Merging Belief Propagation and the Mean Field Approximation: A Free Energy Approach

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Abstract—**We present a joint message passing approach that combines belief propagation and the mean field approximation. Our analysis is based on the region-based free energy approximation method proposed by Yedidia et al., which allows to use the same objective function (Kullback-Leibler divergence) as a starting point. In this method message passing fixed point equations (which correspond to the update rules in a message passing algorithm) are then obtained by imposing different region-based approximations and constraints on the mean field and belief propagation parts of the corresponding factor graph. Our results can be applied, for example, to algorithms that perform joint channel estimation and decoding in iterative receivers. This is demonstrated in a simple example.**

I. Introduction

Variational techniques have been used for decades in quantum and statistical physics, where they are referred to as *mean field* (MF) approximation [1]. They are also applied for statistical inference, see, e.g., [2]–[5]. The basic idea of variational inference is to derive the statistics of "hidden" random variables given the knowledge of "visible" random variables of a certain probability density function (pdf). This is done by approximating the pdf by some "simpler," e.g., (fully) factorized function in an iterative (message passing like) way. Typically, such a function has to fulfill additional constraints. For example, [4] imposes additionally exponential conjugacy constraints in order to derive simple update rules for the messages that propagate along the edges in a Bayesian network. Variational inference methods were recently applied in [6] to the *channel state estimation/interference cancellation part* of a class of MIMO-OFDM receivers that iterate between detection, channel estimation, and decoding.

A different approach is *belief propagation* (BP) [7]. Roughly speaking, with BP one tries to find *local* approximations, which are—exactly or approximately—the marginals of a certain pdf. This can also be done in an iterative way, where messages are passed along the edges in a factor graph [8]. A typical application of BP is *decoding* of turbo codes.

An obvious question that arises is the following: Can we combine both approaches and develop a *unified message passing algorithm* that combines BP and the MF approach, and how do the two types of messages influence each other? The main contribution of this work is to shed light on this open problem using the free energy approach proposed in [9] and to *derive the message passing fixed point equations for a joint approach, where BP is applied to a subset of factor nodes and the MF approximation is employed to the remaining factor nodes of a factor graph.*

The paper is organized as follows. Section II is devoted to the introduction of the region-based free energy approximations proposed by [9]. We briefly summarize the main steps to derive the message passing fixed point equations for BP in Section III. In Section IV, we show how the MF approximation can be included in the free energy framework. Our main result—the combined BP/MF fixed point equations—is presented in Section V. Section VI is devoted to a discussion of a simple example and shows simulation results. Finally, we conclude in Section VII.

II. Region-based free energy approximations

In the following two sections, we follow the presentation and main results given in [9]. Let $p(\mathbf{x})$ be a certain pdf that factorizes as

$$p(\mathbf{x}) = \prod_a f_a(\mathbf{x}_a),$$

where $\mathbf{x} \triangleq \{x_i \mid i \in \mathbf{I}\}$, $\mathbf{I} \triangleq \{1, \ldots, N\}$, $\mathbf{x}_a \subseteq \mathbf{x}$, and $a \in \mathbf{A} \triangleq \{1, \ldots, M\}$. Such a factorization can be visualized in a *factor graph* [8]. We assume that $p(\mathbf{x})$ is a positive function and that $\mathbf{x}$ is a set of discrete random variables. Our analysis can be extended to continuous random variables by simply replacing sums by integrals. Now define the sets of indices

$$N(a) \triangleq \{i \mid x_i \in \mathbf{x}_a\} \quad \text{and} \quad N(i) \triangleq \{a \mid x_i \in \mathbf{x}_a\}.$$

A *region* $R \triangleq \{\mathbf{x}_R, \mathbf{A}_R\}$ consists of a subset $\mathbf{x}_R \subseteq \mathbf{x}$ of variables and a subset $\mathbf{A}_R \subseteq \mathbf{A}$ of indices with the restriction that $a \in \mathbf{A}_R$ implies that $\mathbf{x}_a \subseteq \mathbf{x}_R$. To each region R we associate a *counting number* $c_R \in \mathbb{Z}$. A set $\mathcal{R} \triangleq \{R\}$ of regions is called *valid* if

$$\sum_{R \in \mathcal{R}} c_R I_{\mathbf{A}_R}(a) = \sum_{R \in \mathcal{R}} c_R I_{\mathbf{x}_R}(x_i) = 1 \quad \forall a \in \mathbf{A}, i \in \mathbf{I},$$

where $I_{\_}(\_)$ is the indicator function.

