Low Complexity Equalization for Doubly Selective Channels Modeled by a Basis Expansion

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***Abstract*—We propose a novel equalization method for doubly selective wireless channels, whose taps are represented by an arbitrary Basis Expansion Model (BEM). We view such a channel in the time domain as a sum of product-convolution operators created from the basis functions and the BEM coefficients. Equivalently, a frequency-domain channel can be represented as a sum of convolution-products. The product-convolution representation provides a low-complexity, memory efficient way to apply the channel matrix to a vector. We compute a regularized solution of a linear system involving the channel matrix by means of the GMRES and the LSQR algorithms, which utilize the product-convolution structure without ever explicitly creating the channel matrix. Our method applies to all cyclic-prefix transmissions. In an OFDM transmission with K subcarriers, each iteration of GMRES or LSQR requires only $\mathcal{O}(K \log K)$ flops and $\mathcal{O}(K)$ memory. Additionally, we further accelerate convergence of both GMRES and LSQR by using the single-tap equalizer as a preconditioner. We validate our method with numerical simulations of a WiMAX-like system (IEEE 802.16e) in channels with significant delay and Doppler spreads. The proposed equalizer achieves BERs comparable to those of MMSE equalization, and noticeably outperforms low-complexity equalizers using an approximation by a banded matrix in the frequency domain. With preconditioning, the lowest BERs are obtained within 3–16 iterations. Our approach does not use any statistical information about the wireless channel.**

***Index Terms*—Basis expansion model, doubly selective channels, equalization, OFDM, time-varying channels.**

I. Introduction

A. Motivation and Previous Work

In the last two decades, there has been a steady increase in the number of applications operating in rapidly varying wireless communication channels. Such channels occur due to user mobility in systems like DVB-T and WiMAX, which have been originally designed for fixed receivers. Rapidly varying channels lead to significant intercarrier interference (ICI) in multicarrier communication systems, which must be mitigated by an appropriate equalization method. Moreover, several applications have short symbol durations, and therefore require fast equalization algorithms. One such application is Mobile WiMAX (IEEE 802.16e) with a symbol duration of 102.9 μs.

At present, the most accurate approximations of doubly selective wireless channels with large delay and Doppler spreads are obtained via the basis expansion model (BEM). Consequently, we assume in this paper that the channel is represented in terms of a basis expansion model (BEM), which approximates the channel taps by linear combinations of prescribed basis functions; see [1]–[4]. Choosing an appropriate basis is crucial for accuracy, especially at high Doppler spreads. For example, discrete prolate sequences (DPS) provide a superb approximation, while complex exponentials (CE) give a poor approximation for several reasons; see, e.g., [5]. There also exist methods using differential encoding, which avoid an intermediate estimation step; see [6] and [7].

In the framework of the BEM, channel estimation amounts to an approximate computation of coefficients for the basis functions. There exist several methods for estimating the BEM coefficients of doubly selective channel taps, especially with an orthogonal frequency-division multiplexing (OFDM) transmission setup; see [2]–[4] and [8]. Usually, the channel matrix is reconstructed from estimated BEM coefficients and subsequently used in equalization; see, e.g., [9].

For frequency-selective channels, the conventional single-tap equalization in the frequency domain is a method of choice. However, in the presence of severe ICI, single-tap equalization is unreliable; see [10]–[12]. Several other approaches have been proposed to mitigate ICI in transmissions over rapidly varying channels. For example, [13] describes minimum mean-square error (MMSE) and successive interference cancellation equalizers, which use all subcarriers simultaneously. A similar equalizer is presented in [14]. However, with K OFDM subcarriers, it has the complexity of $\mathcal{O}(K^2)$, and storage requirements of the same order of magnitude. Alternatively, only a few selected subcarriers are used in equalization, which amounts to approximating the frequency-domain channel matrix by a banded matrix. This approach has been exploited for design of low-complexity equalizers; see [9] and [15]–[19]. It is well known that approximation with a channel matrix banded in the frequency domain is equivalent to using a BEM with complex exponentials (CE-BEM); see Theorem 1. However, the CE-BEM [17] provides a poor approximation to the channel matrix [5], [20], so this approach leads to a significant loss of information about the channel.

A related approach is adopted in [9], where the channel matrix is first estimated using the DPS-BEM, and then re-estimated

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by a matrix banded in the frequency domain, which amounts to a transition to the CE-BEM. However, the DPS-BEM is more accurate than the CE-BEM [5], [20], so during the re-estimation some information about the channel is lost.

A time-domain equalizer based on the LSQR algorithm is introduced in [21].

The methods presented in [22]–[24] require a channel matrix vector multiplication at some point in the process of equalization. In upcoming broadband wireless communication systems with significant delay spreads, the number of discrete multipaths is proportional to the bandwidth. For example, in OFDM systems as used with Mobile WiMAX (IEEE 802.16e), a typical number of the discrete multipaths $L = (K/8)$, where K is the number of subcarriers. Thus, for such a system, a matrix-vector multiplication with the channel matrix requires $\mathcal{O}(K^2)$ operations, while our method has a complexity of $\mathcal{O}(K \log K)$. As noticed by the authors of [22], their low-complexity method is currently limited to a CE-BEM. Our proposed approach can be used in the settings discussed in the three papers referenced above, and gives rise to low-complexity equalization with arbitrary bases.

B. Contributions

We introduce a product-convolution (PC) decomposition of a doubly selective time-domain channel matrix as an entirely novel approach to basis expansion models. Equivalently, we view a frequency-domain channel matrix as a sum of convolution-product operators. Either of these decompositions leads to a fast matrix-vector multiplication with the channel matrix. Subsequently, we develop the first low-complexity equalization method that can be used with an arbitrary basis expansion model (BEM). Our method is a significant improvement over its nearest competitor—fast equalization of a channel matrix banded in the frequency domain.

Our algorithm only uses estimated BEM coefficients and the receive signal. We represent the time-domain channel matrix as a sum of product-convolution operators without ever constructing the channel matrix itself. The product operators are diagonal matrices with the basis functions as diagonals. The corresponding BEM coefficients are used as filters in the convolution operators. This particular structure of the channel matrix permits a fast application of the matrix and its conjugate transpose. Therefore, the PC representation combined with classical iterative methods like GMRES [25] and LSQR [26], gives rise to a low-complexity equalizer. Additionally, we further accelerate convergence of both GMRES and LSQR by preconditioning them with the single-tap equalizer.

The proposed equalizer has low complexity, and can be used with an *arbitrary* basis. We demonstrate in Section V, that the product-convolution representation using a well-chosen basis leads to significant improvements in the BER after equalization.

Our main contributions can be summarized as follows.

- We introduce a product-convolution decomposition of a doubly selective wireless channel matrix represented via a basis expansion model.
- We introduce a class of low-complexity equalization methods based on the product-convolution representation. We use standard iterative methods, like GMRES and LSQR, for equalization without reconstructing the channel matrix. In an OFDM system with K subcarriers, each iteration requires $\mathcal{O}(K \log K)$ flops and $\mathcal{O}(K)$ memory, and achieves BERs comparable with those of MMSE equalization.
- We propose the single-tap equalizer as an efficient preconditioner for both GMRES and LSQR.

Since the number of discrete multipaths is proportional to the bandwidth, broadband transmissions suffer from a large number of multipaths. For example, Mobile WiMAX (IEEE 802.16e) with K subcarriers typically exhibits a discrete path delay of $(K/4), (K/8)$; see [27]. In such regimes, reducing the complexity and memory requirements by a factor close to K has significant practical benefits. Moreover, for such transmissions an explicit reconstruction of the channel matrix requires $\mathcal{O}(K^2)$ memory and $\mathcal{O}(K^2)$ flops, which is prohibitive in several practical applications.

Both GMRES [25] and LSQR [26] are well-known iterative methods for the numerical solution of a system of linear equations; see Appendix A for detailed descriptions. LSQR has excellent regularization properties, and achieves BERs comparable to those of MMSE equalization; see [21], [28], and Section V.

Our equalization method applies to any communication systems as long as:

— the wireless channel is estimated using a basis expansion model;

— the transmission uses a cyclic prefix (CP).

The method applies to, for example, cyclic-prefix based orthogonal frequency-division multiplexing (CP-OFDM) and single-carrier frequency-division multiplexing (SC-FDM) with a cyclic-prefix. Our method is quite practical, and may be readily implemented in hardware.

In this paper, we describe basic, deterministic versions of the algorithms. Further modifications using statistical information, a turbo loop (see, e.g., [9]) or decision feedback can be combined with the proposed product-convolution decomposition. For the sake of clarity of presentation, we describe the methods in their simplest form, without relying on any specific probabilistic assumptions. We emphasize that we do not use any approximation of the channel matrix by a matrix banded in the frequency domain.

We validate our method with numerical simulations of a WiMAX-like system in channels with severe Doppler shifts. However, our method applies to any cyclic-prefix communication systems, as long as the wireless channel is modeled by a basis expansion. The proposed equalizer noticeably outperforms current low-complexity equalizers, which are based on an approximation by a banded matrix in the frequency domain.

The paper is organized as follows. In Section II, we introduce our transmission setup and an assumed model for the wireless channel. The proposed iterative equalization methods and preconditioners are described in Section III. A detailed description of the algorithm is provided in Section IV. We present our simulation results in Section V, and conclusions in Section VI.

II. System Model

A. Transmission Model

Our equalization method applies to any wireless transmission scheme with a cyclic prefix (CP). For example, our method applies to CP-OFDM and SC-FDM with a cyclic prefix. In this paper, we focus on CP-OFDM systems operating in doubly selective channels. We consider an equivalent baseband representation of a single-antenna OFDM system with K subcarriers, although our method can be adapted to a MIMO setup in a straightforward manner. We assume a sampling period of $T_{\rm s} = 1/B$, where B denotes the transmit bandwidth. A cyclic prefix of length $L_{\rm cp}$ is used in every OFDM symbol. We choose $L_{\rm cp}$ so large, that $L_{\rm cp}T_{\rm s}$ exceeds the channel's maximum delay, so that we avoid intersymbol interference (ISI). Consequently, throughout this paper we deal with one OFDM symbol at a time, and all further models and formulations refer to one OFDM symbol.

Each subcarrier is used to transmit a symbol $A[k](k = 0,\dots,K-1)$ from a finite symbol constellation (e.g., 4-QAM, PSK, 64-QAM). Depending on the transmission setup, some of these symbols may serve as pilots for channel estimation. The OFDM modulator uses the inverse discrete Fourier transform (IDFT) to map the frequency-domain transmit symbols $A[k]$ into the time-domain transmit signal $x[n]$

$$x[n] = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} A[k]\, e^{j2\pi\frac{nk}{K}}, \tag{1}$$

$n = -L_{\rm cp},\dots,K-1$.

