Design of Robust Equalizers

Markus Rupp

Contact email: mrupp@nt.tuwien.ac.at
web: http://www.nt.tuwien.ac.at
Outline

- Interference scenarios: the task
- Wireless channels: the challenge
- Equalizer on wireless channels: the impact
- Reference models: the hope
- Robust designs of adaptive equalizers: the solution
Interference Scenarios

\[ \text{SINR}_{be} = \frac{P_s}{P_{ISI} + P_{SP-ISI} + P_{MUI} + N_o} \]
Interference Scenarios

- SINR situation before equalizer
  \[ \text{SINR}_{\text{be}} = \frac{P_s}{P_{\text{ISI}} + P_{\text{SP-ISI}} + P_{\text{MUI}} + N_o} \]

- What can a linear equalizer achieve, by applying a function onto the received signal?
  \[ \text{SINR}_{\text{ae}} = \max_{\text{Equalizer strategy}} \quad \text{SINR}_{\text{be}} = \frac{P'_s}{P'_{\text{ISI}} + P'_{\text{SP-ISI}} + P'_{\text{MUI}} + N'_o} \leq \frac{P_s + P_{\text{ISI}} + P_{\text{SP-ISI}}}{N_o} \]

- Make use of ISI and SP-ISI, get rid of MUI.
Outline

- Interference scenarios: the task
- Wireless channels: the challenge
- Equalizer on wireless channels: the impact
- Reference models: the hope
- Robust designs of adaptive equalizers: the solution
**Wireless Channels**

- The most generic model is simply given by

\[ r_k = Hs_k + v_k \]

- 1) frequency selective SISO channel

\[
\begin{bmatrix}
  r_k \\
  r_{k-1} \\
  \vdots \\
  r_{k-R+1}
\end{bmatrix} =
\begin{bmatrix}
  h_0 & h_1 & \cdots & h_{L-1} \\
  \vdots & \ddots & \ddots & \vdots \\
  h_0 & h_1 & \cdots & h_{L-1}
\end{bmatrix}
\begin{bmatrix}
  s_k \\
  s_{k-1} \\
  \vdots \\
  s_{k-S+1}
\end{bmatrix} +
\begin{bmatrix}
  v_k \\
  v_{k-1} \\
  \vdots \\
  v_{k-R+1}
\end{bmatrix}
\]
2) Frequency flat MIMO channel

\[
\begin{bmatrix}
  h_{1,1} & \cdots & h_{1,T} \\
  \vdots & & \vdots \\
  h_{R,1} & \cdots & h_{R,T}
\end{bmatrix}
\begin{bmatrix}
  s_{k,1} \\
  \vdots \\
  s_{k,T}
\end{bmatrix}
+ \begin{bmatrix}
  v_{k,1} \\
  \vdots \\
  v_{k,R}
\end{bmatrix}
\]

3) Multi-user frequency flat MIMO channel

\[
r_k = \sum_{m=1}^{M} H_{m} s_{k,m} + v_k = H s_k + v_k
\]
Wireless Channels

- 4) Multi-user MIMO over frequency selective channel

\[
\mathbf{r}_k = \sum_{m=1}^{M} \begin{bmatrix} \mathbf{H}_{1,m} & \cdots & \mathbf{H}_{T,1,m} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{1,R,m} & \cdots & \mathbf{H}_{T,R,m} \end{bmatrix} \begin{bmatrix} \mathbf{s}_{k,1,m} \\ \vdots \\ \mathbf{s}_{k,T,m} \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{k,1} \\ \vdots \\ \mathbf{v}_{k,R} \end{bmatrix} \\
= \sum_{m=1}^{M} \begin{bmatrix} \mathbf{H}_{1,m} \\ \vdots \\ \mathbf{H}_{T,m} \end{bmatrix} \begin{bmatrix} \mathbf{s}_{k,1,m} \\ \vdots \\ \mathbf{s}_{k,T,m} \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{k,1} \\ \vdots \\ \mathbf{v}_{k,R} \end{bmatrix}
\]

\[
\mathbf{r}_k = \sum_{t=1}^{N_T} \sum_{m=1}^{M} \mathbf{H}_{t,m} \mathbf{s}_{k,t,m} + \mathbf{v}_k = \mathbf{H} \mathbf{s}_k + \mathbf{v}_k
\]
Outline

