Gentzen-type Refutation Systems for Three-Valued Logics*

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Abstract. While the purpose of a conventional proof calculus is to axiomatise the set of valid sentences of a given logic, a *refutation system*, or *complementary calculus*, is concerned with axiomatising the invalid sentences. Instead of exhaustively searching for counter models for some sentence, refutation systems establish invalidity by deduction and thus in a purely syntactic way. Such systems are relevant not only for proof-theoretic reasons but also for realising deductive systems for nonmonotonic logics. In this paper, we introduce Gentzen-type refutation systems for two basic three-valued logics that allow to embed well-known three-valued logics relevant for AI and logic programming like that of Kleene, Łukasiewicz, Gödel, as well as three-valued paraconsistent logics. As an application of our calculus, we provide derived rules for Gödel's three-valued logic, allowing to decide strong equivalence of logic programs under the answer-set semantics.

1 Introduction

In contrast to conventional proof calculi that axiomatise the valid sentences of a logic, *refutation systems*, or *complementary calculi*, are concerned with axiomatising the invalid sentences. Hence, the inference rules of such systems formalise the propagation of *refutability* instead of validity.

While the traditional method to show that a formula is not valid is exhaustive search for counter models, refutation systems establish invalidity by deduction and thus in a purely syntactic way. In fact, already the forefather of modern logic, Aristotle, studied rules that allow to reject assertions based on already rejected ones [1]. Refutation systems have been studied for different families of logics including classical logic, intuitionistic logic, modal logics, and many-valued logics (for more details, we refer to two overview articles on axiomatic refutation [2, 3]). Such systems showed to be relevant not only for proof-theoretic reasons but also for realising deductive systems for nonmonotonic logics. In particular, Bonatti and Olivetti [4] studied sequent-type calculi for default logic, autoepistemic logic, and propositional circumscription which combine a standard sequent-type calculus for classical logic and a dedicated complementary calculus [5]. Moreover, an explicit notion of a refutation proof allows for proof-theoretic

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investigations concerned with proof complexity, i.e., the size of proof representations. Certain speed-up results for nonmonotonic reasoning rely on explicit notions of refutation proofs [6].

In this paper, we introduce analytic Gentzen-type refutation systems for two functionally complete three-valued logics. In general, three-valued logics are relevant for AI since they naturally lend themselves for knowledge representation and reasoning involving vagueness and incomplete information [7]. Moreover, it is well-known that there are also interesting relations between three-valued logics and logic programming. Of particular note in this regard is the result of Lifschitz, Pearce, and Valverde [8], showing that strong equivalence between logic programs under the answer-set semantics coincides with equivalence in the logic of here-and-there [9], the latter logic being equivalent to Gödel's three-valued logic [10]. Moreover, similar to the work of Bonatti and Olivetti [4], our refutation systems can be used to axiomatise three-valued extensions of the major nonmonotonic formalisms due to Przymusinski [11]. Related to our approach, Bryll and Maduch [12] axiomatised the invalid sentences of Łukasiewicz's many-valued logics, including the three-valued case, by a Hilbert-type calculus. However, the calculus of Bryll and Maduch is not analytic and thus rather unsuitable for proof search in practice.

This paper is organised as follows. In Section 2, we provide some background on three-valued logics and present the two logics $\mathcal{L}$ and $\mathcal{P}$, which will be the central logics of our investigation. In Section 3, containing the main contribution of our paper, we introduce Gentzen-type refutation calculi for $\mathcal{L}$ and $\mathcal{P}$. Then, in Section 4, we discuss an application of our refutation systems in the realm of logic programming and nonmonotonic reasoning, and we conclude in Section 5.

2 Three-valued Logics

Unlike classical two-valued logic, three-valued logics admit a further truth value besides true and false. Let $\mathbf{t}$ and $\mathbf{f}$ stand for the classical truth values true and false and $\mathbf{i}$ for the third one. The intended meaning of $\mathbf{i}$ depends on the considered logic and can express, e.g., truth-value gaps as in Kleene logic [13] or indeterminate statements as in the logic of Łukasiewicz [14].

Notably, three-valued logics typically do not extend classical logic but are rival systems since classical theorems like $\psi \vee \neg\psi$, representing the principle of *tertium non datur*, are in general not preserved. The three-valued logics studied in the literature differ in the intended meaning of the truth values and the connectives that are considered. However, semantically, there are only two major classes of three-valued logics: those where $\mathbf{i}$ is *designated*, i.e., associated with truth, and those where $\mathbf{i}$ is not designated. In this paper, we are concerned with two logics, $\mathcal{L}$ and $\mathcal{P}$. Logic $\mathcal{L}$ can be considered as a prototypical logic where $\mathbf{i}$ is not designated, whilst $\mathcal{P}$ is a prototypical logic where $\mathbf{i}$ is designated. Both logics are fully expressive, meaning that they allow to embed any three-valued logic from the literature in it.