After discarding the cyclic prefix at the receiver, the receive signal equals

$$y[n] = \sum_{l=0}^{L-1} h_l[n]\, x[n-l] + w[n], \quad n = 0,\dots,K-1. \tag{2}$$

Here, $w[n]$ denotes complex additive noise of variance N_0, $h_l[n]$ is the complex channel tap associated with delay l, and L is the channel length (maximum discrete-time delay). Consequently, the channel's maximum delay equals $(L-1)T_s$. For simplicity, we make the worst-case assumption that $L = L_{\rm cp}$. In order to simplify notation, throughout this paper we consider the signals to be periodically extended with period K. Therefore, (2) represents the cyclic convolution of length K. We do not use any acyclic convolutions in this paper.

Our equalization method applies to any transmission scheme, as long as the time-domain transmit-receive relation can be modeled as (2). Equivalently, the transmit-receive relation (2) can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \tag{3}$$

where $\mathbf{y} \equiv (y[0],\dots,y[K-1])^T$ is the time-domain receive signal, $\mathbf{x} \equiv (x[0],\dots,x[K-1])^T$ is the time-domain transmit signal, $\mathbf{w} \equiv (w[0],\dots,w[K-1])^T$ is an additive noise process in the time domain, and $\mathbf{H}$ is the time-domain channel matrix.

A generic OFDM demodulator at the receiver's end performs the following tasks with the sampled time-domain receive signal: channel estimation, equalization, demodulation by means of the DFT, quantization, decoding, and deinterleaving. In this paper, we assume that a channel estimate in terms of the BEM coefficients is already provided. In the next section, we develop methods for equalization of the receive signal using the estimated BEM coefficients.

B. Wireless Channel Representation With BEM

We assume a basis expansion model (BEM) for the channel taps. With the BEM, each channel tap $\mathbf{h}_l$ is modeled as a linear combination of suitable basis functions $\mathbf{B}_m$. Several bases are proposed in literature, including complex exponentials [29]–[32], complex exponentials oversampled in the frequency domain [33], discrete prolate spheroidal functions [20], polynomials [34], and the Legendre polynomials [8].

With a specific set of basis functions, the channel tap h_l is represented as follows:

$$h_l[n] = \sum_{m=0}^{M-1} b_{lm}\, B_m[n], \qquad l = 0,\dots,L-1 \tag{4}$$

where b_{lm} is the mth basis coefficient of the lth channel tap, $\mathbf{B}_m$ is the mth basis function, and M is the BEM model order. Moreover, $B_m[n] = \mathbf{B}_m(nT_s), h_l[n] = \mathbf{h}_l(nT_s)$. Relation (4) is correct up to a modeling error, which can be reduced by increasing the model order M. On the other hand, in pilot-based estimation methods increasing M typically decreases the transmission capacity.

Combining (2) and (4), the time-domain receive signal $\mathbf{y}$ is expressed as

$$y[n] = \sum_{l=0}^{L-1} \left(\sum_{m=0}^{M-1} b_{lm}\, B_m[n] \right) x[n-l] + w[n] \tag{5}$$

where $n = 0,\dots,K-1$, and $\mathbf{w}$ is an additive error, which consists of random noise and a systematic modeling error.

C. Equivalence of BEM and Product-Convolution Representation

Changing the order of summation in (5), we obtain

$$y[n] = \underbrace{\sum_{m=0}^{M-1} \underbrace{B_m[n]}_{\rm product} \underbrace{\left(\sum_{l=0}^{L-1} b_{lm}\, x[n-l] \right)}_{\rm cyclic-convolution}}_{\text{sum of product}-\text{convolutions}} + w[n]. \tag{6}$$

Equivalently, the time-domain channel matrix $\mathbf{H}$ can be expressed as a sum of product-convolutions as follows:

$$\mathbf{H} = \sum_{m=0}^{M-1} \mathbf{P}_m \mathbf{C}_m \tag{7}$$

where $\mathbf{P}_m$ is a diagonal matrix with $P_m(i,i)$ equal to $B_m(i)$, and $\mathbf{C}_m$ is a circulant matrix representing the cyclic convolution with the mth set of the BEM coefficients $\{b_{lm}\}_{l=0}^{M-1}$. The above formulas lead to a prescription for a fast matrix vector multiplication, and are fundamental to our equalization algorithm.

TABLE I
CHARACTERISTICS OF KRYLOV SUBSPACE METHODS GMRES AND LSQR APPLIED TO THE TIME-DOMAIN CHANNEL MATRIX $\mathbf{H}$, AND THE TIME-DOMAIN RECEIVE SIGNAL $\mathbf{y}$, WITH i ITERATIONS

Methods	GMRES	LSQR
Krylov subspace	$\mathcal{K}(\mathbf{H}, \mathbf{y}, i)$	$\mathcal{K}(\mathbf{H}^H\mathbf{H}, \mathbf{H}^H\mathbf{y}, i)$
Storage	$i+1$ vectors	4 vectors
Work per iteration	One application of $\mathbf{H}$ and other linear operations.	One application of $\mathbf{H}$, one application of $\mathbf{H}^H$, and linear operations on vectors.

III. EQUALIZATION

A. Iterative Equalization Methods

It is well known that the conventional single-tap equalization in the frequency domain is inaccurate for doubly selective channels with severe ICI; see [10]–[12]. Direct methods, like MMSE equalization, have high computational complexity and memory usage. Low-complexity methods that rely on approximation by a banded matrix in the frequency domain, are equivalent to using the CE-BEM (see Theorem 1), and correct only relatively modest ICI.

In their stead, we propose equalization with iterative methods for the regularized solution of linear systems. In this paper, we use two standard iterative methods, namely GMRES [25] and LSQR [26]. The iterative nature of our method is strictly limited to the algorithms that we use as linear solvers; see Appendix A for their detailed descriptions. In particular, we do not use a feedback loop with partially equalized signal symbols in this paper. Both GMRES and LSQR are Krylov subspace methods, i.e., each approximate solution is sought within an increasing family of Krylov subspaces. Specifically, at the ith iteration GMRES constructs an approximation within the subspace

$$\mathcal{K}(\mathbf{H}, \mathbf{y}, i) = \operatorname{Span}\left\{\mathbf{y}, \mathbf{H}\mathbf{y}, \mathbf{H}^2\mathbf{y}, \ldots, \mathbf{H}^{(i-1)}y\right\} \quad (8)$$

whereas LSQR within the subspace

$$\mathcal{K}(\mathbf{H}^H\mathbf{H}, \mathbf{H}^H\mathbf{y}, i) = \operatorname{Span}\left\{\mathbf{H}^H\mathbf{y}, (\mathbf{H}^H\mathbf{H})\mathbf{H}^H\mathbf{y}, \ldots, (\mathbf{H}^H\mathbf{H})^{(i-1)}\mathbf{H}^H\mathbf{y}\right\}. \quad (9)$$

For a comparison of GMRES and LSQR, see Table I. Both methods use the number of iterations as a regularization parameter.

At each iteration, GMRES and LSQR require the computation of matrix-vector products of the form $\mathbf{Hv}$, together with vector additions, scalar multiplications, and finding the 2-norms of vectors. Additionally, LSQR needs matrix-vector products of the form $\mathbf{H}^H\mathbf{v}$. Since the most expensive part is the computation of the matrix-vector products, the complexity of one iteration of LSQR is approximately twice that of one iteration of GMRES. In Appendix A, we provide detailed descriptions of both GMRES and LSQR. With the product-convolution structure of the channel matrix $\mathbf{H}$, computational complexity is reduced dramatically; see Section IV and Table II.

B. Regularizing Properties of LSQR

In exact arithmetic, the LSQR algorithm is equivalent to the conjugate gradient method applied to the normal equations [26]. Consequently, within K iterations LSQR computes an exact solution of a $K \times K$ system, which amounts to Zero Forcing. In practice, the inputs of LSQR are known only approximately, and using all K iterations visibly amplifies the modeling errors and noise. However, LSQR has a built-in regularization mechanism, with the number of iterations as a regularization parameter. Consequently, an early termination of iterations effectively prevents the amplification of errors and noise. The error obtained using LSQR with the optimal number of iterations is comparable to that of MMSE equalization with the optimal Tikhonov regularization parameter. An excellent survey of regularization methods is given in [28]. A detailed treatment of regularization with LSQR is given in [35, Sec. 7.6]. Combining these two references, one can conclude that LSQR achieves the minimum error possible for a certain class of regularization methods. Our numerical results also confirm that equalization with LSQR is equivalent to MMSE equalization.

Another regularization method consists of applying LSQR to the matrix $\underline{\mathbf{H}}$ and the vector $\underline{\mathbf{y}}$ defined as follows:

$$\underline{\mathbf{H}} = \begin{pmatrix} \mathbf{H} \\ \sigma^2\mathbf{I} \end{pmatrix}, \quad \underline{\mathbf{y}} = \begin{pmatrix} \mathbf{y} \\ \mathbf{0} \end{pmatrix} \quad (10)$$

in order to solve the following least square problem:

$$\tilde{\mathbf{x}} = \arg\max_{\mathbf{x}} \left\{\|\mathbf{Hx} - \mathbf{y}\|_2^2 + \sigma^2\|\mathbf{x}\|_2^2\right\} \quad (11)$$

$$= \arg\max_{\mathbf{x}} \left\{\|\underline{\mathbf{H}}\mathbf{x} - \underline{\mathbf{y}}\|_2^2\right\}. \quad (12)$$

This approach is commonly known as damped LSQR; see Appendix A. The choice of the parameter σ depends on ambient noise and the modeling error. Damped LSQR combines LSQR with the Tikhonov regularization, and it has two regularization parameters: the number of iterations and the (continuous) damping parameter. Our numerical simulations show of that the minimum achievable BERs are similar for LSQR and damped LSQR, even if the noise parameters are known exactly at the receiver. However, a major advantage of damped LSQR is that semi-convergence is much milder, that is the BER as a function of the number of iterations grows very slowly after reaching its minimum.

The optimal number of LSQR iterations depends directly on the noise level, and the distribution of the singular values of the channel matrix, which in turn depends on the maximum Doppler spread and the maximum delay spread. However, we have observed experimentally, that the number of iterations does not essentially depend on the number of OFDM subcarriers K.

C. Preconditioning

It is well-known that the rate of convergence of the conjugate gradient method depends on the condition number of the underlying matrix. Specifically, for a matrix with condition number κ, the error of CG iterations decreases exponentially at the rate of

TABLE II
THE COMPLEX FLOP COUNT FOR THE PROPOSED ALGORITHM PER OFDM SYMBOL WITH i ITERATIONS OF GMRES, LSQR OR DAMPED LSQR

step	*description*	*complex flop count*
1	computing $\mathbf{D}_m$ from the BEM coefficients b_{lm}	***MK logK***
2	computing $\tilde{\mathbf{D}}_m$ from $\mathbf{D}_m$	$(M-1)K$
3	solving $\tilde{\mathbf{x}}$ iteratively using:	
	a. GMRES	***(i+1)MK logK*** $+(i^2+3i+Mi+M)K+5i^2+12i+7$
	b. LSQR	***(2i+1)MK logK*** $+(10i+3+2Mi+2M)K+14i+1$
	c. damped LSQR	***(2i+1)MK logK*** $+(12i+3+2Mi+2M)K+14i+1$
4	equalizing $\mathbf{A}$ as $\mathbf{D}_0^{-1}\mathbf{F}\tilde{\mathbf{x}}$	***K logK*** $+K$
5	quantizing to symbol constellation	K

$(\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)$; see [36, Theorem 10.2.6]. Consequently, CG typically converges much faster if applied to a matrix with a smaller condition number. This observation is routinely used to accelerate convergence of CG (and LSQR) by means of preconditioning. Preconditioning accelerates convergence of CG by effectively replacing a given matrix with one that has a smaller condition number [36, Sec. 10.3].