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We define the *variational free energy* [9]

$$F(b) \triangleq \sum_{\mathbf{x}} b(\mathbf{x}) \ln \frac{b(\mathbf{x})}{p(\mathbf{x})} = \underbrace{\sum_{\mathbf{x}} b(\mathbf{x}) \ln b(\mathbf{x})}_{\triangleq -H(b)} - \underbrace{\sum_{\mathbf{x}} b(\mathbf{x}) \ln p(\mathbf{x})}_{\triangleq -U(b)}. \tag{1}$$

In (1), $H(b)$ denotes entropy and $U(b)$ is called average energy. Note that $F(b)$ is the Kullback-Leibler divergence [10, p. 19] between b and p, i.e., $F(b) = D(b \,||\, p)$. For a set $\mathcal{R}$ of regions, the *region-based variational free energy* is defined as [9] $F_{\mathcal{R}} \triangleq U_{\mathcal{R}} - H_{\mathcal{R}}$ with

$$\begin{aligned} U_{\mathcal{R}} &\triangleq \sum_{R\in\mathcal{R}} c_R U_R, \\ H_{\mathcal{R}} &\triangleq \sum_{R\in\mathcal{R}} c_R H_R, \\ U_R &\triangleq -\sum_{a\in\mathbf{A}_R} \sum_{\mathbf{x}_R} b_R(\mathbf{x}_R) \ln f_a(\mathbf{x}_a), \\ H_R &\triangleq -\sum_{\mathbf{x}_R} b_R(\mathbf{x}_R) \ln b_R(\mathbf{x}_R). \end{aligned}$$

Here, $b_R(\mathbf{x}_R)$ is defined locally on the region R. Instead of minimizing F with respect to b, we minimize $F_{\mathcal{R}}$ with respect to all b_R ($R \in \mathcal{R}$), where the b_R have to fulfill certain constraints. The quantities b_R are called *beliefs*. We give two examples of valid sets of regions.

Example II.1 The trivial example $\mathcal{R} = \{R = (\mathbf{x}, A)\}$.

Example II.2 We define two types of regions:

1) *large regions:* $R_a \triangleq (\mathbf{x}_a, \{a\})$ with $c_{R_a} = 1 \;\forall\, a \in \mathbf{A}$;
2) *small regions:* $R_i \triangleq (\{x_i\}, \emptyset)$ with $c_{R_i} = 1 - |N(i)| \;\forall\, i \in \mathbf{I}$.

Here, $|N(i)|$ denotes the cardinality of the set $N(i)$ for all $i \in \mathbf{I}$. The region-based variational free energy corresponding to the valid set of regions $\mathcal{R} = \{R_i \mid i \in \mathbf{I}\} \cup \{R_a \mid a \in \mathbf{A}\}$ is called the *Bethe free energy* [9], [11]. The exact variational free energy is equal to the Bethe free energy when the factor graph has no cycles [9].

III. Belief propagation fixed point equations

The fixed point equations for BP can be obtained from the Bethe free energy by imposing additional marginalization constraints and computing the stationary points. The Bethe free energy reads

$$F_{\mathcal{R}} = \sum_{a\in\mathbf{A}} \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln \frac{b_a(\mathbf{x}_a)}{f_a(\mathbf{x}_a)} - \sum_{i\in\mathbf{I}} (|N(i)| - 1) \sum_{x_i} b_i(x_i) \ln b_i(x_i), \tag{2}$$

with $b_a(\mathbf{x}_a) \triangleq b_{R_a}(\mathbf{x}_a) \;\forall\; a \in \mathbf{A}$ and $b_i(x_i) \triangleq b_{R_i}(\{x_i\}) \;\forall\, i \in \mathbf{I}$. The summation over the index set $\mathbf{I}$ in (2) can be restricted to indices with $|N(i)| > 1$ (the dependence on beliefs $b_i(x_i)$ with $|N(i)| = 1$ drops out). In addition, we impose marginalization constraints on the beliefs