Both $\mathcal{L}$ and $\mathcal{P}$ are formulated over a countably infinite universe $\mathcal{U}$ of atoms including the truth constants T, F, and I. Based on the connectives $\neg$, $\vee$, $\wedge$, and $\supset$ for negation, disjunction, conjunction, and implication, respectively, the set of well-formed formulae is inductively defined as usual. Upper case Greek letters will denote sets of formulae whilst lower case Greek letters will stand for single formulae. A set of formulae is *consistent* if it does not contain both a formula and its negation.

In $\mathcal{P}$, $\mathbf{t}$ and $\mathbf{i}$ are designated, while in $\mathcal{L}$, the only designated truth value is $\mathbf{t}$. An interpretation is a mapping from $\mathcal{U}$ into $\{\mathbf{t}, \mathbf{f}, \mathbf{i}\}$. For any interpretation I, $I(\mathrm{T}) = \mathbf{t}$, $I(\mathrm{F}) = \mathbf{f}$, and $I(\mathrm{I}) = \mathbf{i}$. As usual, a *valuation* is a mapping from formulae into truth values. We assume the ordering $\mathbf{f} < \mathbf{i} < \mathbf{t}$ on the truth values in what follows. The valuation $v_{\mathcal{L}}^{I}$ of a formula in $\mathcal{L}$ given an interpretation I is is inductively defined as follows:

- $v_{\mathcal{L}}^{I}(\psi) = I(\psi)$, if ψ is an atomic formula,
- $v_{\mathcal{L}}^{I}(\neg\psi) = \mathbf{t}$ if $v_{\mathcal{L}}^{I}(\psi) = \mathbf{f}$, $\mathbf{f}$ if $v_{\mathcal{L}}^{I}(\psi) = \mathbf{t}$, and $\mathbf{i}$ otherwise,
- $v_{\mathcal{L}}^{I}(\psi \wedge \varphi) = \min(v_{\mathcal{L}}^{I}(\psi), v_{\mathcal{L}}^{I}(\varphi))$,
- $v_{\mathcal{L}}^{I}(\psi \vee \varphi) = \max(v_{\mathcal{L}}^{I}(\psi), v_{\mathcal{L}}^{I}(\varphi))$, and
- $v_{\mathcal{L}}^{I}(\psi \supset \varphi) = v_{\mathcal{L}}^{I}(\varphi)$ if $v_{\mathcal{L}}^{I}(\psi) = \mathbf{t}$, and $\mathbf{t}$ otherwise.

The valuation $v_{\mathcal{P}}^{I}$ of a formula in $\mathcal{P}$ given an interpretation I is defined analogously to $v_{\mathcal{L}}^{I}$ except for the implication:

- $v_{\mathcal{P}}^{I}(\psi \supset \varphi) = v_{\mathcal{P}}^{I}(\varphi)$ if $v_{\mathcal{P}}^{I}(\psi) = \mathbf{t}$ or $v_{\mathcal{P}}^{I}(\psi) = \mathbf{i}$, and $\mathbf{t}$ otherwise.

While the semantics of negation, conjunction, and disjunction as defined above coincides with that of the respective connectives in many three-valued logics, as, e.g., that of Kleene [13] and Łukasiewicz [14], the valuations for the implication are those of Avron [15].

Given some interpretation I, a formula ψ is *true* for I in $\mathcal{L}$, in symbols $I \models_{\mathcal{L}} \psi$, if $v_{\mathcal{L}}^{I}(\psi) = \mathbf{t}$, and ψ is *true* for I in $\mathcal{P}$, $I \models_{\mathcal{P}} \psi$, if $v_{\mathcal{P}}^{I}(\psi) = \mathbf{t}$ or $v_{\mathcal{P}}^{I}(\psi) = \mathbf{i}$. If ψ is true for I, I is a *model* of ψ. For a set Γ of formulae, I is a model of Γ if I is a model for each formula in Γ. A formula is *valid* if it is true for each interpretation.

We emphasise that the connectives of both $\mathcal{L}$ and $\mathcal{P}$ are functional complete, i.e., any truth function can be expressed by a formula in $\mathcal{L}$ and $\mathcal{P}$ [16]. Hence, any truth-functional three-valued logic can be embedded in $\mathcal{L}$ or $\mathcal{P}$. We review some prominent logics and how their connectives, if differing, can be defined. Representatives of logics where $\mathbf{i}$ is not designated are the three-valued logics of Kleene [13], Łukasiewicz [14], and Gödel [10]; a logic were $\mathbf{i}$ is designated is the paraconsistent logic P_3 [17].

The intuitive meaning of $\mathbf{i}$ in Kleene logic is "undecided" and expresses, if assigned to a formula, ignorance regarding the actual truth value of that formula in the sense that a formula is undecided iff it is not known whether it is either true or false. Kleene logic is a subsystem of $\mathcal{L}$ defined over the connectives $\wedge$, $\vee$, $\neg$, and $\rightarrow_K$, where $\rightarrow_K$ is the Kleene implication defined as $\psi \rightarrow_K \varphi = \neg\psi \vee \varphi$.

In the logic of Łukasiewicz, $\mathbf{i}$ stands for indeterminate and expresses the truth state of a formula that cannot be assigned true or false because it represents a future contingent statement and does not have a truth value yet. The connectives of Łukasiewicz logic are $\wedge$, $\vee$, $\neg$, and $\rightarrow_L$, where $\wedge$, $\vee$, $\neg$ are the connectives from $\mathcal{L}$, and the Łukasiewicz implication $\rightarrow_L$ can be defined in $\mathcal{L}$ as $\psi \rightarrow_L \varphi = (\psi \supset \varphi) \wedge (\neg\varphi \supset \neg\psi)$ [15].

The connectives of three-valued Gödel logic are $\wedge$, $\vee$, $\sim$, and $\rightarrow_G$. Thus, both negation and implication differ from the respective connectives in $\mathcal{L}$. However, the Gödel negation $\sim$ can be defined as $\sim\psi = \neg(\neg\psi \supset \psi)$ and the Gödel implication $\rightarrow_G$ as $\psi \rightarrow_G \varphi = ((\neg\varphi \supset \neg\psi) \supset \psi) \supset \varphi$.

Typical logics were $\mathbf{i}$ is designated are *paraconsistent* logics, i.e., logics were entailment from inconsistent theories does not become trivial. Logic $\mathcal{P}$ corresponds to the maximal paraconsistent logic P_3 [17] where the interpretation of $\mathbf{i}$ is both true and false.

3 Refutation Calculus

Bryll and Maduch [12] axiomatised the invalid sentences of Łukasiewicz's many-valued logics including the three-valued case. In particular, they used a Hilbert-type calculus with only one rejected axiom and two rules of inference. Since the calculus of Bryll and Maduch is not analytic, its usefulness for proof search in practice is rather limited. In this paper, we aim at an *analytic* Gentzen-style refutation calculus for three-valued logics.

To the best of our knowledge, the first sequential refutation systems for classical propositional logic was introduced by Tiomkin [18]. Later, equivalent systems were independently considered by Goranko [19] and Bonatti [5]. We pursue this work towards similar sequential refutation systems for the above defined three-valued logics $\mathcal{L}$ and $\mathcal{P}$.

By an *anti-sequent* we understand an ordered pair of form $\Gamma \dashv \Delta$, where Γ and Δ are finite sets of formulae. Given a theory Γ and a formula ψ, as usual, we write "Γ, ψ" for $\Gamma \cup \{\psi\}$. An interpretation I *refutes* $\Gamma \dashv \Delta$ iff I is a model of Γ and all formulae in Δ are false under I. An anti-sequent is *refutable* iff it is refuted by some interpretation.

We next introduce sequential refutation calculi for $\mathcal{L}$ and $\mathcal{P}$, denoted by $\mathbf{SRCL}$ and $\mathbf{SRCP}$, respectively. Let us note that these calculi are, in a certain sense, refutational pendants of the Gentzen-type calculi of Avron [20] for axiomatising the valid sentences of $\mathcal{L}$ and $\mathcal{P}$, much in the same way as the complementary systems of Tiomkin [18], Goranko [19], and Bonatti [5] are pendants of standard Gentzen systems for classical logic.

Axioms. Let Γ and Δ be two disjoint sets of literals such that $\neg\mathrm{T}, \mathrm{F} \notin \Gamma$ and $\mathrm{T}, \neg\mathrm{F} \notin \Delta$. Then, the anti-sequent $\Gamma \dashv \Delta$ is an axiom of $\mathbf{SRCL}$ iff $\{\mathrm{I}, \neg\mathrm{I}\} \cap \Gamma = \emptyset$ and Γ is consistent, and $\Gamma \dashv \Delta$ is an axiom of $\mathbf{SRCP}$ iff $\{\mathrm{I}, \neg\mathrm{I}\} \cap \Delta = \emptyset$ and Δ is consistent.

Logical Rules. As in the sequential calculus of Avron [20], we distinguish between two types of rules: The first type is standard in the sense that they introduce one occurrence of $\wedge$, $\vee$, or $\supset$ at a time. We note that the standard rules coincide with the respective introduction rules in the refutation systems for classical logic [18, 19, 5]. The second group contains rules that are non-standard since they introduce two occurrences of a connective at the same time, in particular this concerns negation in combination with all other connectives. The standard rules of $\mathbf{SRCL}$ and $\mathbf{SRCP}$ are depicted in Fig. 1; the non-standard ones are given in Fig. 2. Note that the logical rules of $\mathbf{SRCL}$ and $\mathbf{SRCP}$ coincide, so the only difference lies in the form of the axioms of the respective systems.