Similarly, convergence of GMRES can be accelerated if preconditioning can group the matrix eigenvalues in a cluster away from the point $z=0$ in the complex plane; see [37, Corollary 6.33].

For both LSQR and GMRES, an approximate inverse of a matrix is commonly used as a preconditioner, since the condition number of the so preconditioned matrix is reduced, and its eigenvalues are clustered around the point $z=1$ in the complex plane.

The first term of the product-convolution representation (7) equal to $\mathbf{P}_0\mathbf{C}_0$ may be regarded as a crude approximation to the channel matrix $\mathbf{H}$. Consequently, $(\mathbf{P}_0\mathbf{C}_0)^{-1}$ is a suitable choice for a preconditioner. Some bases, for example, the Legendre polynomials, or complex exponential basis, contain a constant function. In such cases, $\mathbf{P}_0$ is a constant multiple of the identity matrix, and we simply use $\mathbf{C}_0^{-1}$ as a preconditioner. Thus, we actually use the single-tap equalizer in the frequency domain as a preconditioner; see (21). The matrix $\mathbf{C}_0^{-1}$ is the exact inverse of the channel matrix for a purely frequency selective channel, and serves as an approximate inverse for a doubly selective channel matrix with a moderate Doppler shift. However, for channels with large Doppler shifts, $\mathbf{C}_0^{-1}$ is not a useful preconditioner.

In order to describe our preconditioner, we introduce a new variable

$$\tilde{\mathbf{x}} = \mathbf{C}_0\mathbf{x} \tag{13}$$

and substitute it into (3) in the following manner

$$\begin{aligned}\mathbf{y} &= \mathbf{H}\mathbf{x}+\mathbf{w} &(14)\\ &= \mathbf{H}\mathbf{C}_0^{-1}\mathbf{C}_0\mathbf{x}+\mathbf{w} &(15)\\ &= \mathbf{H}\mathbf{C}_0^{-1}\tilde{\mathbf{x}}+\mathbf{w} &(16)\\ &= \tilde{\mathbf{H}}\tilde{\mathbf{x}}+\mathbf{w} &(17)\end{aligned}$$

where $\tilde{\mathbf{H}}=\mathbf{H}\mathbf{C}_0^{-1}$. In view of (7), we have

$$\begin{aligned}\tilde{\mathbf{H}} &= \mathbf{H}\mathbf{C}_0^{-1} &(18)\\ &= \sum_{m=0}^{M-1}\mathbf{P}_m\mathbf{C}_m\mathbf{C}_0^{-1} &(19)\\ &= \sum_{m=0}^{M-1}\mathbf{P}_m\tilde{\mathbf{C}}_m &(20)\end{aligned}$$

where $\tilde{\mathbf{C}}_m=\mathbf{C}_m\mathbf{C}_0^{-1}$ for $m=0,1,\ldots,M-1$. Clearly, the transformed time-domain channel matrix $\tilde{\mathbf{H}}$ is also a sum of product-convolutions, so both matrices $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{H}}^H$ can be applied at a cost $\mathcal{O}(K\log K)$. Algebraically, replacing (3) by (17) is classified as right preconditioning. For some bases, e.g., that of discrete prolates, $\mathbf{P}_0$ is not a constant multiple of the identity matrix. In such cases, one should use both left and right preconditioning; see Appendix B for details.

The eigenvalues of a representative time-domain channel matrix $\mathbf{H}$, and its preconditioned version $\tilde{\mathbf{H}}=\mathbf{H}\mathbf{C}_0^{-1}$, are shown in Fig. 1. We notice that the eigenvalues of the preconditioned matrix $\tilde{\mathbf{H}}$ are clustered near the point $z=1$ in the complex plane. We have observed experimentally, that preconditioning with the single-tap equalizer is not effective for channels whose Doppler shift exceeds 25% of the intercarrier frequency spacing. Such channels are far away from being frequency selective, and the single-tap equalizer is not a reliable approximate inverse.

IV. DESCRIPTION OF THE ALGORITHM

A. Decomposition of Channel Matrix

The proposed equalization uses only the BEM coefficients of the channel taps and the time-domain receive signal. We assume that estimates of the BEM coefficients are known, for example they are provided by one of the estimation methods mentioned in the introduction. In this subsection, we derive algebraic formulas for the algorithms presented in the next subsection.

It is well known (for example, see [36, p. 202]) that conjugating a circulant matrix by the discrete Fourier transform

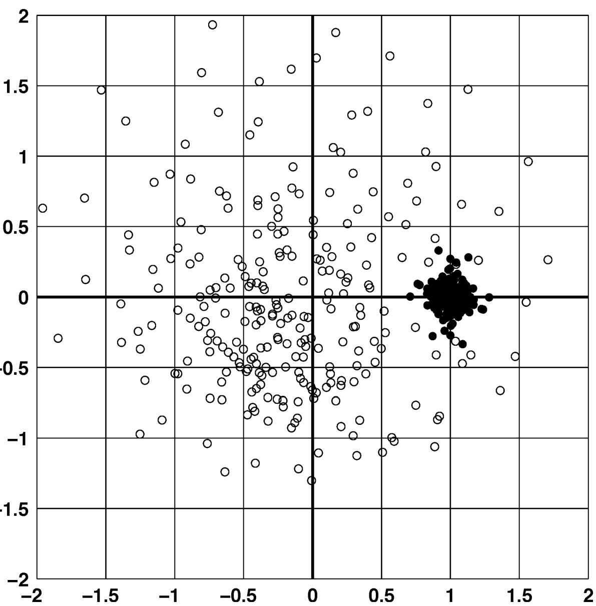

Fig. 1. The eigenvalues of the time-domain channel matrix with a Doppler shift equal to 17% of the inter carrier frequency spacing without preconditioning '∘' and with preconditioning '•'.

(DFT) results in a diagonal matrix. The cyclic convolution matrices $\mathbf{C}_m$ are thus expressed as

$$\mathbf{C}_m = \mathbf{F}^H \mathbf{D}_m \mathbf{F}, \qquad m = 0, \ldots, M-1 \tag{21}$$

where $\mathbf{D}_m$ are diagonal matrices, and $\mathbf{F}$ is the matrix of the DFT in K dimensions. The diagonal of the matrix $\mathbf{D}_m$ coincides with the DFT of the BEM coefficients $b_{\cdot m}$ zero-padded to length K, as follows:

$$\mathbf{D}_m(i,i) = (\mathbf{F}[b_{\cdot m}, 0, \ldots, 0]^T)(i), \tag{22}$$

for $i = 1, \ldots, K$, and $(\,\cdot\,)^T$ denotes the transpose operation.

Substituting relation (21) into (19), we get

$$\tilde{\mathbf{H}} = \sum_{m=0}^{M-1} \mathbf{P}_m \mathbf{F}^H \mathbf{D}_m \mathbf{F} \mathbf{F}^H \mathbf{D}_0^{-1} \mathbf{F} \tag{23}$$

$$= \sum_{m=0}^{M-1} \mathbf{P}_m \mathbf{F}^H \tilde{\mathbf{D}}_m \mathbf{F} \tag{24}$$

where

$$\tilde{\mathbf{D}}_m = \mathbf{D}_m \mathbf{D}_0^{-1} \tag{25}$$

is a diagonal matrix. Similarly, substituting relation (21) into (13), we get

$$\tilde{\mathbf{x}} = \mathbf{C}_0 \mathbf{x} = \mathbf{F}^H \mathbf{D}_0 \mathbf{F} \mathbf{x}. \tag{26}$$

Combining (24) and (17), we express the time-domain receive signal $\mathbf{y}$ in the following form:

$$\mathbf{y} = \left(\sum_{m=0}^{M-1} \mathbf{P}_m \mathbf{F}^H \tilde{\mathbf{D}}_m \mathbf{F} \right) \tilde{\mathbf{x}} + \mathbf{w}. \tag{27}$$

In the frequency domain, (27) has the form

$$\mathbf{Y} = \mathbf{F}\mathbf{y} \tag{28}$$

$$= \mathbf{F} \left(\sum_{m=0}^{M-1} \mathbf{P}_m \mathbf{F}^H \tilde{\mathbf{D}}_m \mathbf{F} \right) \tilde{\mathbf{x}} + \mathbf{W} \tag{29}$$

$$= \mathbf{F} \left(\sum_{m=0}^{M-1} \mathbf{P}_m \mathbf{F}^H \tilde{\mathbf{D}}_m \right) \mathbf{F}\mathbf{F}^H \mathbf{D}_0 \mathbf{F} \mathbf{x} + \mathbf{W} \tag{30}$$

$$= \underbrace{\mathbf{F} \left(\sum_{m=0}^{M-1} \mathbf{P}_m \mathbf{F}^H \tilde{\mathbf{D}}_m \right)}_{\text{ICI}} \underbrace{\mathbf{D}_0}_{\text{FS}} \mathbf{A} + \mathbf{W} \tag{31}$$

where $\mathbf{W}$ is the noise in the frequency domain,

$$\mathbf{A} = \mathbf{F}\mathbf{x} = \mathbf{D}_0^{-1} \mathbf{F} \tilde{\mathbf{x}} \tag{32}$$

is the frequency-domain transmit signal as used in (1), and $\mathbf{Y}$ is the receive signal in the frequency domain. Equation (31) demonstrates that a doubly selective frequency-domain channel matrix is the product of a frequency selective (FS) operator $\mathbf{D}_0$, and an operator accounting for ICI. Equivalently, a frequency-domain channel can be represented as a sum of convolution-products.

B. Algorithm

The proposed equalization algorithm in the time domain is based on (24), and can be summarized as follows: given the time-domain receive signal $\mathbf{y}$ and the BEM coefficients b_{lm}, we solve for $\tilde{\mathbf{x}}$ using iterative solvers GMRES or LSQR, and then we approximate $\mathbf{A}$ with $\mathbf{D}_0^{-1}\mathbf{F}\tilde{\mathbf{x}}$. Specifically, we perform the following steps.

Step 1) Compute the diagonal matrices $\mathbf{D}_m$ from the BEM coefficients b_{lm}; see (22).

Step 2) Compute the diagonal matrices $\tilde{\mathbf{D}}_m = \mathbf{D}_m \mathbf{D}_0^{-1}$; see (25).

Step 3) Solve (27) for $\tilde{\mathbf{x}}$ using GMRES or LSQR.