- Interference scenarios: the task
- Wireless channels: the challenge
- Equalizer on wireless channels: the impact
- Reference models: the hope
- Robust designs of adaptive equalizers: the solution
Equalizers on Wireless Channels

- Let us consider a SISO frequency selective channel with simple ISI (no noise).

\[
\mathbf{r}_k = \begin{bmatrix}
  h_0 & h_1 \\
  h_0 & h_1 \\
  \vdots & \vdots \\
  h_0 & h_1 \\
\end{bmatrix}
\begin{bmatrix}
  s_k \\
  s_{k-\tau} \\
  \vdots \\
  s_{k-S+1} \\
\end{bmatrix}
= \mathbf{Hs}_k
\]

\[
\mathbf{e}_k^T = \begin{bmatrix}
  0 & \cdots & 1 & 0
\end{bmatrix}
\]

\[
\mathbf{e}_k^T \mathbf{s}_k = s_{k-\tau}
\]

\[
\mathbf{He}_k = \begin{bmatrix}
  h_1 \\
  h_0 \\
\end{bmatrix}
\]
Equalizers on Wireless Channels

- Let us consider a SISO frequency selective channel with simple ISI (no noise).

\[
\begin{bmatrix}
    h_0 & h_1 & s_k \\
    h_0 & h_1 & s_{k-\tau} \\
    \vdots & \vdots & \vdots \\
    h_0 & h_1 & s_{k-S+1}
\end{bmatrix}
= Hs
\]

\[
He_{k-\tau} = \begin{bmatrix}
    h_1 \\
    h_0
\end{bmatrix}
\]

signal part

\[
f^HHe_{k-\tau} \propto s_{k-\tau}
\]

\[
\max_f \text{SIR} = \max_f \frac{P_S}{P_I} = \max_f \frac{\|e_{\tau}^T H^H f\|^2}{f^H H H^H f - \|e_{\tau}^T H^H f\|^2} = \max_f \frac{f^H H e_{\tau} e_{\tau}^T H^H f}{f^H H (I - e_{\tau} e_{\tau}^T) H^H f}
\]
Equalizers on Wireless Channels

- While this expression is complicated, we can apply a monotone mapping:

\[
\max_f \frac{P_S}{P_S + P_I} = \max_f \frac{\text{SIR}}{\text{SIR} + 1} = \gamma^{\text{SIR}} \leq 1
\]

- Correspondingly:

\[
\max_f \gamma^{\text{SIR}} = \max_f \frac{f^H H e_\tau e_\tau^T H^H f}{f^H H H^H f}
\]

- Which leads to the well-known overdetermined ZF solution:

\[
f^\text{max SIR}_\tau = \alpha (H H^H)^{-1} H e_\tau = \alpha f^\text{ZF,o}_\tau
\]
Equalizers on Wireless Channels

- Correspondingly, including noise

\[
\max_{f} \frac{P_{S}}{P_{I} + N_{0}} = \max_{f} \frac{f^{H} H e_{T} e_{T}^{T} H^{H} f}{f^{H} [H (I - e_{T} e_{T}^{T}) H^{H} + N_{0} I] f}.
\]

\[
\max_{f} \frac{P_{S}}{P_{S} + P_{I} + N_{0}} = \max_{f} \frac{\text{SINR}}{\text{SINR} + 1} = \gamma^{\text{SINR}} \leq 1.
\]

\[
\max_{f} \gamma^{\text{SINR}} = \max_{f} \frac{f^{H} H e_{T} e_{T}^{T} H^{H} f}{f^{H} [H H^{H} + N_{0} I] f}.
\]

- results in the (also well-known) MMSE solution

\[
f_{\tau}^{\text{max SINR}} = \alpha (H H^{H} + N_{0} I)^{-1} H e_{T} = \alpha f_{\tau}^{\text{MMSE}}
\]
Equalizers on Wireless Channels

\[ 0 \leq \gamma^{SIR,\text{max}}, \gamma^{SINR,\text{max}} \leq 1 \]

- What is the maximum SIR (SINR) achievable then?

\[
\begin{align*}
\text{SIR}^{\text{max}} &= \frac{\mathbf{e}_\tau^T \mathbf{H}^H \left[ \mathbf{H} \mathbf{H}^H \right]^{-1} \mathbf{H} \mathbf{e}_\tau}{1 - \gamma^{SIR,\text{max}}} \\
&= \frac{\gamma^{SIR,\text{max}}}{1 - \gamma^{SIR,\text{max}}}
\end{align*}
\]

\[
\begin{align*}
\text{SINR}^{\text{max}} &= \frac{\mathbf{e}_\tau^T \mathbf{H}^H \left[ \mathbf{H} \mathbf{H}^H + \sigma_v^2 \mathbf{I} \right]^{-1} \mathbf{H} \mathbf{e}_\tau}{1 - \gamma^{SINR,\text{max}}} \\
&= \frac{\gamma^{SINR,\text{max}}}{1 - \gamma^{SINR,\text{max}}}
\end{align*}
\]

- Thus „any“ positive value seems possible.
Equalizers on Wireless Channels

\[ SIR_{\text{max}}^{\text{max}} = \frac{\gamma^{SIR,\text{max}}}{1 - \gamma^{SIR,\text{max}}} \]

\[ \text{SINR}_{\text{max}}^{\text{max}} = \frac{\gamma^{\text{SINR,\max}}}{1 - \gamma^{\text{SINR,\max}}} \]

\[ 0 \leq \gamma^{SIR,\text{max}}, \gamma^{\text{SINR,\max}} \leq 1 \]
Underdetermined ZF

- In this case we have \( R_{N_R} > S_{N_T M} \)

- The ZF solution reads \( f_{\tau}^{\text{max SIR}} = \alpha H(H^H H)^{-1} e_{\tau} \)

\[
SIR_{\text{max}} = \frac{e_{\tau}^T H^H H [H^H H]^{-1} e_{\tau}}{e_{\tau}^T \{I - H^H H [H^H H]^{-1}\} e_{\tau}} = \frac{e_{\tau}^T e_{\tau}}{1 - e_{\tau}^T e_{\tau}} = \frac{1}{0}
\]

- This degree of freedom allows for perfect interference cancellation or interference alignment techniques!
Equalizer Example

- 1) Underdetermined flat MIMO, $N_R=4$, $N_T=3$  $\sigma_v^2=1$
  - $1.0933 + 1.0891i$  $1.1093 - 0.0326i$  $-0.8637 - 0.5525i$
  - $0.0774 - 1.1006i$  $-1.2141 - 1.5442i$  $-1.1135 - 0.0859i$
  - $-0.0068 + 1.4916i$  $1.5326 + 0.7423i$  $-0.7697 + 1.0616i$
  - $0.3714 - 2.3505i$  $-0.2256 + 0.6156i$  $1.1174 - 0.7481i$
  - ZF: Equalizer $\mathbf{f}_1= 0.2580 + 0.0578i$  $0.0151 - 0.0447i$  $-0.0256 - 0.1077i$  $0.1189 - 0.3068i$
  - SIR$_1=oo$ ($oo$, $oo$)
  - MMSE: Equalizer $\mathbf{f}_1= 0.1845 + 0.0547i$  $0.0053 - 0.0464i$  $-0.0163 - 0.0417i$  $0.0872 - 0.2441i$
  - SINR$_1=5.99$ (5.04, 4.29)  ZF: 5.19 (4.55, 3.97)

Large values, can reconstruct all three streams!
Equalizer Example

- 1) Overdetermined flat MIMO, $N_R=3$, $N_T=4$  $\sigma_v^2=1$
  - $\begin{bmatrix} 1.0933 - 1.0891i & 0.0774 + 1.1006i & -0.0068 - 1.4916i & 0.3714 + 2.3505i \\ 1.1093 + 0.0326i & -1.2141 + 1.5442i & 1.5326 - 0.7423i & -0.2256 - 0.6156i \\ -0.8637 + 0.5525i & -1.1135 + 0.0859i & -0.7697 - 1.0616i & 1.1174 + 0.7481i \end{bmatrix}$
  - $ZF$: Equalizer $f_1= 0.2580 - 0.0578i -0.0072 - 0.1438i -0.2359 + 0.2187i$
  - $SIR_1=1.91$ (2.2, 3.0, 8.9)
  - $MMSE$: Equalizer $f_1= 0.1845 - 0.0547i 0.0210 - 0.0876i -0.1715 + 0.1599i$
  - $SINR_1 = 1.07$, (1.56,1.58,3.19) $ZF:1.02$ (1.55,1.48,2.99)

Small values
May reconstruct only one stream!
**Equalizer in Wireless Channels**

- What works for SISO frequency selective channels, equally works for more complicated transmissions, as long as the model holds: \( r_k = Hs_k + v_k \)

- For example, MU MIMO on freq. selective channels:

\[
\max_f \gamma_{\tau,t,m}^{\text{SINR}} = \max_f \frac{f^H H_{t,m} e_{\tau,t,m} e_{\tau,t,m}^T H_{t,m}^H f}{f^H \left[ \sum_{t'='}^{N_T} \sum_{m'=1}^{M} H_{t',m'} H_{t',m'}^H + N_0 I \right] f}
\]

\[
f_{\tau,t,m}^{\text{max SINR}} = \alpha \left( \sum_{t'=1}^{N_T} \sum_{m=1}^{M} H_{t',m} H_{t',m}^H + N_0 I \right)^{-1} H_{t,m} e_{\tau,t,m} = \alpha f_{\tau,t,m}^{\text{MU-MMSE}}
\]
Outline

- Interference scenarios: the task
- Wireless channels: the challenge
- Equalizer on wireless channels: the impact
- Reference models: the hope
- Robust designs of adaptive equalizers: the solution
Open problems (at this stage) are
- need to know the channels
- need to know noise variance
- need to invert a (large) matrix

Thus
- complexity
- numerical stability
- Filling the backbone network with data (in MU case)
Reference Models

- What (cost function) does a linear equalizer minimize?
  - Answer 1: maximize SINR or SIR
  - Answer 2: minimize MMSE or ISI energy (ZF)
  - Answer 3: anything else that leads to the desired result

but helps interpreting the equalizer as: ????
Reference Model

- Interpret (MU-MIMO) ZF as LS problem:

\[ f_{\tau,t,m}^{ZF} = \arg\min_f \|H^H f - e_{\tau,t,m}\|_2^2 = \arg\min_f \|H^H [f - (HH^H)^{-1}He_{\tau,t,m}]\|_2^2 \]

- Two potential solutions:

  1) overdetermined LS solution, if \( RN_R < SN_{TM} \)

  \[ f_{\tau,t,m}^{ZF,o} = (HH^H)^{-1}He_{\tau,t,m} \]

  2) underdetermined LS solution, if \( RN_R > SN_{TM} \)

  \[ f_{\tau,t,m}^{ZF,u} = H(H^H H)^{-1} e_{\tau,t,m} \]
This also works for MMSE ($\mathbb{E}[|s|^2]=1$):

$$f_{\tau,t,m}^{\text{MMSE}} = \arg\min_f \left( \|H^H f - e_{\tau,t,m}\|_2^2 + N_o \|f\|_2^2 \right)$$
$$= \arg\min_f \left\| \left(HH^H + N_o I\right)^{\frac{1}{2}} \left[f - \left(HH^H + N_o I\right)^{-1}He_{\tau,t,m}\right] \right\|_2^2 + \text{MMSE}$$

$$\text{MMSE} = e_{\tau,t,m}^T \left[I - H^H (HH^H + N_o I)^{-1}H\right] e_{\tau,t,m}$$

Only a single solution in this case:

$$f_{\tau,t,m}^{\text{MMSE}} = \left(HH^H + N_o I\right)^{-1}He_{\tau,t,m}$$
Reference Model

- Model equalizer as

\[ e_{\tau,t,m} = H^H f_{\tau,t,m}^{\text{MMSE}} + v_{\tau,t,m}^{\text{MMSE}} \]

\[ v_{\tau,t,m}^{\text{MMSE}} = (I - H^H (HH^H + N_0 I)^{-1} H) e_{\tau,t,m} \]

- Apply equalizer on observation \( r_k \):

\[ f_{\tau,t,m}^{\text{MMSE},H} r_k = s_{k-\tau,t,m} - v_{\tau,t,m}^{\text{MMSE},H} s_k + f_{\tau,t,m}^{\text{MMSE},H} v_k = s_{k-\tau,t,m} + v_{k,t,m}^{\text{MMSE}} \]

- For ZF similar result!
Outline

- Interference scenarios: the task
- Wireless channels: the challenge
- Equalizer on wireless channels: the impact
- Reference models: the hope
- Robust designs of adaptive equalizers: the solution
Robust Designs

- Consider classical adaptive parameter estimation (identification) problem:

    - Under what condition can we design an (iterative, recursive) algorithm that ensures the MMSE/ZF solution under any signals (symbol, noise, channels..)?
Robust Design

- **Iterative algorithm:**
  \[
  \hat{f}_l = \hat{f}_{l-1} + \mu H B (e_\tau - H^H \hat{f}_{l-1})
  \]
  \[
  = \hat{f}_{l-1} + \mu H B (H^H f^{\text{ref}} + \bar{v} - H^H \hat{f}_{l-1})
  \]
  \[
  = \hat{f}_{l-1} + \mu H B (H^H [f^{\text{ref}} - \hat{f}_{l-1}] + \bar{v})
  \]
  - Works on a fixed data set \(X\) until the fixed point \(f^{\text{ref}}\) is achieved (to some extent).

- **Recursive algorithm:**
  \[
  \hat{f}_k = \hat{f}_{k-1} + \mu Y_k (s_k - X_k^H \hat{f}_{k-1})
  \]
  \[
  = \hat{f}_{k-1} + \mu Y_k (X_k^H [f^{\text{ref}} - \hat{f}_{k-1}] + \bar{v}_k)
  \]
  - Works on an instantaneous data set \(X_k, Y_k\) until the fixed point \(f^{\text{ref}}\) is achieved (to some extent).
Robust Design

- **Iterative algorithms**
  - typically ZF problem,
  - solves the matrix inversion, given the channel

- **Theorem:** The iterative algorithm converges to zero, if the following conditions are satisfied:
  
  1) All eigenvalues of $\mathbf{I} - \mu \mathbf{XBX}^H$ are in magnitude smaller than one, or correspondingly

     $$0 < \mu < \frac{2}{\max \lambda \{\mathbf{XBX}^H\}}.$$

  2) The noise $\bar{\nu} = 0.$

- → Such conditions are typically satisfied!
Robust Design: Fast Speed Learning

- The free parameter matrix $B$ offers interesting options for speeding up convergence.

- The optimally fast algorithm

\[ \hat{f}_l = \hat{f}_{l-1} + \mu \left[ H H^H \right]^{-1} H (e_{\tau,t,m} - H^H \hat{f}_{l-1}) \quad ; l = 1, 2, ... \]

- Requires a matrix inverse. But for a set of channels with a priori known information (PedB, VehA), it can be replaced by a fixed matrix:

\[ \hat{f}_l = \hat{f}_{l-1} + \mu R_{HH}^{-1} H (e_{\tau,t,m} - H^H \hat{f}_{l-1}) \quad ; l = 1, 2, ... \]
Robust Design

- **Recursive algorithm:**
  \[ \tilde{f}_k = \tilde{f}_{k-1} - \mu Y_k \left( X_k^H \tilde{f}_{k-1} + \tilde{v}_k \right) \quad ; \ k = 1, 2, \ldots, \]

- **Theorem:** for \( X_k = r_k, Y_k = B r_k \) and \( \tilde{v}_k = \tilde{u}_k r_k \)
  
  the recursive algorithm converges, if the following conditions hold:

1) The algorithm is called robust and \( l_2 \)-stable for step-sizes \( 0 < \mu < \max_k \frac{1}{r_k H B r_k} \) (alternatively for a time variant step-size \( 0 < \mu_k < \frac{2}{r_k H B r_k} \)). Consequently there does not exist any sequence (noise or data) that causes the algorithm to diverge.

2) Furthermore, if the noise \( \tilde{v}_k \) is \( l_2 \) bounded, i.e., \( \sum \mu_k |\tilde{v}_k|^2 < S_v < \infty \), the undistorted a priori estimation error \( \sqrt{\mu_k e_{a,k}} = \sqrt{\mu_k r_k H \tilde{f}_{k-1}} \) turns to zero.

3) If additionally, the regression vector \( r_k \) is of persistent excitation, the parameter error vector turns to zero \( \lim_{k \to \infty} \tilde{f}_{k-1} = 0 \).
Robust Design

- Most well-known member of this group is the **LMS algorithm for equalization**:

\[
\hat{f}_k = \hat{f}_{k-1} + \mu_k r_k \left[ s_{k-T,t,m}^* - r_k^H \hat{f}_{k-1} \right] \quad ; \quad k = 1, 2, \ldots
\]

\[
= \hat{f}_{k-1} + \mu_k r_k \left[ r_k^H (f_{T,t,m}^{\text{MMSE}} - \hat{f}_{k-1}) + \tilde{v}_{k,t,m}^{\text{MMSE}} \right].
\]

- With the step-size condition:

\[
0 < \mu_k < \frac{2}{\|r_k\|_2^2}
\]
Robust Design

- Problematic is the never ending noise energy, resulting in a steady-state mismatch of size:

\[ M_{\text{MMSE}} = \frac{\alpha}{2 - \alpha} \left( \| v_{\tau, t, m}^{\text{MMSE}} \|_2^2 + N_o \| f_{\tau, t, m}^{\text{MMSE}} \|_2^2 \right) \]

- assuming a normalized step-size \( \mu_k = \alpha / \| r_k \|_2^2 \)

- Speed-up variants are simply designed by \( B = (R_{HH} + N_o I)^{-1} \)

- Block update variants are also possible.
Blind Channel Estimation (for equalization)

- Basic idea of blind estimation:

\[ h_1, h_2, s_k \]

\[ r_k^{(1)}, g_1 = h_2, s_k^{(1)} \]

\[ r_k^{(2)}, g_2 = h_1, s_k^{(2)} \]

\[ 0 \]
Robust Design: Blind Channel Estimation

- The concept can straightforwardly be extended to the blind channel estimation (for equalization) case.

- Consider this LS problem:

\[
H^B = \arg \min_{h} \left\| H_1^H h_2 - H_2^H h_1 \right\|_2^2
\]

\[
h = \begin{bmatrix}
h_2 \\
h_1
\end{bmatrix}
\]

\[
= \arg \min_h \left\| [H_1^H, -H_2^H] h \right\|_2^2
\]

\[
= \arg \min_h \left\| H^H h \right\|_2^2.
\]

- Drawback: need to know the channel in order to estimate it!
Robust Design: Blind Channel Estimation

- Take observations: \( r_k^T = [r_k^{(1)^T}, -r_k^{(2)^T}] \)

- Stack them into matrices: \( R_k^H = [R_k^{(1)^H}, -R_k^{(2)^H}] \)

- Design recursive update

\[
\hat{h}_k = \hat{h}_{k-1} - \mu_k R_k R_k^H \hat{h}_{k-1} \quad ; k = 1, 2, \ldots
\]

\[
\tilde{h}_k = \tilde{h}_{k-1} - \mu_k R_k \left( R_k^H h^B - V_k^H h^B - R_k^H \hat{h}_{k-1} \right) = 0
\]

\[
= \tilde{h}_{k-1} - \mu_k R_k (R_k^H \tilde{h}_{k-1} - V_k^H h^B).
\]

- Reference system is simply zero!

- Stability for

\[
0 < \mu_k < \frac{1}{\max \lambda\{R_k R_k^H\}}
\]
Conclusion

- Maximizing SINR is equivalent to classical equalizer design criteria (MMSE, ZF).
- By formulating the linear equalizer problems as LS problem, we obtain a reference model.
- Due to the reference model, many well known results from adaptive filters can be applied to adaptive equalizer designs, resulting in robust designs.
Not all adaptive algorithms are robust.

Consider the well-known adaptive ZF algorithm from Lucky (1965):

\[ \hat{f}_k = \hat{f}_{k-1} + \mu s_k (s_k - \hat{f}_{k-1}^H r_k) \]

This algorithm has recently been proven to be non-robust!
Thank you for your attention


- Slides can be downloaded from http://publik.tuwien.ac.at/files/PubDat_199101.pdf