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Convergence of Adaptive FEM for Elliptic Obstacle Problems

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Outline

We treat the convergence of adaptive lowest-order FEM for some elliptic obstacle problem with affine obstacle. For error estimation, we use a residual error estimator which is an extended version of the estimator from [1] that additionally controls the data oscillations. Our main result states that an appropriately weighted sum of energy error, edge residuals, and data oscillations satisfies a contraction property that leads to convergence. In addition, we discuss the generalization to the case of inhomogeneous Dirichlet data and non-affine obstacles $\chi \in H^2(\Omega)$ and obtain similar results.

Error Analysis

- Edge based residual-type error estimator

$$\eta_\ell^2 := \sum_{E \in \mathcal{E}_\ell^\Omega} \rho_\ell(E)^2 + \sum_{E \in \mathcal{E}_\ell} \mathrm{osc}_\ell(E)^2, \text{ with}$$

$$\rho_\ell(E)^2 := h_E \| [\partial_n U_\ell] \|_{L^2(E)}^2 \qquad E \in \mathcal{E}_\ell^\Omega$$

$$\mathrm{osc}_\ell(E)^2 := \begin{cases} |\Omega_{\ell,E}| \, \| f - f_{\Omega_{\ell,E}} \|_{L^2(\Omega_{\ell,E})}^2 & E \in \mathcal{E}_\ell^\Omega \\ |T| \, \| f \|_{L^2(T)}^2 & E \in \mathcal{E}_{\ell,\Gamma} \end{cases}$$

- Reliability estimate

$$\| u - U_\ell \|_{H^1(\Omega)}^2 \lesssim \mathcal{J}(U_\ell) - \mathcal{J}(u) \lesssim \eta_\ell^2$$

Obstacle Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\Gamma = \partial\Omega$. An obstacle is defined by the affine function χ with $\chi \geq 0$ on Γ. By $\mathcal{A} := \{ v \in H_0^1(\Omega) \ : \ v \geq \chi \text{ a.e. in } \Omega \}$, we denote the set of admissible functions. The obstacle problem then reads: *Find $u \in \mathcal{A}$ such that*

$$\mathcal{J}(u) = \min_{v \in \mathcal{A}} \mathcal{J}(v),$$

with the energy functional $\mathcal{J}(v) = \frac{1}{2} (\nabla v, \nabla v)_{L^2(\Omega)} - (f, v)_{L^2(\Omega)}$.

Adaptive Algorithm and Convergence

- Using the Dörfler strategy in the marking step of an adaptive algorithm

`Solve` ⟼ `Estimate` ⟼ `Mark` ⟼ `Refine`

we prove that the combined error quantity $\Delta_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u) + \gamma \eta_\ell^2$ satisfies the contraction property

$$\Delta_{\ell+1} \leq \kappa \Delta_\ell \qquad \text{for all } \ell \in \mathbb{N},$$

cf. [2]. The constants $0 < \gamma, \kappa < 1$ depend only on the adaptivity parameter θ and the shape regularity constant. In particular, there holds $\lim_{\ell \to \infty} \mathcal{J}(U_\ell) = \mathcal{J}(u)$ as well as $\lim_{\ell \to \infty} \| u - U_\ell \|_{H^1(\Omega)} = 0 = \lim_{\ell \to \infty} \eta_\ell$.