Theorem 1 (Soundness). *For any anti-sequent* $\Gamma \dashv \Delta$*,*

(i) *if* $\Gamma \dashv \Delta$ *is provable in* $\mathbf{SRCL}$*, then* $\Gamma \dashv \Delta$ *is refutable in* $\mathcal{L}$*, and*

(ii) *if* $\Gamma \dashv \Delta$ *is provable in* $\mathbf{SRCP}$*, then* $\Gamma \dashv \Delta$ *is refutable in* $\mathcal{P}$*.*

$$\frac{\Gamma \dashv \Delta, \psi}{\Gamma, \psi \supset \varphi \dashv \Delta}\ (\supset l)_1 \qquad \frac{\Gamma, \varphi \dashv \Delta}{\Gamma, \psi \supset \varphi \dashv \Delta}\ (\supset l)_2 \qquad \frac{\Gamma, \psi \dashv \Delta, \varphi}{\Gamma \dashv \Delta, \psi \supset \varphi}\ (\supset r)$$

$$\frac{\Gamma, \psi, \varphi \dashv \Delta}{\Gamma, \psi \wedge \varphi \dashv \Delta}\ (\wedge l) \qquad \frac{\Gamma \dashv \Delta, \psi}{\Gamma \dashv \Delta, \psi \wedge \varphi}\ (\wedge r)_1 \qquad \frac{\Gamma \dashv \Delta, \varphi}{\Gamma \dashv \Delta, \psi \wedge \varphi}\ (\wedge r)_2$$

$$\frac{\Gamma, \psi \dashv \Delta}{\Gamma, \psi \vee \varphi \dashv \Delta}\ (\vee l)_1 \qquad \frac{\Gamma, \varphi \dashv \Delta}{\Gamma, \psi \vee \varphi \dashv \Delta}\ (\vee l)_2 \qquad \frac{\Gamma \dashv \Delta, \psi, \varphi}{\Gamma \dashv \Delta, \psi \vee \varphi}\ (\vee r)$$

Fig. 1. Standard rules of $\mathbf{SRCL}$ and $\mathbf{SRCP}$.

Proof. By induction on the proof length. For the base case, we show that each axiom of $\mathbf{SRCL}$ is refutable in $\mathcal{L}$, and likewise each axiom of $\mathbf{SRCP}$ is refutable in $\mathcal{P}$. Let $\Gamma \dashv \Delta$ be an axiom of $\mathbf{SRCL}$. Define an interpretation I as $I(a) = \mathbf{t}$, if $a \in \Gamma$, $I(a) = \mathbf{f}$, if $\neg a \in \Gamma$, and $I(a) = \mathbf{i}$, otherwise. I is a model of Γ in $\mathcal{L}$ but no literal in Δ is true under I in $\mathcal{L}$ by construction. Hence, $\Gamma \dashv \Delta$ is refutable in $\mathcal{L}$. On the other hand, if $\Gamma \dashv \Delta$ is an axiom of $\mathbf{SRCP}$, then define $I(a) = \mathbf{f}$, if $a \in \Delta$, $I(a) = \mathbf{t}$, if $\neg a \in \Delta$, and $I(a) = \mathbf{i}$, otherwise. Likewise, I is a model of Γ in $\mathcal{P}$ but no literal in Δ is true under I in $\mathcal{P}$, and thus $\Gamma \dashv \Delta$ is refutable in $\mathcal{P}$.

For the inductive step, we have to show that each rule is *sound* with respect to $\mathcal{L}$ and $\mathcal{P}$, i.e., if the premiss of a rule is refutable in $\mathcal{L}$, then the conclusion is refutable in $\mathcal{L}$ as well, and likewise for $\mathcal{P}$. For $\mathcal{L}$, we first consider Rule $(\neg\wedge l)_1$, and assume its premiss $\Gamma, \neg\psi \dashv \Delta$ is refutable in $\mathcal{L}$. Hence, $I \models_{\mathcal{L}} \Gamma$ and $I \models_{\mathcal{L}} \neg\psi$ but $I \not\models_{\mathcal{L}} \Delta$, for some interpretation I. It is easy to verify that $I \models_{\mathcal{L}} \neg\psi$ implies $I \models_{\mathcal{L}} \neg(\psi \wedge \varphi)$ by inspecting the truth conditions of $\neg$ and $\wedge$. Consequently, the conclusion of Rule $(\neg\wedge l)_1$ is refutable in $\mathcal{L}$ as well. The soundness argument for $(\neg\wedge l)_1$ with respect to $\mathcal{P}$ is essentially the same. Soundness of the remaining rules of the two calculi can be shown *mutatis mutandis*. □

Theorem 2 (Completeness). *For any anti-sequent* $\Gamma \dashv \Delta$,
(i) *if* $\Gamma \dashv \Delta$ *is refutable in* $\mathcal{L}$, *then* $\Gamma \dashv \Delta$ *is provable in* $\mathbf{SRCL}$, *and*
(ii) *if* $\Gamma \dashv \Delta$ *is refutable in* $\mathcal{P}$, *then* $\Gamma \dashv \Delta$ *is provable in* $\mathbf{SRCP}$.