Step 4) Approximate $\mathbf{A}$ as $\mathbf{D}_0^{-1}\mathbf{F}\tilde{\mathbf{x}}$; see (32).

Step 5) Quantize according to the alphabet used (4-QAM, PSK, etc.).

We employ Step 2) only if we do preconditioning, otherwise we take $\tilde{\mathbf{D}}_m$ equal to $\mathbf{D}_m$.

A similar algorithm for equalization in the frequency domain can be formulated using (31), with the frequency-domain channel matrix expressed as a sum of convolution-products. Equalization in the time and in the frequency domain gives identical errors, because the errors are related by a unitary operator.

Other iterative methods for the solution of linear systems can also be used in place of LSQR and GMRES.

C. Computational Complexity

We report operation counts of equalization of one OFDM symbol. We consider the diagonal matrices $\mathbf{P}_m$ to be precomputed. The computation of diagonal matrices $\mathbf{D}_m$ from the BEM coefficients in Step 1) requires $\mathcal{O}(MK \log K)$ operations. Whenever preconditioning is used, we perform Step 2) (creation of the diagonal matrices $\tilde{\mathbf{D}}_m$), which requires $\mathcal{O}(MK)$ operations. In Step 3), we solve for $\tilde{\mathbf{x}}$ using iterative methods

GMRES or LSQR, which requires $\mathcal{O}(K \log K)$ operations per iteration. A typical number of iterations does not exceed 16. In Step 4), we compute the frequency-domain transmit signal $\mathbf{A}$ from $\tilde{\mathbf{x}}$ using (32), which requires $\mathcal{O}(K \log K)$ operations. In Step 5), we quantize the signal $\mathbf{A}$ according to the alphabet used at the cost of $\mathcal{O}(K)$ operations. A detailed breakdown of computational complexity is provided in Table II.

For certain narrowband applications in high Doppler regimes, the number of discrete channel taps L is relatively small compared to K. For such applications, the convolution should be implemented directly with the complexity of LK flops per iteration.

D. Memory

The equalization process begins with the time-domain receive signal $\mathbf{y}$ and the BEM coefficients b_{lm}, which are stored as K and ML floating point complex numbers, respectively. $\mathbf{P}_m$ and $\tilde{\mathbf{D}}_m$ are diagonal matrices, which are stored as K complex numbers each. The matrix-vector multiplications required by GMRES and LSQR are done using pointwise multiplications and the FFT-s of size K; see (24). After the ith iteration, GMRES requires storing $i+1$ vectors of length K, while LSQR requires storing four vectors of length K. Thus, the proposed algorithm requires $\mathcal{O}(K)$ memory.

E. Comparison With Other Low-Complexity Equalizers

Current equalizers achieve a reduced complexity by means of approximation by a banded matrix in the frequency domain [17], [9], [38]. Generally, such methods also require preprocessing with a time domain window, which increases ICI of neighboring subcarriers relative to distant ones. A banded approximation of the frequency domain channel matrix is equivalent to the complex exponential BEM (CE-BEM). A rigorous formulation is given in the following theorem (a related discussion is also provided in [39]).

Theorem 1: Let $\mathbf{H}$ be an arbitrary $K \times K$ time-domain channel matrix with the maximum discrete delay $L-1$, and let $\mathbf{B}$ be the (cyclically) banded truncation of the frequency-domain channel matrix $\mathbf{FHF}^H$ with the bandwidth $2Q+1, 0 \leq Q < (K/2)$. The time-domain matrix $\mathbf{E} = \mathbf{F}^H\mathbf{BF}$ is a CE-BEM matrix with the model order $2Q+1$, and with the same maximum discrete delay $L-1$.

However, the CE-BEM is known to be inaccurate for doubly selective wireless channels; see [2] and [5].

Setting aside the computational complexity, the most accurate equalization is obtained using the entire channel matrix. This is prominent in the results obtained in [17] by using the matched filter bound (MFB) equalization, and in [38] by using the non-banded block linear equalizer (BLE). Moreover, no windowing is used by the two equalizers. However, the equalization results with the full channel matrix in [9] are not significantly better than those of equalization with a banded approximation. This is because preprocessing with a window in the time domain is done before equalization; see Section VI in [9]. We have observed experimentally that such preprocessing increases the condition number of the whole channel matrix by orders of magnitude and degrades the BER after equalization with the entire channel matrix. Low-complexity equalizers in [9], [17], and [38] use a banded approximation of the frequency domain channel matrix. On the other hand, our proposed algorithm achieves low complexity by means of a representation of the channel matrix as a sum of product-convolution operators, which is inherent in any BEM. Thus, we use sparsity of the channel matrix in a different way than a banded approximation of the channel matrix in frequency domain. It is observed in [21] that for equalization with LSQR is as accurate as one with MMSE. We confirm this observation experimentally in Section V.

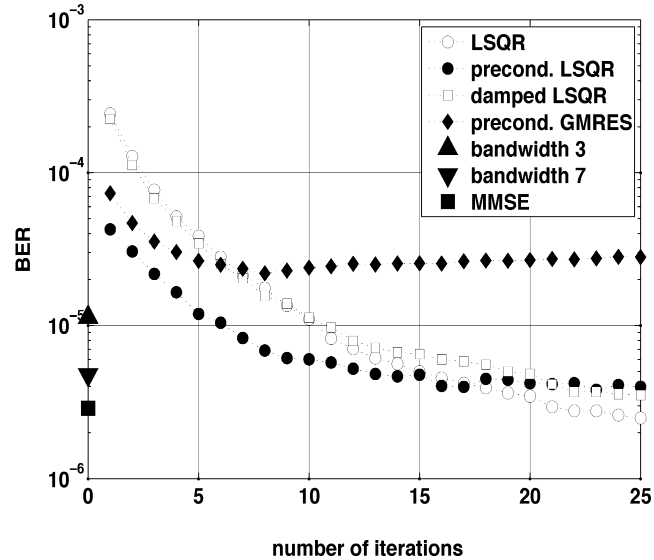

Fig. 2. The BER as a function of the number of iterations at the receiver velocity of 175 km/h and the SNR of $E_b/N_0 = 20$ dB using exact CSI.

The computation of the optimal windows, as suggested in [17], [38], requires the second order channel statistics, which are not easy to obtain. In this paper we discuss methods which do not require any statistical information. Thus, when comparing the proposed equalization method with banded equalizers, we perform a linear preprocessing with a fixed window before applying the banded equalizer. Specifically, we use the Blackman window, which is close to the window proposed in [38]. Figs. 2–4 compare the BERs of the proposed method with equalization using a banded frequency-domain channel matrix, which is preprocessed with a Blackman window. The latter equalization with bandwidth D requires $\mathcal{O}(D^2K)$ operations. The proposed equalizer does not use any windowing. We note that using a time domain window on the receiver's side is equivalent to using a BEM with a modified basis, namely the one obtained by multiplying the original basis by the window. Thus, a windowed BEM is also encompassed by our method, which applies to arbitrary bases.

V. Computer Simulations

A. Simulation Setup

Our transmission setup conforms to the IEEE 802.16e specifications. We simulate a coded OFDM system with $K = 256$ subcarriers, utilizing $B = 2.8$ MHz of bandwidth at a carrier frequency of $f_c = 5.8$ GHz. We use a cyclic prefix of length $L_{\rm cp} = 32$ in order to avoid ISI. Consequently, the sampling period is $T_s = 1/B = 0.357$ μs, and the symbol duration is

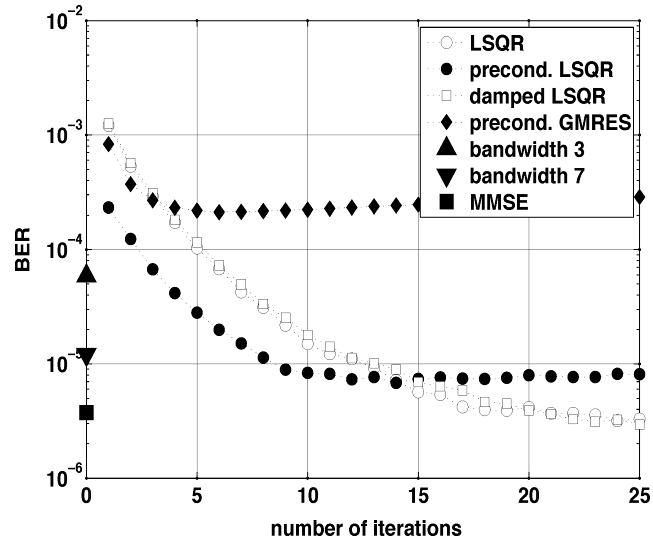

Fig. 3. The BER as a function of the number of iterations at the receiver velocity of 300 km/h and the SNR of $E_b/N_0 = 20$ dB using exact CSI.

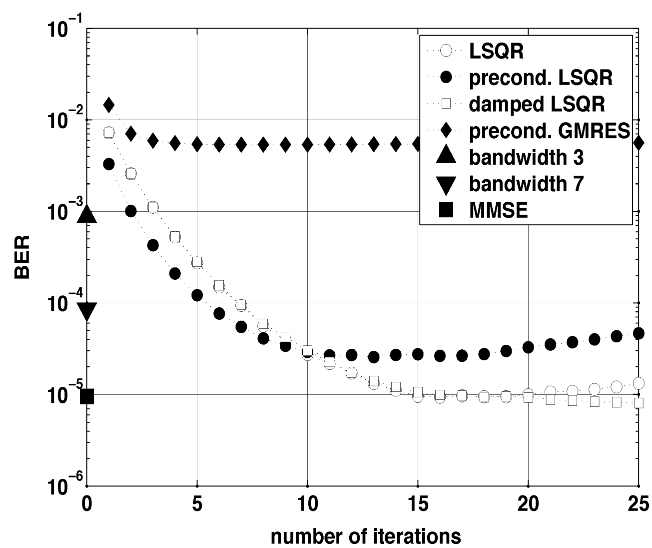

Fig. 4. The BER as a function of the number of iterations at the receiver velocity of 550 km/h and the SNR of $E_b/N_0 = 20$ dB using exact CSI.

$(K + L_{\text{cp}})T_s = (256 + 32) \times 0.357\ \mu\text{s} = 102.9\ \mu\text{s}$. The information bits are encoded using a convolutional code of rate 1/2, passed through an interleaver, and mapped to 4-QAM symbols. For experiments with estimated BEM coefficients, we use a frequency-domain Kronecker delta (FDKD) pilot arrangement in each OFDM symbol, as described in [40], [8]. The pilots are only used for estimation of the BEM coefficients, and do not have any influence on the proposed equalization algorithm. Experiments with exact channel state information (CSI) do not use pilots in transmission.

We simulate a wide sense stationary uncorrelated scattering (WSSUS) Rayleigh fading channel with a maximum delay of $11.4\ \mu$s, which corresponds to the worst case when $L = L_{\text{cp}} = 32$ taps. Each tap has an average path gain of -2 dB and a Jakes Doppler spectrum. We filter the simulated transmit signal through channels with varying maximum Doppler shifts. The maximum Doppler shift $\nu_{\max}$ is related to the receiver velocity v by the formula

$$\nu_{\max} = \frac{v}{c} f_c \tag{33}$$

where c is the speed of light. To the signal filtered through the channel, we add additive white Gaussian noise (AWGN) of varying energy per data bit to noise spectral density (E_b/N_0). The values of $\nu_{\max}, v, E_b/N_0$ are reported for all the experiments. The channel is simulated using the MATLAB Communication Toolbox (version 4.2).

At the receiver, we first compute the BEM coefficients. Specifically, in our experiments we use the basis of the Legendre polynomials [8]. In experiments with estimated channel taps, we use the algorithm described in [8] for estimation of the BEM coefficients. In experiments with the exact channel matrix, we compute the BEM coefficients by projecting the channel taps on the basis functions. Subsequently, we equalize the receive signal using the proposed algorithm. Finally, the equalized signal is quantized and decoded using the BCJR algorithm and deinterleaved. As a measure of performance, we report the bit error rate (BER) averaged over 100 000 OFDM symbols.

B. Discussion of Results

First, we study the dependence of the BER on the number of iterations of GMRES, LSQR, and damped LSQR, with and without preconditioning. In the case of GMRES, we only present the results with preconditioning, since in our case without preconditioning GMRES needs approximately K iterations to achieve a useful BER. We note that one iteration of LSQR requires approximately twice as many flops as that of GMRES; see Table II for details. Figs. 2–4 show the BER as a function of the number of iterations at receiver velocities of 175, 300, and 550 km/h, respectively. We notice, that receiver velocities of 175, 300, and 550 km/h correspond to reflector velocities of 87.5, 150, and 275 km/h, respectively, and are ubiquitous in the modern environment. Additive noise in the channel is simulated for a fixed SNR of $E_b/N_0 = 20$ dB. The exact CSI is used in all these experiments. The BER at iteration number zero corresponds to single-tap equalization, and is shown for comparison.

Fig. 2 presents results for the receiver velocity of 175 km/h, which corresponds to a Doppler shift of 0.94 kHz, or about 8.6% of the subcarrier spacing. The BERs of the banded equalizer with bandwidth 3 and 7 are equal to $1.1 \cdot 10^{-5}$ and $4.8 \cdot 10^{-6}$, respectively. The BER of MMSE equalization equals $2.9 \cdot 10^{-6}$. The BER of preconditioned GMRES decreases from $7.3 \cdot 10^{-5}$ after one iteration to $2.2 \cdot 10^{-5}$ after eight iterations. The BER of LSQR decreases from $2.4 \cdot 10^{-4}$ after one iteration to $2.5 \cdot 10^{-6}$ after 25 iterations. The BER of preconditioned LSQR decreases from $4.3 \cdot 10^{-5}$ after one iteration to $4.0 \cdot 10^{-6}$ after 16 iterations. The BER of damped LSQR decreases from $2.2 \cdot 10^{-4}$ after one iteration to $3.5 \cdot 10^{-6}$ after 25 iterations.

Fig. 3 presents results for the receiver velocity of 300 km/h, which corresponds to a Doppler shift of 1.61 kHz, or about 14.7% of the subcarrier spacing. The BERs of the banded equalizer with bandwidth 3 and 7 are equal to $5.9 \cdot 10^{-5}$ and $1.2 \cdot 10^{-5}$, respectively. The BER of MMSE equalization equals $3.7 \cdot 10^{-6}$. The BER of preconditioned GMRES decreases from $8.3 \cdot 10^{-4}$ after one iteration to $2.1 \cdot 10^{-4}$ after 6 iterations. The BER of LSQR decreases from $1.2 \cdot 10^{-3}$ after one iteration to $3.2 \cdot 10^{-6}$ after 24 iterations. The BER of preconditioned LSQR decreases from $2.3 \cdot 10^{-4}$ after one iteration to $6.8 \cdot 10^{-6}$ after 14 iterations. The BER of damped LSQR decreases from $1.3 \cdot 10^{-3}$ after one iteration to $3.0 \cdot 10^{-6}$ after 25 iterations.

Fig. 4 presents results for the receiver velocity of 550 km/h, which corresponds to a Doppler shift of 2.95 kHz, or about 27% of the subcarrier spacing. The BERs of the banded equalizer with bandwidth 3 and 7 are equal to $8.8 \cdot 10^{-4}$ and $8.6 \cdot 10^{-5}$, respectively. The BER of MMSE equalization equals $9.5 \cdot 10^{-6}$. The BER of preconditioned GMRES decreases from $1.5 \cdot 10^{-2}$ after one iteration to $5.4 \cdot 10^{-3}$ after 5 iterations. The BER of LSQR decreases from $7.3 \cdot 10^{-3}$ after one iteration to $9.4 \cdot 10^{-6}$ after 16 iterations. The BER of preconditioned LSQR decreases from $3.3 \cdot 10^{-3}$ after one iteration to $2.6 \cdot 10^{-5}$ after 13 iterations. The BER of damped LSQR decreases from $7.2 \cdot 10^{-3}$ after one iteration to $8.1 \cdot 10^{-6}$ after 25 iterations.

All iterative methods in Figs. 2–4 display the phenomenon known as semi-convergence. Specifically, the first few iterations provide approximations of increasing accuracy, which is confirmed by the decreasing BERs. The subsequent iterations do not further improve the solution, and sometimes even amplify ambient noise, as evidenced by the slowly increasing BERs. We observe that both LSQR and damped LSQR achieve comparable BERs. Thus, we only consider LSQR in our further experiments.

Figs. 5–7 show the dependence of the BER on the SNR expressed in terms of the energy per data bit to noise spectral density ratio E_b/N_0 for channels simulated with the receiver speed of 175, 300, and 550 km/h. In all experiments, the number of iterations depends on the receiver velocity and the iterative method used, but not on the SNR. Specifically, the number of iterations is optimized for a fixed SNR of $E_b/N_0 = 20$ dB. For example, for the exact channel the iteration numbers are determined from Figs. 2–4. Further optimization of the number of iterations with respect to the SNR is not practical, since the noise level is often unknown. We present our results for channels estimated using a pilot-aided method described in [8], and also the results obtained with the exact channel matrix as a benchmark. We also report the BERs of the banded equalizer with bandwidth 7, and those of the MMSE equalizer.

Fig. 5(a) and (b) shows the BER with the exact and estimated CSI, respectively, corresponding to the receiver velocity of 175 km/h, or 8.6% of the subcarrier spacing. With exact CSI, we use 20 iterations of LSQR, and 10 iterations of preconditioned LSQR, and 8 iterations of preconditioned GMRES. With estimated CSI, we use 14 iterations of LSQR, and 8 iterations of preconditioned LSQR, and 5 iterations of preconditioned GMRES. We report the observed BERs for $E_b/N_0 =$ 15 dB, and E_b/N_0 =25 dB, respectively, which are of high practical interest. With exact CSI, LSQR, respectively, achieves the BERs of $9.0 \cdot 10^{-5}$ and $6.3 \cdot 10^{-7}$, preconditioned LSQR achieves the BERs of $9.8 \cdot 10^{-5}$ and $1.6 \cdot 10^{-6}$, and preconditioned GMRES achieves the BERs of $3.1 \cdot 10^{-4}$ and $1.6 \cdot 10^{-5}$. The banded equalizer achieves the BERs of $2.8 \cdot 10^{-4}$ and $2.3 \cdot 10^{-6}$, and the MMSE equalizer achieves the BERs of $8.6 \cdot 10^{-5}$ and $4.1 \cdot 10^{-7}$, respectively. With estimated CSI, LSQR, respectively, achieves the BERs of $3.5 \cdot 10^{-3}$ and $1.2 \cdot 10^{-5}$, preconditioned LSQR achieves the BERs of $3.9 \cdot 10^{-3}$ and $2.0 \cdot 10^{-5}$, and preconditioned GMRES achieves the BERs of $3.9 \cdot 10^{-3}$ and $3.4 \cdot 10^{-5}$. The banded equalizer achieves the BERs of $4.4 \cdot 10^{-3}$ and $4.0 \cdot 10^{-5}$, and the MMSE equalizer achieves the BERs of $3.0 \cdot 10^{-3}$ and $1.3 \cdot 10^{-5}$, respectively.

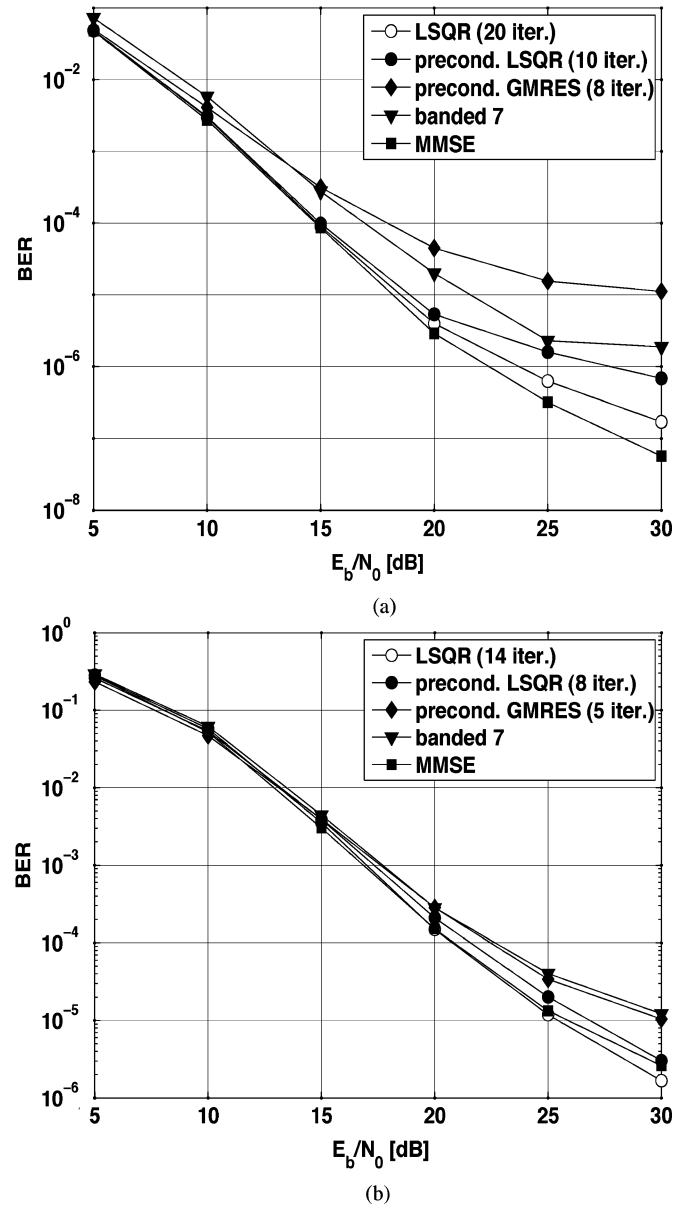

Fig. 5. The BER as a function of the SNR expressed as E_b/N_0 at the receiver velocity of 175 km/h. (a) Using exact CSI. (b) Using estimated CSI.

Fig. 7(a) and (b) shows the BER with the exact and estimated CSI, respectively, corresponding to the receiver velocity of 300 km/h, or 14.7% of the subcarrier spacing. With exact CSI, we use 20 iterations of LSQR, and ten iterations of preconditioned LSQR, and eight iterations of preconditioned GMRES. With estimated CSI, we use ten iterations of LSQR, and six iterations of preconditioned LSQR, and five iterations of preconditioned GMRES. We report the observed BERs for $E_b/N_0 =$ 15 dB, and $E_b/N_0 = 25$ dB, respectively. The banded equalizer

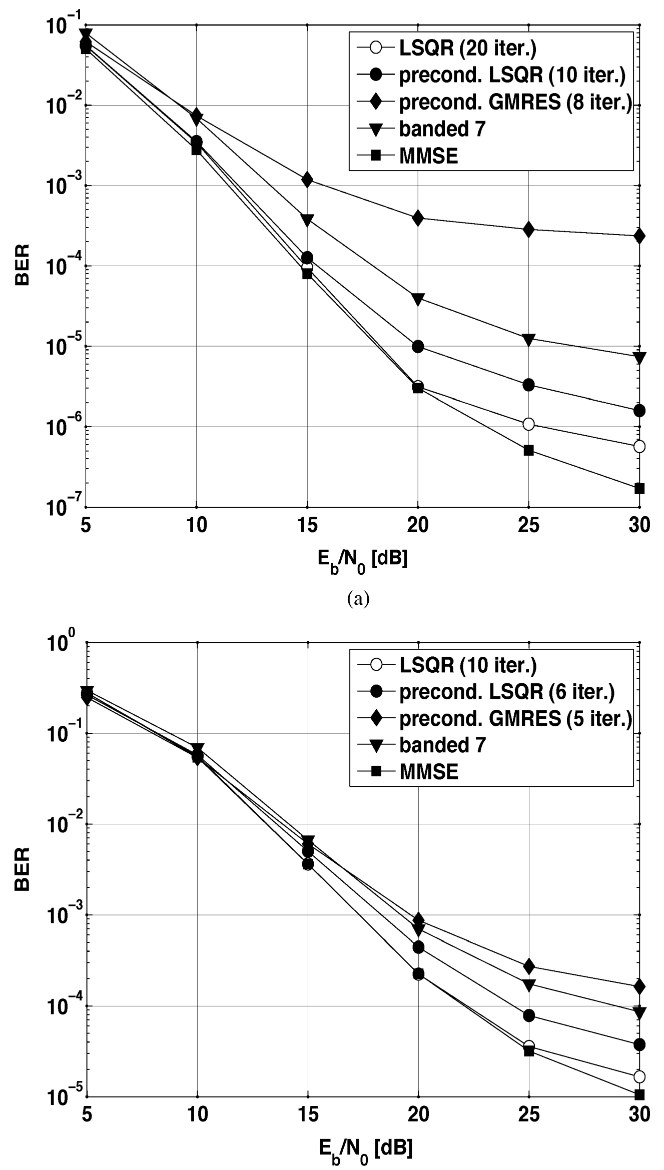

Fig. 6. The BER as a function of the SNR expressed as E_b/N_0 at the receiver velocity of 300 km/h. (a) Using exact CSI. (b) Using estimated CSI.

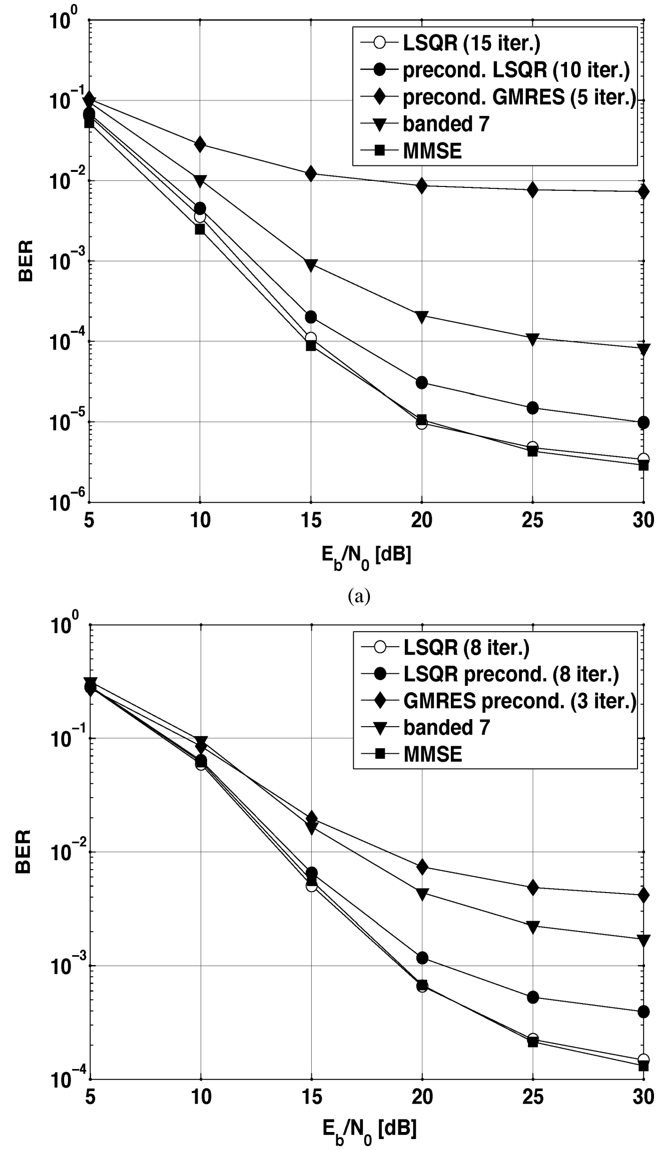

Fig. 7. The BER as a function of the SNR expressed as E_b/N_0 at the receiver velocity of 550 km/h. (a) Using exact CSI. (b) Using estimated CSI.

achieves the BERs of $3.8 \cdot 10^{-4}$ and $1.3 \cdot 10^{-5}$, and the MMSE equalizer achieves the BERs of $8.0 \cdot 10^{-5}$ and $5.1 \cdot 10^{-7}$, respectively. With exact CSI, LSQR, respectively, achieves the BERs of $9.5 \cdot 10^{-5}$ and $1.1 \cdot 10^{-6}$, preconditioned LSQR achieves the BERs of $1.3 \cdot 10^{-4}$ and $3.3 \cdot 10^{-6}$, and preconditioned GMRES achieves the BERs of $1.2 \cdot 10^{-3}$ and $2.8 \cdot 10^{-4}$. With estimated CSI, LSQR, respectively, achieves the BERs of $3.6 \cdot 10^{-3}$ and $3.6 \cdot 10^{-5}$, preconditioned LSQR achieves the BERs of $5.0 \cdot 10^{-3}$ and $7.8 \cdot 10^{-5}$, and preconditioned GMRES achieves the BERs of $6.0 \cdot 10^{-3}$ and $2.7 \cdot 10^{-4}$. The banded equalizer achieves the BERs of $6.7 \cdot 10^{-3}$ and $1.8 \cdot 10^{-4}$, and the MMSE equalizer achieves the BERs of $3.6 \cdot 10^{-3}$ and $3.2 \cdot 10^{-5}$, respectively.

Fig. 6(a) and (b) shows the BER with the exact and estimated CSI, respectively, corresponding to the receiver velocity of 550 km/h, or 27% of the subcarrier spacing. With exact CSI, we use 15 iterations of LSQR, and ten iterations of preconditioned LSQR, and five iterations of preconditioned GMRES. With estimated CSI, we use 14 iterations of LSQR, and 8 iterations of preconditioned LSQR, and five iterations of preconditioned GMRES. We report the observed BERs for $E_b/N_0 = 15$ dB, and $E_b/N_0 = 25$ dB, respectively. The banded equalizer achieves the BERs of $9.2 \cdot 10^{-4}$ and $1.1 \cdot 10^{-4}$, and the MMSE equalizer achieves the BERs of $8.8 \cdot 10^{-5}$ and $4.3 \cdot 10^{-6}$, respectively. With exact CSI, LSQR, respectively, achieves the BERs of $1.1 \cdot 10^{-4}$ and $4.8 \cdot 10^{-6}$, preconditioned LSQR achieves the BERs of $2.0 \cdot 10^{-4}$ and $1.5 \cdot 10^{-5}$, and preconditioned GMRES achieves the BERs of $1.2 \cdot 10^{-2}$ and $7.7 \cdot 10^{-3}$. With estimated CSI, LSQR, respectively, achieves the BERs of $5.0 \cdot 10^{-3}$ and $2.2 \cdot 10^{-4}$, preconditioned LSQR achieves the BERs of $6.5 \cdot 10^{-3}$ and $5.3 \cdot 10^{-4}$, and preconditioned GMRES achieves the BERs of $2.0 \cdot 10^{-2}$ and $4.8 \cdot 10^{-3}$. The banded equalizer achieves the BERs of $1.7 \cdot 10^{-2}$ and $2.2 \cdot 10^{-3}$, and the MMSE equalizer achieves the BERs of $5.6 \cdot 10^{-3}$ and $2.1 \cdot 10^{-4}$, respectively.

In our experiments, we observe that convergence of GMRES is very slow, which renders the method impractical. Preconditioned GMRES converges fast when applied to doubly selective channels with moderate Doppler shifts. LSQR is very effective for doubly selective channels with moderate to large Doppler shifts. Preconditioning LSQR with the single-tap equalizer accelerates convergence by a factor of about 2 in channels with moderate Doppler spreads. However, it is not effective for channels whose Doppler shift exceeds 25% of the intercarrier frequency spacing. Such channels are far away from being purely frequency selective, and the single-tap equalizer is not reliable as an approximate inverse.

The banded equalizers have a complexity of $\mathcal{O}(B^2K)$, where B is the bandwidth of a banded approximation in the frequency domain; see [9] for more details. In the example discussed in Fig. 3, the complexity of an LSQR based equalizer after 8 iterations is approximately the double of the complexity of a banded equalizer with bandwidth 7. On the other hand, the BER achieved with the LSQR based equalizer is equivalent to that of the full-block MMSE equalizer.

LSQR-based equalization outperforms equalization based on approximation by a banded matrix in the frequency domain by approximately a factor of 10 in BER at the normalized Doppler of 27%. Our results indicate that the proposed equalization method can be applied in practice in combination with presently available BEM estimation algorithms.