$$b_i(x_i) = \sum_{\mathbf{x}_a \backslash x_i} b_a(\mathbf{x}_a) \quad \forall i \in \mathbf{I}, a \in N(i), \tag{3}$$

which can be included in the Lagrangian

$$L \triangleq F_{\mathcal{R}} + \sum_{a\in\mathbf{A}} \sum_{i\in N(a)} \sum_{x_i} \lambda_{a,i}(x_i) \left(b_i(x_i) - \sum_{\mathbf{x}_a\backslash x_i} b_a(\mathbf{x}_a) \right), \tag{4}$$

where the $\lambda_{a,i}(x_i)$ are Lagrange multipliers [12, p. 283].

The following theorem gives a connection between the BP fixed points with *positive beliefs* and stationary points of the Lagrangian in (4).

Theorem 1 *[9, Theorem 2] Stationary points of the constrained Bethe free energy must be BP fixed points with positive beliefs and vice versa.*

Note that beliefs with tight nonnegativity constraints can only belong to critical points but not to stationary points. We summarize the main steps in the proof of Theorem 1. The stationary points of the Lagrangian in (4) can then be evaluated as

$$\begin{cases} b_a(\mathbf{x}_a) \propto f_a(\mathbf{x}_a) \exp\left(\sum_{i\in N(a)} \lambda_{a,i}(x_i)\right) & \forall\; a \in \mathbf{A} \\ b_i(x_i) \propto \exp\left(\frac{1}{|N(i)|-1} \sum_{a\in N(i)} \lambda_{a,i}(x_i)\right) & \forall\; i \in \mathbf{I}. \end{cases} \tag{5}$$

Now we apply the following lemma.

Lemma 1 *[9, p. 2292] For each $i \in \mathbf{I}$ (recall that $|N(i)| > 1$) we can reparametrize*

$$\lambda_{a,i}(x_i) = \ln \prod_{c\in N(i)\backslash a} m_{c\to i}(x_i) \quad \forall\; a \in N(i) \tag{6}$$

in an unique way with $m_{a\to i}(x_i) > 0 \;\forall a \in N(i)$. The inverse of this mapping is given by

$$m_{a\to i}(x_i) = \exp\Bigg(\frac{2 - |N(i)|}{|N(i)| - 1} \lambda_{a,i}(x_i) + \frac{1}{|N(i)|-1} \sum_{b\in N(i)\backslash a} \lambda_{b,i}(x_i) \Bigg) \quad \forall\; a \in N(i).$$

The proof of Lemma 1 is based on a simple matrix inversion. Defining

$$n_{i\to a}(x_i) \triangleq \prod_{c\in N(i)\backslash a} m_{c\to i}(x_i) \quad \forall i \in \mathbf{I}, a \in N(i), \tag{7}$$

plugging the reparametrization (6) into (5), and applying the marginalization constraints in (3) yields the following fixed point equations for BP:

$$\begin{cases} m_{a\to i}(x_i) = \sum_{\mathbf{x}_a\backslash x_i} f_a(\mathbf{x}_a) \prod_{j\in N(a)\backslash i} n_{j\to a}(x_j) \\ n_{i\to a}(x_i) = \prod_{c\in N(i)\backslash a} m_{c\to i}(x_i). \end{cases} \tag{8}$$

Remark III.1 This result can be extended to the case where the functions f_a are nonnegative under the assumption that $\sum_{\mathbf{x}_a \setminus x_i} f_a(\mathbf{x}_a) > 0$ for all $i \in N(a)$ (If this expression is zero for one $x_i = \bar{x}_i$ then $p(\mathbf{x}) = 0$ for all $\mathbf{x} \setminus x_i$ and $x_i = \bar{x}_i$ and we can remove $\bar{x}_i$). The key observation is that we must set $b_a(\bar{\mathbf{x}}_a) = 0$ whenever $f_a(\bar{\mathbf{x}}_a) = 0$ for a certain $\mathbf{x}_a = \bar{\mathbf{x}}_a$ if we assume that $F_{\mathcal{R}}$ is *finite*. The beliefs $b_i(x_i)$ are always positive.