Proof. Let us define the *complexity* of an anti-sequent $\Gamma \dashv \Delta$ as two times the sum of the number of occurrences of the connectives $\wedge$, $\vee$, and $\supset$ in $\Gamma \dashv \Delta$ plus the number of all occurrences of $\neg$ in $\Gamma \dashv \Delta$ that do not immediately precede an atom. We show the result by induction on the complexity of an anti-sequent. Thereby, we only prove the case of $\mathcal{L}$; the proof for $\mathcal{P}$ is analogous.

$$\frac{\Gamma, \psi \dashv \Delta}{\Gamma, \neg\neg\psi \dashv \Delta}\ (\neg\neg\, l) \qquad \frac{\Gamma \dashv \Delta, \psi}{\Gamma \dashv \Delta, \neg\neg\psi}\ (\neg\neg\, r)$$

$$\frac{\Gamma, \neg\psi \dashv \Delta}{\Gamma, \neg(\psi \wedge \varphi) \dashv \Delta}\ (\neg\wedge\, l)_1 \qquad \frac{\Gamma, \neg\varphi \dashv \Delta}{\Gamma, \neg(\psi \wedge \varphi) \dashv \Delta}\ (\neg\wedge\, l)_2 \qquad \frac{\Gamma \dashv \Delta, \neg\psi, \neg\varphi}{\Gamma \dashv \Delta, \neg(\psi \wedge \varphi)}\ (\neg\wedge\, r)$$

$$\frac{\Gamma, \neg\psi, \neg\varphi \dashv \Delta}{\Gamma, \neg(\psi \vee \varphi) \dashv \Delta}\ (\neg\vee\, l) \qquad \frac{\Gamma \dashv \Delta, \neg\psi}{\Gamma \dashv \Delta, \neg(\psi \vee \varphi)}\ (\neg\vee\, r)_1 \qquad \frac{\Gamma \dashv \Delta, \neg\varphi}{\Gamma \dashv \Delta, \neg(\psi \vee \varphi)}\ (\neg\vee\, r)_2$$

$$\frac{\Gamma, \psi, \neg\varphi \dashv \Delta}{\Gamma, \neg(\psi \supset \varphi) \dashv \Delta}\ (\neg\supset\, l) \qquad \frac{\Gamma \dashv \Delta, \psi}{\Gamma \dashv \Delta, \neg(\psi \supset \varphi)}\ (\neg\supset\, r)_1 \qquad \frac{\Gamma \dashv \Delta, \neg\varphi}{\Gamma \dashv \Delta, \neg(\psi \supset \varphi)}\ (\neg\supset\, r)_2$$

Fig. 2. Non-Standard rules of $\mathbf{SRCL}$ and $\mathbf{SRCP}$.

For the base case, assume that $\Gamma \dashv \Delta$ is refutable in $\mathcal{L}$ and has complexity 0. Because of the latter, Γ and Δ are sets of literals. Since $\Gamma \dashv \Delta$ is refutable in $\mathcal{L}$, i.e., some model I of Γ falsifies all literals in Δ, it follows that Γ and Δ are disjoint, $\neg\mathrm{T}, \mathrm{F} \notin \Gamma$, and $\mathrm{T}, \neg\mathrm{F} \notin \Delta$. Moreover, neither I nor $\neg$I can be in Γ, and for each atom a in Γ, $\neg a \notin \Gamma$. This proves that $\Gamma \dashv \Delta$ is an axiom of $\mathbf{SRCL}$ which completes the base case.

For the inductive step, we have to show that each refutable anti-sequent in $\mathcal{L}$ of complexity $n > 0$ is provable in $\mathbf{SRCL}$, given the induction hypothesis that each refutable anti-sequent of complexity less than n is provable in $\mathbf{SRCL}$.

Consider a refutable anti-sequent $\Gamma \dashv \Delta$ of complexity n. Since $n > 0$, Γ or Δ contains a non-literal formula α whose top-level connective is $\wedge$, $\vee$, $\supset$, or $\neg$. We distinguish between the different cases concerning the form of α which is one of $\neg\neg\varphi$,$\neg(\varphi \wedge \psi)$, $\neg(\varphi \vee \psi)$, $\neg(\varphi \supset \psi)$, $\varphi \wedge \psi$, $\varphi \vee \psi$, or $\varphi \supset \psi$.