VI. Conclusion

We present a novel, low-complexity equalization method, which uses the BEM coefficients of the wireless channel taps without ever creating the channel matrix. The method is aimed at doubly selective channels with moderate and high Doppler spreads. Equalization is performed with classical iterative methods for linear systems, specifically with GMRES or LSQR. The main idea is to treat the wireless channel modeled by a basis expansion as a sum of product-convolution operators. This special structure permits a fast computation of matrix-vector products. For example, in case of an OFDM system with K subcarriers, each iteration costs $\mathcal{O}(K\log K)$ operations.

Convergence of both GMRES and LSQR can be significantly accelerated by preconditioning with the single-tap equalizer. We typically need 3–16 iterations for convergence. We validate our method by computer simulations, which use existing pilot-aided channel estimation methods. LSQR-based equalization outperforms equalization based on approximation by a banded matrix in the frequency domain by approximately a factor of 10 in BER at the normalized Doppler of 27%.

Appendix A
GMRES and LSQR

In this Appendix, we give a detailed description of GMRES [25] and LSQR [26], two well-known iterative methods for the numerical solution of a system of linear equations. We consider a linear system derived from the time-domain transmit-receive relation (3) by ignoring the noise component

$$\mathbf{H}\mathbf{x} = \mathbf{y}. \tag{34}$$

The matrix $\mathbf{H}$ is of size $K \times K$, while $\mathbf{x}$ and $\mathbf{y}$ are vectors of length K.

GMRES: The ith Krylov subspace for the matrix $\mathbf{H}$ and the vector $\mathbf{y}$ is defined as follows:

$$\mathcal{K}(\mathbf{H}, \mathbf{y}, i) = \operatorname{span}\left\{\mathbf{y}, \mathbf{H}\mathbf{y}, \mathbf{H}^2\mathbf{y}, \ldots, \mathbf{H}^{(i-1)}\mathbf{y}\right\}. \tag{35}$$

GMRES approximates the exact solution of (34) by a vector $\mathbf{x}_i \in \mathcal{K}(\mathbf{H}, \mathbf{y}, i)$ that minimizes the norm $\|r_i\|_2$ of the residual

$$r_i = \mathbf{y} - \mathbf{H}\mathbf{x}_i. \tag{36}$$

The vectors $\mathbf{y}, \mathbf{H}\mathbf{y}, \ldots, \mathbf{H}^{(i-1)}\mathbf{y}$ are not necessarily orthogonal, so the Arnoldi iteration [36, Sec. 9.4] is used to find orthonormal basis vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_i$ for $\mathcal{K}(\mathbf{H}, \mathbf{y}, i)$. Subsequently, the vector $\mathbf{x}_i \in \mathcal{K}(\mathbf{H}, \mathbf{y}, i)$ is written as $\mathbf{x}_i = \mathbf{Q}_i\mathbf{b}_i$, where $\mathbf{Q}_i$ is the $K \times i$ matrix formed by $\mathbf{q}_1, \ldots, \mathbf{q}_i$, and $\mathbf{b}_i \in \mathbb{C}^i$.

The Arnoldi process produces an $(i+1)\times i$ upper Hessenberg matrix $\mathcal{H}_i$ which satisfies

$$\mathbf{H}\mathbf{Q}_i = \mathbf{Q}_{(i+1)}\mathcal{H}_i. \tag{37}$$

Because $\mathbf{Q}_i$ has orthogonal columns, we have

$$\|\mathbf{y} - \mathbf{H}\mathbf{x}_i\|_2 = \|\mathcal{H}_i\mathbf{b}_n - \beta\mathbf{e}_1\|_2 \tag{38}$$

where $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)$, and $\beta = \|\mathbf{y}\|_2$. Therefore, $\mathbf{x}_i$ can be found by minimizing the norm of the residual

$$r_n = \beta\mathbf{e}_1 - \mathcal{H}_i\mathbf{b}_i. \tag{39}$$

This is a linear least squares problem of size $i \times i$, which is solved using the QR factorization. One can summarize the GMRES method as follows.

At every step of the iteration:

1) Do one step of the Arnoldi method.
2) Find the $\mathbf{b}_i$ which minimizes $\|r_i\|_2$ using the QR factorization.
3) Compute $\mathbf{x}_i = \mathbf{Q}_i\mathbf{b}_i$.

The steps are repeated until the residual norm is smaller than a required threshold.

LSQR: LSQR is an iterative algorithm for the approximate solution of the linear system (34). In exact arithmetic, LSQR is equivalent to the conjugate gradient method for the normal equations $\mathbf{H}^H\mathbf{H}\mathbf{x} = \mathbf{H}^H\mathbf{y}$. In the ith iteration, one constructs a vector $\mathbf{x}_i$ in the Krylov subspace

$$\mathcal{K}(\mathbf{H}^H\mathbf{H}, \mathbf{H}^H\mathbf{y}, i) = \operatorname{Span}\left\{\mathbf{H}^H\mathbf{y}, (\mathbf{H}^H\mathbf{H})\mathbf{H}^H\mathbf{y}, \ldots, (\mathbf{H}^H\mathbf{H})^{(i-1)}\mathbf{H}^H\mathbf{y}\right\} \tag{40}$$

that minimizes the norm of the residual $\|\mathbf{y} - \mathbf{H}\mathbf{x}_i\|_2$.

LSQR consists of two steps: the Golub–Kahan bidiagonalization and the solution of a bidiagonal least squares problem. The Golub–Kahan bidiagonalization [36] constructs vectors $\mathbf{u}_i, \mathbf{v}_i$, and positive constants $\alpha_i, \beta_i (i = 1, 2, \ldots)$ as follows:

1. Set $\beta_1 = \|\mathbf{y}\|_2, \mathbf{u}_1 = \mathbf{y}/\beta_1, \alpha_1 = \|\mathbf{H}^H\mathbf{y}\|_2, \mathbf{v}_1 = \mathbf{H}^H\mathbf{y}/\alpha_1$.
2. For $i = 1, 2, \ldots$, set

$$\begin{aligned}\beta_{i+1} &= \|\mathbf{H}\mathbf{v}_i - \alpha_i\mathbf{u}_i\|_2\\ \mathbf{u}_{i+1} &= (\mathbf{H}\mathbf{v}_i - \alpha_i\mathbf{u}_i)/\beta_{i+1}\\ \alpha_{i+1} &= \|\mathbf{H}^H\mathbf{u}_i - \beta_i\mathbf{v}_i\|_2\\ \mathbf{v}_{i+1} &= (\mathbf{H}^H\mathbf{u}_i - \beta_i\mathbf{v}_i)/\alpha_{i+1}.\end{aligned}$$

The process is terminated if $\alpha_{i+1} = 0$ or $\beta_{i+1} = 0$.

In exact arithmetic, the $\mathbf{u}_i$'s are orthonormal, and so are the $\mathbf{v}_i$'s. Therefore, one can reduce the approximation problem over the ith Krylov subspace to the following least square problem:

$$\min_{\mathbf{w}_i} \|\mathbf{B}_i\mathbf{w}_i - [\beta_1, 0, 0, \ldots]^T\|_2 \tag{41}$$

where $\mathbf{B}_i$ is the $(i+1) \times i$ lower bidiagonal matrix with $\alpha_1, \ldots, \alpha_i$ on the main diagonal, and $\beta_2, \ldots, \beta_{i+1}$ on the first subdiagonal. This least squares problem is solved at a negligible cost using the QR factorization of the bidiagonal matrix $\mathbf{B}_i$. Finally, the ith approximate solution is computed as

$$\mathbf{x}_i = \sum_{j=1}^{i} \mathbf{w}_i(j)\mathbf{v}_j. \tag{42}$$

The second LSQR step solves the least squares problem (41) using the QR factorization of $\mathbf{B}_i$. The computational costs of this step are negligible due to the bidiagonal structure of $\mathbf{B}_i$. Furthermore, [26] introduced a simple recursion to compute $\mathbf{w}_i$ and $\mathbf{x}_i$ via a simple vector update from the approximate solution obtained in the previous iteration.

Damped LSQR: The following transmit–receive relation is presented in (3),

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w} \tag{43}$$

where $\mathbf{y}$ is the time domain receive signal, $\mathbf{x}$ is the time domain transmit signal, $\mathbf{H}$ is the time domain channel matrix, and $\mathbf{w}$ is the additive noise. The objective of equalization is to compute the transit signal $\mathbf{x}$, given the received signal $\mathbf{y}$, and an estimate of the channel matrix $\mathbf{H}$.

MMSE equalization is formulated as follows:

$$\begin{aligned}\tilde{\mathbf{x}} &= \arg\min_{\mathbf{x}} \left\{\|\mathbf{H}\mathbf{x} - \mathbf{y}\|_2^2 + \sigma^2\|\mathbf{x}\|_2^2\right\} &(44)\\ &= \arg\min_{\mathbf{x}} \left\{\|\underline{\mathbf{H}}\mathbf{x} - \underline{\mathbf{y}}\|_2^2\right\} &(45)\end{aligned}$$

where

$$\underline{\mathbf{H}} = \begin{pmatrix}\mathbf{H}\\ \sigma^2\mathbf{I}\end{pmatrix}, \quad \underline{\mathbf{y}} = \begin{pmatrix}\mathbf{y}\\ \mathbf{0}\end{pmatrix}. \tag{46}$$

The parameter σ is related to the power of the noise $\mathbf{w}$, and is known as the Tikhonov regularization parameter. We note that a similar approach can be used in the frequency domain.

We can approximately compute $\tilde{\mathbf{x}}$ using LSQR as explained in the previous subsection, but now LSQR is applied to the matrix $\underline{\mathbf{H}}$ and $\underline{\mathbf{y}}$. This variant of LSQR using the regularization parameter σ is known as damped LSQR. We note that setting σ to zero reduces damped LSQR to standard LSQR as described in the previous section. In formula (44), a preconditioned channel matrix $\tilde{\mathbf{H}}$ can also be used to achieve faster convergence.

The most computationally expensive part of each iteration of damped LSQR is one matrix-vector product with the matrix $\underline{\mathbf{H}}$, and another matrix-vector product with the matrix $\underline{\mathbf{H}}^H$. Clearly, the computation of one matrix-vector product with either the matrix $\underline{\mathbf{H}}$ or the matrix $\underline{\mathbf{H}}^H$, requires K more complex flops than one matrix-vector product with the matrix $\mathbf{H}$ or $\mathbf{H}^H$, respectively. We demonstrate that the computation of matrix-vector products with the matrix $\mathbf{H}$ or $\mathbf{H}^H$, requires $\mathcal{O}(K \log K)$ flops, where K is the number of sub-carriers for OFDM transmission; see Section IV. Thus, the computational complexity of damped LSQR is $\mathcal{O}(K \log K)$ per iteration.