IV. Fixed point equations for the mean field approximation

The MF approximation can be interpreted as a message passing algorithm on a factor graph [13]. In this section, we briefly show how the corresponding fixed point equations can be obtained by the free energy approach. To this end we define one region $R \triangleq (\mathbf{x}, \mathbf{A})$ with $c_R = 1$ and impose the constraint that $b(\mathbf{x})$ fully factorizes, i.e.,

$$b(\mathbf{x}) = \prod_{i \in \mathbf{I}} b_i(x_i).$$

This constraint can be directly plugged into the expression for the variational free energy in (1). Doing so we get

$$F = \sum_{i \in \mathbf{I}} \sum_{x_i} b_i(x_i) \ln b_i(x_i) - \sum_{a \in \mathbf{A}} \sum_{\mathbf{x}_a} \prod_{i \in N(a)} b_i(x_i) \ln f_a(\mathbf{x}_a).$$

The stationary points for the MF approximation can easily be evaluated:

$$b_i(x_i) \propto \exp\left(\sum_{a \in N(i)} \sum_{\mathbf{x}_a \setminus x_i} \prod_{j \in N(a) \setminus i} b_j(x_j) \ln f_a(\mathbf{x}_a) \right) \forall\, i \in \mathbf{I}.$$

The updates b_i can be evaluated by iterating over $i \in \mathbf{I}$. At each step the objective function decreases and the algorithm is guaranteed to converge. To derive a particular update b_i we need all previous updates b_j for $j \in \bigcup_{a \in N(i)} N(a) \setminus i$.

A message passing interpretation for the MF approximation can be obtained by setting $n_{i \to N(i)}(x_i) \triangleq b_i(x_i) \ \forall\ i \in \mathbf{I}$, which results in [13]

$$\begin{cases} n_{i \to N(i)}(x_i) = \prod_{a \in N(i)} m_{a \to i}(x_i) \\ m_{a \to i}(x_i) \triangleq \exp\left(\sum_{\mathbf{x}_a \setminus x_i} \prod_{j \in N(a) \setminus i} n_{j \to N(j)}(x_j) \ln f_a(\mathbf{x}_a) \right). \end{cases} \tag{9}$$

Remark IV.1 In the MF approach, we assume that the functions $f_a(\mathbf{x}_a)$ are positive.

V. Combined belief propagation / mean field fixed point equations

We are now in a position to combine BP and the MF approximation. Let

$$p(\mathbf{x}) = \prod_{a \in A_{\text{MF}}} f_a(\mathbf{x}_a) \prod_{a \in A_{\text{BP}}} f_a(\mathbf{x}_a)$$

be a partially factorized pdf. As before we have $\mathbf{x} = \{x_i \mid i \in \mathbf{I}\}$, $\mathbf{I} = \{1, \dots, N\}$, $\mathbf{x}_a \subseteq \mathbf{x}$, and $a \in \mathbf{A} = \{1, \dots, M\}$ with $\mathbf{A} = \mathbf{A}_{\text{MF}} \cup \mathbf{A}_{\text{BP}}$. Furthermore, we set

$$\begin{aligned} \mathbf{I}_{\text{MF}} &\triangleq \{i \in \mathbf{I} \mid \exists a \in \mathbf{A}_{\text{MF}} \text{ with } i \in N(a)\} \\ \mathbf{I}_{\text{BP}} &\triangleq \{i \in \mathbf{I} \mid \exists a \in \mathbf{A}_{\text{BP}} \text{ with } i \in N(a)\}. \end{aligned}$$

Note that $\mathbf{A}_{\text{MF}} \cap \mathbf{A}_{\text{BP}} = \emptyset$ but $\mathbf{I}_{\text{MF}} \cap \mathbf{I}_{\text{BP}} \neq \emptyset$ in general. We define the set $\mathcal{R}$ of valid regions:

1) one MF region $R_{\text{MF}} \triangleq (\mathbf{x}_{\text{MF}}, \mathbf{A}_{\text{MF}})$ with $\mathbf{x}_{\text{MF}} \triangleq \{x_i \mid i \in \mathbf{I}_{\text{MF}}\}$ and $c_{R_{\text{MF}}} = 1$;
2) small regions $R_i \triangleq (\{x_i\}, \emptyset)$ with $c_{R_i} = 1 - |N_{\text{BP}}(i)| - I_{\mathbf{I}_{\text{MF}}}(i)$ for all $i \in \mathbf{I}_{\text{BP}}$;
3) large regions $R_a \triangleq (\mathbf{x}_a, \{a\})$ with $c_{R_a} = 1$ for all $a \in \mathbf{A}_{\text{BP}}$,

with $N_{\text{BP}}(i) \triangleq \{a \in \mathbf{A}_{\text{BP}} \mid a \in N(i)\}$. This yields the region-based variational free energy

$$\begin{aligned} F_{\mathcal{R}} = &\sum_{a \in \mathbf{A}_{\text{BP}}} \sum_{\mathbf{x}_a} b_a(\mathbf{x}_a) \ln \frac{b_a(\mathbf{x}_a)}{f_a(\mathbf{x}_a)} \\ &- \sum_{a \in \mathbf{A}_{\text{MF}}} \sum_{\mathbf{x}_a} \prod_{i \in N(a)} b_i(x_i) \ln f_a(\mathbf{x}_a) \\ &- \sum_{i \in \mathbf{I}} (|N_{\text{BP}}(i)| - 1) \sum_{x_i} b_i(x_i) \ln b_i(x_i). \end{aligned} \tag{10}$$

We can restrict the summation over the index set $\mathbf{I}$ in the last term in (10) to indices $i \in \mathbf{I}$ with $|N_{\text{BP}}(i)| \neq 1$. The constraints for the BP part can be included in a Lagrangian

$$L \triangleq F_{\mathcal{R}} + \sum_{a \in \mathbf{A}_{\text{BP}}} \sum_{i \in N_{\text{BP}}(a)} \sum_{x_i} \lambda_{a,i}(x_i) \left(b_i(x_i) - \sum_{\mathbf{x}_a \setminus x_i} b_a(\mathbf{x}_a) \right).$$

We now derive the stationary points of this Lagrangian. To this end we define the set

$$\Delta \triangleq \{i \in \mathbf{I}_{\text{BP}} \cap \mathbf{I}_{\text{MF}} \mid |N_{\text{BP}}(i)| = 1\},$$

which corresponds to variable nodes that are "dead ends" in the BP part, i.e., there is a *unique* $a_i \in \mathbf{A}_{\text{BP}}$ for each $i \in \Delta$, but are connected to the MF part. The stationary points can be evaluated as

$$\begin{aligned} \lambda_{a_i,i}(x_i) &= \ln(b_i^{\text{MF}}(x_i)) \ \forall\, i \in \Delta \\ b_a(\mathbf{x}_a) &\propto b_a^{\text{BP}}(\mathbf{x}_a) \prod_{i \in N(a)} b_i^{\text{MF}}(x_i) \quad \forall\, a \in \mathbf{A}_{\text{BP}} \\ b_i(x_i) &\propto \begin{cases} b_i^{\text{MF}}(x_i) b_i^{\text{BP}}(x_i) & \forall\, i \in \mathbf{I} \setminus \Delta \\ \sum_{\mathbf{x}_{a_i} \setminus x_i} b_{a_i}(\mathbf{x}_{a_i}) & \forall\, i \in \Delta, \end{cases} \end{aligned}$$

with

$$b_a^{\mathrm{BP}}(\mathbf{x}_a) \triangleq f_a(\mathbf{x}_a)\exp\left(\sum_{i\in N(a)\setminus\Delta}\tilde{\lambda}_{a,i}(x_i)\right) \quad \forall\, a \in \mathbf{A}_{BP}$$

$$b_i^{\mathrm{BP}}(x_i) \triangleq \begin{cases} \exp\left(\frac{1}{|N_{\mathrm{BP}}(i)|-1}\sum_{a\in N_{\mathrm{BP}}(i)}\tilde{\lambda}_{a,i}(x_i)\right) & \forall\, i\in \mathbf{I}_{BP}\setminus\Delta \\ 1 & \forall\, i \in \mathbf{I}\setminus\mathbf{I}_{BP}\end{cases}$$