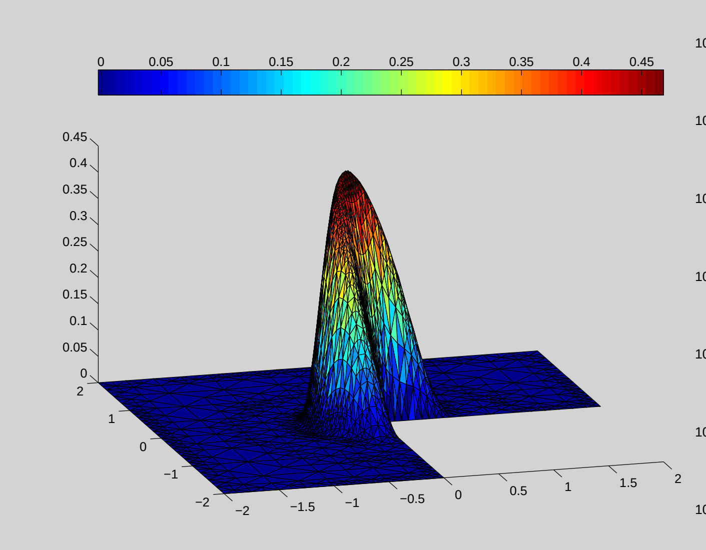
- In the case of a non-affine obstacle $\chi \in H^2(\Omega)$, we get the slightly weaker result

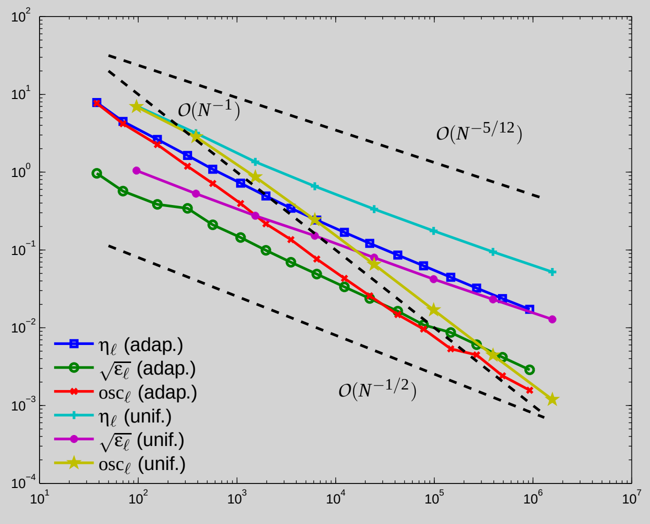
$$\widetilde{\Delta}_{\ell+1} \leq \kappa \widetilde{\Delta}_\ell + \alpha_\ell \qquad \text{for all } \ell \in \mathbb{N}$$

for a certain zero sequence $\alpha_\ell \geq 0$ with $\lim_\ell \alpha_\ell = 0$. Here, $\widetilde{\Delta}_\ell$ denotes a similar combined error quantity that additionally controls the adaptive resolution of the Dirichlet data, see [3].

Numerical Experiment

Model problem with $\chi \equiv 0$ on L-shaped domain





Left: Numerical solution after 8 iterations, $\theta = 0.6$, $\#\mathcal{T}_8 = 3.542$ elements.

- Corner singularity at origin
- Adaptively generated mesh

Right: Error $\varepsilon_\ell := \mathcal{J}(U_\ell) - \mathcal{J}(u)$, estimator η_ℓ, and data oscillationes osc_ℓ plotted over the number of elements $N_\ell := \#\mathcal{T}_\ell$ in the uniform and adaptive case for $\theta = 0.6$.

- Uniform mesh-refinement → convergence order $O(N^{-5/12})$
- Proposed adaptive algorithm → optimal convergence $O(N^{-1/2})$

Innovations

Adaptive Algorithm:
- Contractive error quantity for nonlinear variational inequality
- Improvements of prior work on AFEM for obstacle problems

Error estimator:
- Contraction property of edge-based data oscillation terms

References:

[1] D. Braess, C. Carstensen, and R. Hoppe, *Convergence analysis of a conforming adaptive finite element method for an obstacle problem*, Numer. Math. 107 (2007), 455–471

[2] M. Page and D. Praetorius, *Convergence of adaptive FEM for some elliptic obstacle problem*, ASC Report, 5/2010, Vienna University of Technology, 2010

[3] M. Feischl, M. Page, and D. Praetorius *Convergence of adaptive FEM for some elliptic obstacle problem with inhomogeneous Dirichlet data*, ASC Report, 33/2010, Vienna University of Technology, 2010