Assume that α can be written as $\neg(\psi \wedge \varphi)$. We distinguish whether $\alpha \in \Gamma$ or $\alpha \in \Delta$. If $\alpha \in \Gamma$, $\Gamma \dashv \Delta$ can be written as $\Gamma', \neg(\psi \wedge \varphi) \dashv \Delta$, where $\Gamma' = \Gamma \setminus \{\alpha\}$. $\Gamma', \neg(\psi \wedge \varphi) \dashv \Delta$ is refutable in $\mathcal{L}$ only if $\Gamma', \neg\psi \dashv \Delta$ is refutable in $\mathcal{L}$ or $\Gamma', \neg\varphi \dashv \Delta$ is refutable $\mathcal{L}$. Since the complexity of both $\Gamma', \neg\psi \dashv \Delta$ as well as that of $\Gamma', \neg\varphi \dashv \Delta$ is strictly less than n, it follows from the induction hypothesis that $\Gamma', \neg\psi \dashv \Delta$ or $\Gamma', \neg\varphi \dashv \Delta$ is provable in $\mathbf{SRCL}$. If $\Gamma', \neg\psi \dashv \Delta$ is provable, then $\Gamma', \neg(\psi \wedge \varphi) \dashv \Delta$ is provable by Rule $(\neg\wedge\, l)_1$, and if $\Gamma', \neg\varphi \dashv \Delta$ is provable, then $\Gamma', \neg(\psi \wedge \varphi) \dashv \Delta$ is provable by Rule $(\neg\wedge\, l)_2$.

If $\alpha \in \Delta$, we can write $\Gamma \dashv \Delta$ as $\Gamma \dashv \Delta', \neg(\psi \wedge \varphi)$, where $\Delta' = \Delta \setminus \{\alpha\}$. $\Gamma \dashv \Delta', \neg(\psi \wedge \varphi)$ is provable in $\mathcal{L}$ only if $\Gamma \dashv \Delta', \neg\psi, \neg\varphi$ is refutable in $\mathcal{L}$. As in the argument before, the complexity of $\Gamma \dashv \Delta', \neg\psi, \neg\varphi$ is strictly smaller than n. Thus, by induction hypothesis, $\Gamma \dashv \Delta', \neg\psi, \neg\varphi$ is provable in $\mathbf{SRCL}$. Hence, $\Gamma \dashv \Delta', \neg(\psi \wedge \varphi)$ is provable as well due to Rule $(\neg\wedge\, r)$. The remaining cases can be shown by similar arguments. $\square$

Corollary 1. *A formula ψ is not valid in $\mathcal{L}$ iff $\dashv \psi$ is provable in* $\mathbf{SRCL}$*. Likewise, ψ is not valid in $\mathcal{P}$ iff $\dashv \psi$ is provable in* $\mathbf{SRCP}$.

Clearly, $\mathbf{SRCL}$ and $\mathbf{SRCP}$ are analytic calculi. Note that our systems contain no structural rules and no cut rule.

As mentioned above, there is a close relation between our refutation systems and the respective sequent systems of Avron [20] for proving valid sequents. In fact, for each rule of form

$$\frac{\Gamma' \vdash \Delta'}{\Gamma \vdash \Delta}$$

in Avron's systems, our system contains a respective rule were $\vdash$ is replaced by $\dashv$. Moreover, for each rule of form

$$\frac{\Gamma' \vdash \Delta' \quad \Gamma'' \vdash \Delta''}{\Gamma \vdash \Delta}$$

of Avron, our systems contain two rules

$$\frac{\Gamma' \dashv \Delta'}{\Gamma \dashv \Delta} \quad \text{and} \quad \frac{\Gamma'' \dashv \Delta''}{\Gamma \dashv \Delta}\,.$$

Hence, as already remarked by Bonatti [5] for the sequential refutation systems for classical logic, exhaustive search in the standard system becomes non-determinism in the refutation system—a property that often allows for quite concise proofs. While all rules in the systems of Avron are invertible, the non-deterministic rules in our systems lose this property.

Contrary to standard sequential systems, proofs in our systems are not trees but sequences, thus each proof has a single axiom. Axioms in proofs and counter models for refutable formulae are closely related:

Theorem 3. *Let ψ be some formula that is not valid in $\mathcal{L}$, and let $\Gamma \dashv \Delta$ be the unique axiom in a derivation of $\dashv \psi$ in* $\mathbf{SRCL}$*. Then, for each interpretation I that refutes the axiom $\Gamma \dashv \Delta$ in $\mathcal{L}$, it holds that $I \not\models_{\mathcal{L}} \psi$.*

Likewise, let φ be some formula that is not valid in $\mathcal{P}$, and let $\Gamma \dashv \Delta$ be the unique axiom in a derivation of $\dashv \varphi$ in $\mathbf{SRCP}$*. Then, for each interpretation I that refutes the axiom $\Gamma \dashv \Delta$ in $\mathcal{P}$, it holds that $I \not\models_{\mathcal{P}} \varphi$.*

The significance of the above proposition is that a proof does not represent a single counter model, rather it represents an entire class of counter models.