$$b_i^{\mathrm{MF}}(x_i) \triangleq \begin{cases} \exp\left(\sum_{a\in N_{\mathrm{MF}}(i)}\sum_{\mathbf{x}_a\setminus x_i}\prod_{j\in N(a)\setminus i} b_j(x_j)\ln f_a(\mathbf{x}_a)\right) & \forall\, i\in\mathbf{I}_{MF} \\ 1 & \forall\, i\in\mathbf{I}\setminus\mathbf{I}_{MF},\end{cases}$$

where we defined $N_{\mathrm{MF}}(i) \triangleq \{a \in \mathbf{A}_{\mathrm{MF}} \mid a \in N(i)\}$ and

$$\tilde{\lambda}_{a,i}(x_i) \triangleq \lambda_{a,i}(x_i) - \ln b_i^{\mathrm{MF}}(x_i)\ \forall\, i\in\mathbf{I}_{\mathrm{BP}}\setminus\Delta, a\in N_{\mathrm{BP}}(i).$$

The messages for the BP part can now be introduced in a similar way as for solely BP. Applying Lemma 1 to $\tilde{\lambda}_{a,i}(x_i)$ for all $i \in \mathbf{I}_{\mathrm{BP}}\setminus\Delta$ gives the reparametrization

$$\tilde{\lambda}_{a,i}(x_i) = \ln\prod_{c\in N_{\mathrm{BP}}(i)\setminus a} m_{c\to i}^{\mathrm{BP}}(x_i) \quad \forall\, a\in N_{\mathrm{BP}}(i).$$

Defining the messages

$$\begin{aligned} n_{i\to N(i)}^{\mathrm{MF}}(x_i) &\triangleq b_i^{\mathrm{MF}}(x_i) \qquad \forall\, i\in\mathbf{I}_{\mathrm{BP}} \\ n_{i\to a}^{\mathrm{BP}}(x_i) &\triangleq \prod_{c\in N_{\mathrm{BP}}(i)\setminus a} m_{c\to i}^{\mathrm{BP}}(x_i)\ \forall\, i\in\mathbf{I}_{\mathrm{BP}}\setminus\Delta, a\in N_{\mathrm{BP}}(i)\end{aligned}$$

yields

$$\begin{aligned} b_a^{\mathrm{BP}}(\mathbf{x}_a) &= f_a(\mathbf{x}_a)\prod_{i\in N(a)\setminus\Delta} n_{i\to a}^{\mathrm{BP}}(x_i) \quad \forall\, a\in\mathbf{A}_{\mathrm{BP}} \\ b_a(\mathbf{x}_a) &\propto b_a^{\mathrm{BP}}(\mathbf{x}_a)\prod_{i\in N(a)} n_{i\to N(i)}^{\mathrm{MF}}(x_i) \quad \forall\, a\in\mathbf{A}_{\mathrm{BP}} \\ b_i(x_i) &\propto n_{i\to N(i)}^{\mathrm{MF}}(x_i)\underbrace{\prod_{a\in N_{\mathrm{BP}}(i)} m_{a\to i}^{\mathrm{BP}}(x_i)}_{=b_i^{\mathrm{BP}}(x_i)} \quad \forall\, i\in\mathbf{I}\setminus\Delta.\end{aligned}$$

Using the marginalization constraints, we end up with the fixed point equations for the BP part

$$\begin{cases} m_{a\to i}^{\mathrm{BP}}(x_i) = \sum_{\mathbf{x}_a\setminus x_i} f_a(\mathbf{x}_a)\prod_{j\in N(a)\setminus(\{i\}\cup\Delta)} n_{j\to a}^{\mathrm{BP}}(x_j)\prod_{j\in N(a)\setminus i} n_{j\to N(j)}^{\mathrm{MF}}(x_j) \\ n_{i\to a}^{\mathrm{BP}}(x_i) = \prod_{c\in N_{\mathrm{BP}}(i)\setminus a} m_{c\to i}^{\mathrm{BP}}(x_i)\end{cases} \tag{11}$$

for all $a \in \mathbf{A}_{\mathrm{BP}}, i\in\mathbf{I}_{\mathrm{BP}}\setminus\Delta$. The beliefs $b_i(x_i)$ for $i\in\Delta$ can be evaluated from the marginalization constraints, i.e.,

$$b_i(x_i) \propto n_{i\to\{a_i\}}^{\mathrm{MF}}(x_i)\underbrace{\sum_{\mathbf{x}_{a_i}\setminus x_i} b_{a_i}^{\mathrm{BP}}(\mathbf{x}_{a_i})\prod_{j\in N(a_i)\setminus i} n_{j\to N(j)}^{\mathrm{MF}}(x_j)}_{\triangleq b_i^{\mathrm{BP}}(x_i)}$$

for all $i\in\Delta$.

It remains to introduce the remaining messages for the MF part

$$\begin{cases} n_{i\to N(i)}^{\mathrm{MF}}(x_i) = \prod_{a\in N_{\mathrm{MF}}(i)} m_{a\to i}^{\mathrm{MF}}(x_i) \\ m_{a\to i}^{\mathrm{MF}}(x_i) \triangleq \exp\left(\sum_{\mathbf{x}_a\setminus x_i}\prod_{j\in N(a)\setminus i} b_j^{\mathrm{BP}}(x_j)\, n_{j\to N(j)}^{\mathrm{MF}}(x_j)\ln f_a(\mathbf{x}_a)\right)\end{cases} \tag{12}$$

for all $a\in\mathbf{A}_{\mathrm{MF}}, i\in\mathbf{I}_{\mathrm{MF}}$. All these steps are reversible. Thus, we have proved the following theorem.

Theorem 2 *Stationary points of the constrained variational free energy in the combined BP/MF approach must be fixed points with positive beliefs and vice versa. The corresponding fixed point equations are* (11) *and* (12).

Remark V.1 The inclusion of hard constraints in the *BP part* can be done in the same fashion as for solely BP propagation.

VI. A SIMPLE EXAMPLE

Assume a frequency-flat time-varying channel with input-output relationship

$$\mathbf{y} = \mathbf{X}\mathbf{h} + \mathbf{z},$$

where $\mathbf{z} \in \mathcal{CN}(\mathbf{0}, \gamma^{-1}\mathbf{I}), \mathbf{X} \triangleq \mathrm{diag}(x_i \mid i = 1,\ldots,n)$, and $\mathbf{y}\in\mathbb{C}^n$. The symbols $x_i\in\mathbb{C}$ belong to a certain modulation alphabet. Rewriting

$$p(\mathbf{y},\mathbf{X},\gamma,\mathbf{h}) \propto p(\mathbf{y}|\mathbf{X},\gamma,\mathbf{h})p(\gamma)p(\mathbf{h})p(\mathbf{X}),$$

where we used the fact that γ, $\mathbf{h}$, and $\mathbf{x}$ are independent, gives a factorization where we wish to apply BP for $p(\mathbf{X})$ and the MF approximation for the remaining factors. Notice that $p(\mathbf{X})$ includes modulation and the code constraints. We assume that the prior distributions of γ and $\mathbf{h}$ are of the form

$$\begin{aligned} p(\gamma) &\propto \gamma^{k^P-1}\exp(-\gamma\theta^P) \\ p(\mathbf{h}) &\propto \exp(-(\mathbf{h}-\bar{\mathbf{h}}^P)^H\mathbf{\Lambda}_{\mathbf{h}}^P(\mathbf{h}-\bar{\mathbf{h}}^P)).\end{aligned}$$