APPENDIX B
LEFT PRECONDITIONING

If a constant function is the first basis function of a BEM, then the matrix $\mathbf{C}_0^{-1}$ is typically useful as a right preconditioner. With extensive numerical simulations, we have found no difference in BER between using $\mathbf{C}_0^{-1}$ as a left or as a right preconditioner. Using $\mathbf{C}_0^{-1}$ as a right preconditioner preserves the product-convolution structure of the transformed time-domain channel matrix. On the other hand, using $\mathbf{C}_0^{-1}$ as a left preconditioner transforms the time-domain channel matrix into a sum of convolution-product-convolution operators. Consequently, preconditioning by $\mathbf{C}_0^{-1}$ on the left is computationally more expensive than preconditioning on the right. For a BEM which does not include a constant function, $\mathbf{P}_0^{-1}$ is often a suitable left preconditioner. Left preconditioning can be introduced into the time-domain transmit-receive relation (3) as follows:

$$\begin{aligned}\tilde{\mathbf{y}} = \mathbf{P}_0^{-1}\mathbf{y} &= \mathbf{P}_0^{-1}\mathbf{H}\mathbf{x} + \mathbf{P}_0^{-1}\tilde{\mathbf{w}} &(47)\\ &= \mathbf{P}_0^{-1}\mathbf{H}\mathbf{x} + \tilde{\mathbf{w}} &(48)\end{aligned}$$

where $\tilde{\mathbf{y}}$ is the transformed time-domain receive signal, and $\tilde{\mathbf{w}}$ is the transformed noise. Substituting (7) in (47), we obtain

$$\begin{aligned}\tilde{\mathbf{y}} &= \mathbf{P}_0^{-1}\left(\sum_{m=0}^{M-1}\mathbf{P}_m\mathbf{C}_m\right)\mathbf{x} + \tilde{\mathbf{w}} &(49)\\ &= \left(\sum_{m=0}^{M-1}\mathbf{P}'_m\mathbf{C}_m\right)\mathbf{x} + \tilde{\mathbf{w}} &(50)\\ &= \tilde{\mathbf{H}}_L\mathbf{x} + \tilde{\mathbf{w}} &(51)\end{aligned}$$

where $\tilde{\mathbf{H}}_L = \mathbf{P}_0^{-1}\mathbf{H}$ is the transformed time-domain channel matrix. Clearly, $\tilde{\mathbf{H}}_L$ is also a sum of product-convolution operators, which allows a fast computation of matrix-vector products with $\tilde{\mathbf{H}}_L$ and $\tilde{\mathbf{H}}_L^H$.

REFERENCES

[1] T. Zemen and C. F. Mecklenbraeuker, "Time-variant channel estimation using discrete prolate spheroidal sequences," *IEEE Trans. Signal Process.*, vol. 53, no. 9, pp. 3597–3607, Sep. 2005.

[2] Z. Tang, R. C. Cannizzaro, G. Leus, and P. Banelli, "Pilot-assisted time-varying channel estimation for OFDM systems," *IEEE Trans. Signal Process.*, vol. 55, no. 5, pp. 2226–2238, May 2007.

[3] Z. Tang and G. Leus, "Pilot schemes for time-varying channel estimation in OFDM systems," in *Proc. IEEE Workshop Signal Processing Advances Wireless Communications (SPAWC)*, Helsinki, Finland, Jun. 2007, pp. 1–5.

[4] C. Shin, J. G. Andrews, and E. J. Powers, "An efficient design of doubly selective channel estimation for OFDM systems," *IEEE Trans. Wireless Commun.*, vol. 6, no. 10, pp. 3790–3802, Oct. 2007.

[5] T. Zemen, C. F. Mecklenbrauker, and R. R. Müller, "Time variant channel equalization for MC-CDMA via Fourier basis functions," in *Proc. MC-SS Workshop 2003*, Oberpaffenhofen, Germany, 2003, pp. 451–458.

[6] X. Ma, G. B. Giannakis, and B. Lu, "Block differential encoding for rapidly fading channels," *IEEE Trans. Commun.*, vol. 52, no. 3, pp. 416–425, 2004.

[7] A. Cano, X. Ma, and G. B. Giannakis, "Block differential modulation over doubly selective wireless fading channels," *IEEE Trans. Commun.*, vol. 53, no. 12, pp. 2157–2166, 2005.

[8] T. Hrycak, S. Das, G. Matz, and H. G. Feichtinger, "Practical estimation of rapidly varying channels in OFDM systems," in preparation.

[9] K. Fang, L. Rugini, and G. Leus, "Low-complexity block turbo equalization for OFDM systems in time-varying channels," *IEEE Trans. Signal Process.*, vol. 56, no. 11, pp. 5555–5566, Nov. 2008.

[10] P. Robertson and S. Kaiser, "The effects of Doppler spreads in OFDMA mobile radio systems," in *Proc. 50th IEEE VTS Vehicular Technology Conf. (VTC) 1999—Fall*, Amsterdam, The Netherlands, Sep. 1999, vol. 1, pp. 329–333.

[11] Y. Li and L. J. Cimini, "Bounds on the interchannel interference of OFDM in time-varying impairments," *IEEE Trans. Commun.*, vol. 49, no. 3, pp. 401–404, Mar. 2001.

[12] M. Russell and G. L. Stuber, "Interchannel interference analysis of OFDM in a mobile environment," in *Proc. 45th IEEE Vehicular Technology Conf.*, Chicago, IL, Jul. 1995, vol. 2, pp. 820–824.

[13] Y.-S. Choi, P. J. Voltz, and F. A. Cassara, "On channel estimation and detection for multicarrier signals in fast and selective Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 49, no. 8, pp. 1375–1387, Aug. 2001.

[14] X. Cai and G. B. Giannakis, "Bounding performance and suppressing intercarrier interference in wireless mobile OFDM," *IEEE Trans. Commun.*, vol. 51, no. 12, pp. 2047–2056, Dec. 2003.

[15] A. Gorokhov and J.-P. Linnartz, "Robust OFDM receivers for dispersive time-varying channels: Equalization and channel acquisition," *IEEE Trans. Commun.*, vol. 52, no. 4, pp. 572–583, Apr. 2004.

[16] L. Rugini, P. Banelli, and G. Leus, "Simple equalization of time-varying channels for OFDM," *IEEE Commun. Lett.*, vol. 9, no. 7, pp. 619–621, Jul. 2005.

[17] P. Schniter, "Low-complexity equalization of OFDM in doubly selective channels," *IEEE Trans. Signal Process.*, vol. 52, no. 4, pp. 1002–1011, Apr. 2004.

[18] G. Tauböck, M. Hampejs, G. Matz, F. Hlawatsch, and K. Gröchenig, "LSQR-based ICI equalization for multicarrier communications in strongly dispersive and highly mobile environments," presented at the 8th IEEE Workshop Signal Processing Advances Wireless Communications, Helsinki, Finland, Jun. 2007.

[19] S. U. Hwang, J. H. Lee, and J. Seo, "Low complexity iterative ICI cancellation and equalization for OFDM systems over doubly selective channels," *IEEE Trans. Broadcast.*, vol. 55, no. 1, pp. 132–139, Mar. 2009.

[20] T. Zemen and C. F. Mecklenbrauker, "Time-variant channel equalization via discrete prolate spheroidal sequences," in *Conf. Rec. 37th Asilomar Conf. Signals, Systems, Computers*, Nov. 2003, vol. 2, pp. 1288–1292.

[21] T. Hrycak and G. Matz, "Low-complexity time-domain ICI equalization for OFDM communications over rapidly varying channels," in *Proc. Asilomar Conf. Signals, Systems, Computers*, Pacific Grove, CA, Oct./Nov. 2006, pp. 1767–1771.

[22] I. Barhumi and M. Moonen, "MLSE and MAP equalization for transmission over doubly selective channels," *IEEE Trans. Veh. Technol.*, vol. 58, no. 8, pp. 4120–4128, Oct. 2009.

[23] I. Barhumi, G. Leus, and M. Moonen, "Equalization for OFDM over doubly selective channels," *IEEE Trans. Signal Process.*, vol. 54, no. 4, pp. 1445–1458, Apr. 2006.

[24] L. Song and J. K. Tugnait, "Doubly-selective fading channel equalization: A comparison of the Kalman filter approach with the basis expansion model-based equalizers," *IEEE Trans. Wireless Commun.*, vol. 8, no. 1, pp. 60–65, 2009.

[25] Y. Saad and M. Schultz, "GMRES: A generalized minimum residual algorithm for solving non-symmetric linear systems," *SIAM J. Scientif. Stat. Comput.*, vol. 7, pp. 856–869, 1986.

[26] C. C. Paige and M. A. Saunders, "LSQR: An algorithm for sparse linear equations and sparse least square problems," *ACM Trans. Math. Softw.*, vol. 8, pp. 43–71, 1982.

[27] *Draft IEEE Standard for Local and Metropolitan Area Networks Part 16: Air Interface for Fixed and Mobile Broadband Wireless Access Systems*, IEEE Draft Standard 802.16e/D7, 2005.

[28] A. Neumaier, "Solving ill-conditioned and singular linear systems: A tutorial on regularization," *SIAM Rev.*, vol. 40, pp. 636–666, 1998.

[29] M. K. Tsatsanis and G. B. Giannakis, "Modelling and equalization of rapidly fading channels," *Int. J. Adapt. Control Signal Process.*, vol. 10, pp. 159–176, 1996.

[30] G. B. Giannakis and C. Tepedelenlioglu, "Basis expansion models and diversity techniques for blind identification and equalization of time-varying channels," in *Proc. IEEE*, 1998, pp. 1969–1986.

[31] H. A. Cirpan and M. K. Tsatsanis, "Maximum likelihood blind channel estimation in the presence of Doppler shifts," *IEEE Trans. Signal Process.*, vol. 47, no. 6, pp. 1559–1569, Jun. 1999.

[32] M. Guillaud and D. T. M. Slock, "Channel modeling and associated inter-carrier interference equalization for OFDM systems with high Doppler spread," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, Apr. 2003, vol. 4, pp. 237–240.

[33] G. Leus, "On the estimation of rapidly varying channels," in *Proc. Eur. Signal Processing Conf. (EUSIPCO)*, Sep. 2004, vol. 4, pp. 2227–2230.

[34] D. K. Borah and B. T. Hart, "Frequency-selective fading channel estimation with a polynomial time-varying channel model," *IEEE Trans. Commun.*, vol. 47, no. 6, pp. 862–873, Jun. 1999.

[35] A. Bjorck, *Numerical Methods for Least Squares Problems*, 1st ed. Philadelphia, PA: SIAM, 1995.

[36] G. Golub and C. F. van Loan, *Matrix Computations*, 3rd ed. Baltimore, MD: The Johns Hopkins Univ. Press, 1996.

[37] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd ed. Philadelphia, PA: SIAM, 2003.

[38] L. Rugini, P. Banelli, and G. Leus, "Low-complexity banded equalizers for OFDM systems in Doppler spread channels," *EURASIP J. Appl. Signal Process.*, pp. 1–13, 2006.

[39] Z. Tang, "OFDM transmission over rapidly changing channels," Ph.D. dissertation, Faculty of Electr. Eng., Math., and Comp. Sci., Delft Univ. Technol., Delft, The Netherlands, 2007.

[40] A. P. Kannu and P. Schniter, "MSE-optimal training for linear time-varying channels," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP)*, Mar. 2005, vol. 3, pp. 789–792.

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