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Convergent Geometric Integrator for the Landau-Lifshitz-Gilbert Equation

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Outline

In physics, the Landau-Lifshitz-Gilbert Equation (LLG) is a well-accepted model to describe the dynamics of micromagnetic phenomena. We generalize the approach of [1] to the total magnetic field, including exchange energy, anisotropy energy, magnetostatic energy, as well as Zeeman energy. Since the computation of the demagnetization field is the most time and memory consuming part of the simulation, the proposed time integrator is split into an implicit part and an explicit part. The first one deals with the higher-order term stemming from the exchange energy, whereas the lower-order terms are treated explicitly. Our extension still guarantees the side constraint $|\mathbf{m}(t,\mathbf{x})| = 1$ to be fulfilled as well as unconditional convergence. In contrast to previous works, another benefit of our scheme is that only one linear system per time-step has to be solved. Finally, our analysis allows to replace the operator $\mathcal{P}$ which maps $\mathbf{m}$ onto the corresponding demagnetization field by a discrete operator $\mathcal{P}_h$.

LLG Equation

Let Ω denote a magnetic body and $\mathbf{m} : (0,\tau_{\text{end}}) \times \Omega \to \mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ be the magnetization. With $\alpha > 0$ the damping parameter, LLG reads

$$\begin{aligned} \mathbf{m}_\tau &= \frac{-1}{1+\alpha^2}\mathbf{m} \times \mathbf{h}_{\text{eff}} - \frac{\alpha}{1+\alpha^2}\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}}) \\ \mathbf{m}(0) &= \mathbf{m}_0 \quad \text{in } H^1(\Omega;\mathbb{S}^2) \\ \partial_n \mathbf{m} &= 0 \quad \text{on } (0,\tau) \times \partial\Omega, \end{aligned}$$

where the total magnetic field is given as variation of the Gibbs Free energy

$$\mathbf{h}_{\text{eff}}(\mathbf{m}) = -\frac{\delta e}{\delta \mathbf{m}} = \Delta \mathbf{m} + D\Phi(\mathbf{m}) + \mathcal{P}\mathbf{m} - \mathbf{f}.$$

Here, $\mathcal{P}\mathbf{m}$ refers to the demagnetization field which is induced by the magnetostatic Maxwell's equations, Φ is the anisotropy density, and $\mathbf{f}$ the applied field.

Convergence Result

Assumptions:

- Fix $1/2 < \theta \le 1$
- Let initial triangulation $\mathcal{T}_0$ satisfy certain angle condition
- Let $\mathcal{T}_h$ be a family of regular triangulations with mesh-sizes $h \searrow 0$
- Let $\mathbf{m}_h^0 \to \mathbf{m}_0$ in $H^1(\Omega;\mathbb{R}^3)$ as $h \searrow 0$
- Let the approximate stray-field operator $\mathcal{P}_h$ satisfy

$$\|\mathcal{P}_h \mathbf{m}_h^j\|_{L^2(\Omega)} \le C\,\|\mathbf{m}_h^j\|_{L^2(\Omega)} \text{ and } \|\mathcal{P}\mathbf{m} - \mathcal{P}_h \mathbf{m}\|_{L^2(\Omega)} \xrightarrow{h\to 0} 0 \quad \text{a.e. in } (0,\tau_{\text{end}})$$

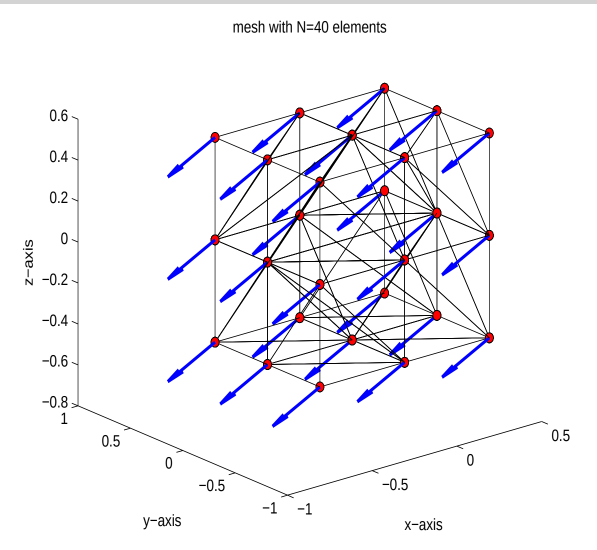
Result:

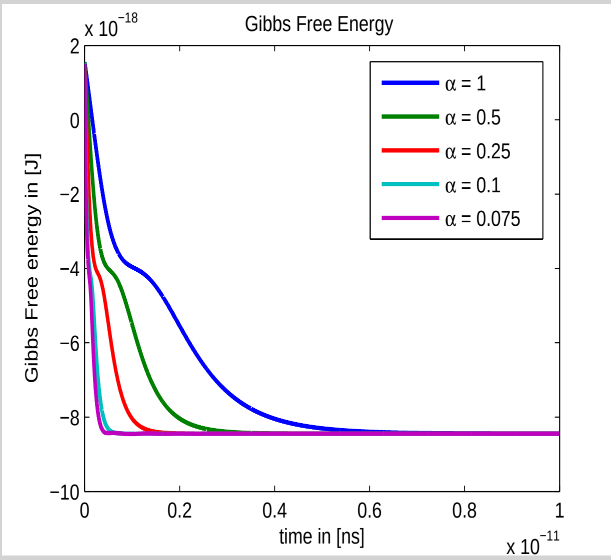
As $h,k \searrow 0$, the approximate magnetization $\mathbf{m}_{hk}$ admits a subsequence which converges weakly in $H^1(\Omega_\tau;\mathbb{R}^3)$ to a weak solution $\mathbf{m}$ of LLG. ■

Algorithm

- **Input**: initial $\mathbf{m}_h^0 \in M_h$, damping parameter α, parameter $0 < \theta \le 1$
  1. Find $\mathbf{v}_h^j \in K_{\mathbf{m}_h^j}$ such that for all $\psi_h \in K_{\mathbf{m}_h^j}$
  $$\begin{aligned} &\alpha \int_\Omega \mathbf{v}_h^j \cdot \psi_h + \int_\Omega (\mathbf{m}_h^j \times \mathbf{v}_h^j) \cdot \psi_h \\ &= -\int_\Omega \nabla(\mathbf{m}_h^j + \theta k \mathbf{v}_h^j) \cdot \nabla \psi_h + \int_\Omega (D\Phi(\mathbf{m}_h^j) + \mathcal{P}_h(\mathbf{m}_h^j) - \mathbf{f}) \cdot \psi_h. \end{aligned}$$
  2. Define nodewise $\mathbf{m}_h^{j+1}(\mathbf{z}) = \frac{\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})}{|\mathbf{m}_h^j(\mathbf{z}) + k\mathbf{v}_h^j(\mathbf{z})|}$ and iterate.
- **Output**: discrete solutions $\mathbf{v}_h^j \in K_{\mathbf{m}_h^j}, \mathbf{m}_h^j \in M_h$

Numerical Experiment





Numerical Experiment: We consider a micromagnetic cube with edge-length of $10nm$ and random initial magnetization. Furthermore, the material considered is uniaxial with easy axis along the z-axis. The applied external field is constant and proportional to $(-1,0,-1)$.

Left: Micromagnetic body Ω with alignment of magnetization at each node of the triangulation $\mathcal{T}_h$ in equilibrium state.

Right: Variation of the Gibbs Free energy from the beginning of the simulation until equilibrium state is reached under consideration of various values for the damping parameter α.

Innovations

Numerical Scheme:

- Including total magnetic field
- Time-splitting for more effective computation

Analytical Result:

- Convergence result for introduced numerical scheme
- Effective treatment of magnetostatic energy

References:

[1] F. ALOUGES, *A new finite element scheme for Landau-Lifchitz equations*, Discrete Contin. Dyn. Syst. Ser. S, 2008

[2] P. GOLDENITS and D. PRAETORIUS, *Effective Simulation of the Dynamics of Ferromagnetism*, work in progress, 2011

[3] P. GOLDENITS, *PhD Thesis*, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2012