4 An Application for Answer-Set Programming

Refutation systems have notable applications in nonmonotonic reasoning: Bonatti and Olivetti [4] introduced analytic sequent-type calculi for the major nonmonotonic logics, viz. for default logic, autoepistemic logic, and propositional circumscription, which combine a standard sequent-type calculus for classical propositional logic and a dedicated complementary calculus [5]. Following that work, similar combined calculi for the three-valued extensions of circumscription, autoepistemic logic, closed-world assumption, and default logic due to Przymusinski [11] can be envisaged as a promising application field of our refutation systems for three-valued logics. We note here in passing the introduction of a sequent-type calculus for intuitionistic default logic along these lines [21].

In what follows, we outline a different application scenario that is concerned with logic programs under the answer-set semantics [22], a prominent nonmonotonic approach to logic programming. The considered application illustrates the relevance of the three-valued logic of Gödel [10] for analysing certain program properties.

In a nutshell, a *disjunctive logic program* is a set of rules of form

$$a_1 \vee \cdots \vee a_l \leftarrow a_{l+1}, \ldots, a_m, \mathrm{not}\ a_{m+1}, \ldots \mathrm{not}\ a_n ,$$

where all a_i are atoms over the universe $\mathcal{U}$ and "not" denotes *default negation*. The *head* of r, $\mathrm{H}(r)$, is $\{a_1, \ldots, a_l\}$, the *positive body* of r, $\mathrm{B}^+(r)$, is $\{a_{l+1}, \ldots, a_m\}$, and the *negative body* of r, $\mathrm{B}^-(r)$, is $\{a_{m+1}, \ldots, a_n\}$. An interpretation I for some program is identified with the set of true atoms. I is a *model* of a rule r if $\mathrm{H}(r) \cap I \neq \emptyset$ whenever $\mathrm{B}^+(r) \subseteq I$ and $\mathrm{B}^-(r) \cap I = \emptyset$, and I is a model of a program P iff I is a model of each rule in P. An interpretation I is an *answer set* of a program P iff I is a minimal model of the *reduct* P^I that is defined as the program that results from P by (i) deleting all rules $r \in P$ with $\mathrm{B}^-(r) \cap I \neq \emptyset$, and (ii) deleting all default negated atoms from the remaining rules [22].

For instance, $P = \big\{ a \leftarrow \mathrm{not}\ b, b \leftarrow \mathrm{not}\ a \big\}$ is a logic program with the intuitive meaning that a is true if there is no evidence for b, and likewise b is true if there is no evidence for a. This program has two answer sets, $\{a\}$ and $\{b\}$.

Two logic programs are *equivalent* if they have the same answer sets. In contrast to classical logic, equivalence between programs fails to yield a replacement property. However, the notion of *strong equivalence* [8] circumvents this problem, basically by definition: two programs P and Q are strongly equivalent iff, for each program R, $P \cup R$ and $P \cup R$ have the same answer sets.

For instance, consider the program $Q = \{a \vee b\}$, expressing that either a or b is true. Like for P above, the answer sets of Q are $\{a\}$ and $\{b\}$. Thus, P and Q are equivalent. However, they are not strongly equivalent as we will show with the help of our refutation calculus $\mathbf{SRCL}$. In fact, our method will also provide a counterexample showing the failure of strong equivalence.

The central observation connecting strong equivalence with three-valued logics is the well-known result that strong equivalence between two programs P and Q holds iff P and Q, interpreted as theories, are equivalent in the (monotonic) logic of here-and-there [8], which is equivalent to the three-valued logic of Gödel [10]. As the Gödel connectives $\sim$ and $\rightarrow_G$ are definable in $\mathcal{L}$, we can extend $\mathbf{SRCL}$ by the derived rules depicted in Fig. 3 (redundant rules are eliminated).

To verify that P and Q are not strongly equivalent, it suffices to give a proof of one of the anti-sequents

$$P' \dashv Q' \quad \text{or} \quad Q' \dashv P'$$

$$\frac{\Gamma, \neg\psi \dashv \Delta}{\Gamma, {\sim}\psi \dashv \Delta}\ ({\sim}\, l) \qquad \frac{\Gamma \dashv \Delta, \neg\psi}{\Gamma \dashv \Delta, {\sim}\psi}\ ({\sim}\, r)$$

$$\frac{\Gamma \dashv \Delta, \psi}{\Gamma, \psi \to_G \varphi \dashv \Delta}\ (\to_G\ l)_1 \qquad \frac{\Gamma, \psi \dashv \Delta, \varphi}{\Gamma \dashv \Delta, \psi \to_G \varphi}\ (\to_G\ r)_1$$

$$\frac{\Gamma, \varphi \dashv \Delta}{\Gamma, \psi \to_G \varphi \dashv \Delta}\ (\to_G\ l)_2 \qquad \frac{\Gamma, \neg\varphi \dashv \Delta, \neg\psi}{\Gamma \dashv \Delta, \psi \to_G \varphi}\ (\to_G\ r)_2$$

$$\frac{\Gamma \dashv \Delta, \neg\psi}{\Gamma, \neg{\sim}\psi \dashv \Delta}\ (\neg{\sim}\, l) \qquad \frac{\Gamma, \neg\psi \dashv \Delta}{\Gamma \dashv \Delta, \neg{\sim}\psi}\ (\neg{\sim}\, r)$$

$$\frac{\Gamma \dashv \Delta, \neg\varphi}{\Gamma \dashv \Delta, \neg(\psi \to_G \varphi)}\ (\neg{\to_G}\ r)_1$$

$$\frac{\Gamma, \neg\varphi \dashv \Delta, \neg\psi}{\Gamma, \neg(\psi \to_G \varphi) \dashv \Delta}\ (\neg{\to_G}\ l) \qquad \frac{\Gamma, \neg\psi \dashv \Delta}{\Gamma \dashv \Delta, \neg(\psi \to_G \varphi)}\ (\neg{\to_G}\ r)_2$$

Fig. 3. Derived rules for three-valued Gödel logic.

in **SRCL**, where P' and Q' are the theories corresponding to P and Q, respectively, given by

$$P' = \{({\sim}b \to_G a) \wedge ({\sim}a \to_G b)\} \text{ and}$$
$$Q' = \{a \vee b\}\,.$$

While $Q' \dashv P'$ is not provable in **SRCL**, a derivation for $P' \dashv Q'$ exists:

$$\cfrac{\cfrac{\cfrac{\cfrac{\dashv a, b, \neg a, \neg b}{\dashv a, b, {\sim}a, {\sim}b}\ ({\sim}\,r), ({\sim}\,r)}{{\sim}a \to_G b, {\sim}b \to_G a \dashv a, b}\ (\to_G\ r)_1, (\to_G\ r)_1}{{\sim}a \to_G b, {\sim}b \to_G a \dashv a \vee b}\ (\vee\, r)}{({\sim}a \to_G b) \wedge ({\sim}b \to_G a) \dashv a \vee b}\ (\wedge\, l)$$

Hence, P and Q are indeed not strongly equivalent. In fact, as detailed in a moment, a concrete program R such that $P \cup R$ and $Q \cup R$ have different answer sets, i.e., a *witness* that P and Q are not strongly equivalent, can be immediately constructed from the axiom $\dashv a, b, \neg a, \neg b$ of the above proof:

$$R = \{a \leftarrow b,\ b \leftarrow a\}\,.$$

Indeed, $P \cup R$ has no answer set while $Q \cup R$ yields $\{a, b\}$ as its unique answer set.

The general method to obtain a witness program (as R above) from an axiom in **SRCL** is as follows. Given an axiom $\Gamma \dashv \Delta$, construct some interpretation I that refutes $\Gamma \dashv \Delta$. Such an interpretation I can be obtained as follows:

Theorem 4. *Let ψ be a formula that is not valid in $\mathcal{L}$, $\Gamma \dashv \Delta$ the unique axiom of a proof of ψ, and define interpretation I as $I(a) = \mathbf{t}$ if $a \in \Gamma$, $I(a) = \mathbf{f}$ if $\neg a \in \Gamma$ or $\{a, \neg a\} \cap (\Gamma \cup \Delta) = \emptyset$, and $I(a) = \mathbf{i}$ otherwise. Then, I refutes $\Gamma \dashv \Delta$.*

Based on such an interpretation I that refutes the axiom $\Gamma \dashv \Delta$, a witness program R can always be constructed as follows:

$$R = \{a \leftarrow \mid a \in \Gamma \cup \Delta, I(a) = \mathbf{t}\} \cup \{a \leftarrow b \mid a, b \in \Gamma \cup \Delta, I(a) = I(b) = \mathbf{i}\}.$$

The correctness of the this construction follows from the proof of the main theorem by Lifschitz, Pearce, and Valverde [8].

5 Conclusion

We introduced two analytic sequential refutation calculi, **SRCL** and **SRCP**, that are sound and complete for two fully expressive three-valued logics. While the more conventional method to show that a formula is not valid is by exhaustive search for counterexamples, refutation systems allow to show that a formula is not valid in a purely deductive way. Also, a proof in a refutation system is very different from a failed proof of validity in traditional systems: On the one hand, an explicit notion of a refutation proof allows for proof-theoretic mediations concerned with the size of proof representations as exemplified in earlier work [6]. On the other hand, refutation systems can be combined with traditional systems into single proof systems with, as pointed out by Goranko [23], a potentially greater efficiency than traditional systems.

Combined systems were already considered by Bonatti and Olivetti [4] for some major nonmonotonic logics. It seems natural to apply the systems introduced in this paper to study similar systems for three-valued extensions of such logics [11]. Besides that, there are interesting direct applications of three-valued logics for nonmonotonic reasoning, like the relation between strong equivalence of logic programs and the three-valued Gödel logic [8], were our calculi can be readily applied for program analysis.

A Prolog implementation of **SRCL** and **SRCP** is available at

`www.kr.tuwien.ac.at/research/projects/mmdasp.`

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