Convergence of Adaptive FEM for Elliptic Obstacle Problems

Michael Feischl, Marcus Page, and Dirk Praetorius

Outline

We treat the convergence of adaptive lowest-order FEM for some elliptic obstacle problem with affine obstacle. For error estimation, we use a residual error estimator which is an extended version of the estimator from [1] that additionally controls the data oscillations. Our main result states that an appropriately weighted sum of energy error, edge residuals, and data oscillations satisfies a contraction property that leads to convergence. In addition, we discuss the generalization to the case of inhomogeneous Dirichlet data and non-affine obstacles $\chi \in H^2(\Omega)$ and obtain similar results.

Error Analysis

- Edge based residual-type error estimator
  \[ \eta^2 := \sum_{E \in \mathcal{E}_h} \rho(E)^2 + \sum_{E \in \mathcal{T}_h} \text{osc}(E)^2, \]
  with
  \[ \rho(E)^2 := h_E \| \Delta u_E \|_{L^2(E)}^2, \quad E \in \mathcal{T}_h, \]
  \[ \text{osc}(E)^2 := \sum_{T \in \mathcal{T}_h \cap \partial E} \| T \| \| f \|_{L^2(T)}^2, \quad E \in \mathcal{T}_h, \]
- Reliability estimate
  \[ \| u - u_h \|_{H^1(\Omega)} \leq J(U) - J(u) \leq \eta^2 \]

Obstacle Problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary $\Gamma = \partial \Omega$. An obstacle is defined by the affine function $\chi$ with $\chi \geq 0$ on $\Gamma$. By $\mathcal{A} := \{ v \in H^1(\Omega) : v \geq \chi \text{ a.e. in } \Omega \}$, we denote the set of admissible functions. The obstacle problem then reads: Find $u \in \mathcal{A}$ such that $f(u) = \min_{v \in \mathcal{A}} f(v)$, with the energy functional $f(v) = \frac{1}{2} (\nabla v, \nabla v)_{L^2(\Omega)} - (f, v)_{L^2(\Omega)}$.

Adaptive Algorithm and Convergence

- Using the Dörfler strategy in the marking step of an adaptive algorithm

\[ \text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine} \]

we prove that the combined error quantity $\Delta := J(U_{\ell}) - J(u) + \eta^2$ satisfies the contraction property
\[ \Delta_{\ell+1} \leq \kappa \Delta_{\ell} \quad \text{for all } \ell \in \mathbb{N}, \]
where $\kappa < 1$ depends only on the adaptivity parameter $\theta$ and the shape regularity constant. Moreover, there holds $\lim_{\ell \to \infty} J(U_{\ell}) = J(u)$.

- In the case of a non-affine obstacle $\chi \in H^2(\Omega)$, we get the slightly weaker result
  \[ \Delta_{\ell+1} \leq \kappa \Delta_{\ell} + \alpha_{\ell} \quad \text{for all } \ell \in \mathbb{N} \]
  for a certain zero sequence $\alpha_{\ell} \geq 0$ with $\lim_{\ell \to \infty} \alpha_{\ell} = 0$. Here, $\Delta_{\ell}$ denotes a similar combined error quantity that additionally controls the adaptive resolution of the Dirichlet data, see [3].

Numerical Experiment

Model problem with $\chi = 0$ on L-shaped domain

Left: Numerical solution after 8 iterations, $\theta = 0.6$, $\# \mathcal{T} = 3,542$ elements.

- Corner singularity at origin
- Adaptively generated mesh

Right: Error $\varepsilon_\ell := J(U_{\ell}) - J(u)$, estimator $\eta$, and data oscillations $\text{osc}$; plotted over the number of elements $N_\ell := \# \mathcal{T}_\ell$ in the uniform and adaptive case for $\theta = 0.6$.

- Uniform mesh-refinement $\rightarrow$ convergence order $O(N^{-1/2})$
- Proposed adaptive algorithm $\rightarrow$ optimal convergence $O(N^{-5/12})$

Innovations

Adaptive Algorithm:
- Contractive error quantity for nonlinear variational inequality
- Improvements of prior work on AFEM for obstacle problems

Error estimator:
- Contraction property of edge-based data oscillation terms

References: