.831
3	3.440	3.658	2.530	5.002	5.468
4	3.277	3.475	2.434	3.882	5.468
5	5.495	5.653	3.244	5.101	9.942

Conclusions

- In a GM-Model with stochastic constraints, *reproducing* estimates have been studied that are "variance preserving" for the constraints.
- Starting with the LESS and with an unconstrained solution that is based on Helmert's Transformation (HT), *two reproducing variants* have previously been proposed in the literature: the "ad-hoc" modification of the LESS and the Forgetful Helmert Transformation (FHT).
- It could be shown that neither solution is *optimal* among the reproducing Linear Uniformly Unbiased Estimates (repro-BLUUE), in the sense of minimizing the (average) Mean Square Error of the estimates.
- Practical formulas for the *repro-BLUUE* have successfully been derived and tested in a simple example ("levelling network").