Let

$$\bar{\mathbf{h}} \triangleq \mathrm{E}_{b_{\mathbf{h}}}(\mathbf{h}) \qquad \mathbf{R}_{\mathbf{h}} \triangleq \mathrm{Cov}_{b_{\mathbf{h}}}(\mathbf{h}) \qquad \bar{\gamma} \triangleq \mathrm{E}_{b_\gamma}(\gamma)$$
$$\bar{\mathbf{X}} \triangleq \mathrm{E}_{\{b_i\}}(\mathbf{X}) \qquad \mathbf{\Sigma} \triangleq \mathrm{Var}_{\{b_i\}}(\mathbf{X}),$$

with $b_i = b_i^{\mathrm{BP}} b_i^{\mathrm{MF}}$ $(i = 1,\ldots,n)$, $b_{\mathbf{h}} = b_{\mathbf{h}}^{\mathrm{MF}}$, and $b_\gamma = b_\gamma^{\mathrm{MF}}$. Then we get the following message passing update equations:
Update for γ*:*

$$\begin{aligned} n_{\gamma\to N(\gamma)}(\gamma) &= m_{p(\gamma)\to\gamma}^{\mathrm{MF}}(\gamma)\, m_{p(\mathbf{y}|\mathbf{x},\mathbf{h},\gamma)\to\gamma}^{\mathrm{MF}}(\gamma) \\ &= \gamma^{k^P+N-1}\exp(-\gamma(\theta^P+\underline{\theta})),\end{aligned}$$

with

$$\begin{aligned}\underline{\theta} &\triangleq \mathrm{E}_{\{b_i\}}\mathrm{E}_{b_{\mathbf{h}}}\|\mathbf{y}-\mathbf{X}\mathbf{h}\|^2 \\ &= \|\mathbf{y}\|^2 + \mathrm{Tr}((\mathbf{R}_{\mathbf{h}}+\bar{\mathbf{h}}\bar{\mathbf{h}}^H)(\mathbf{\Sigma}+\bar{\mathbf{X}}\bar{\mathbf{X}}^H)) - 2\Re(\mathbf{y}^H\bar{\mathbf{X}}\bar{\mathbf{h}}).\end{aligned}$$

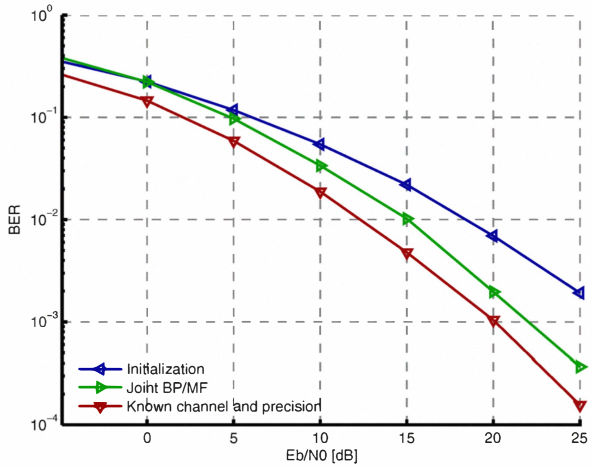

Fig. 1. Average BER versus Eb/N0 for a time-varying channel with a square Doppler spectrum. The channel code is a turbo code with polynomial $(1, 1/3)$ and codeword length of 196, the modulation scheme is 16-QAM, and the interleaver is random. A pilot based LMMSE estimate yields the initialization of $\bar{\mathbf{h}}$; two QPSK modulated pilot symbols are employed for this purpose. The channel covariance matrix is assumed to be perfectly known at the receiver.

Update for $\mathbf{h}$*:*

$$\begin{aligned} n_{\mathbf{h}\to N(\mathbf{h})}(\mathbf{h}) &= m^{\mathrm{MF}}_{p(\mathbf{h})\to\mathbf{h}}(\mathbf{h}) m^{\mathrm{MF}}_{p(\mathbf{y}|\mathbf{x},\mathbf{h},\gamma)\to\mathbf{h}}(\mathbf{h}) \\ &\propto \exp\left(-(\mathbf{h}-\bar{\mathbf{h}})^H \mathbf{R}_{\mathbf{h}}^{-1}(\mathbf{h}-\bar{\mathbf{h}})\right), \end{aligned}$$

with

$$\begin{aligned} \mathbf{R}_{\mathbf{h}}^{-1} &= (\mathbf{\Lambda}_{\mathbf{h}}^P + \underline{\mathbf{\Lambda}}) & \bar{\mathbf{h}} &= \mathbf{R}_{\mathbf{h}}(\mathbf{\Lambda}_{\mathbf{h}}^P \bar{\mathbf{h}}^P + \underline{\bar{\mathbf{h}}}) \\ \underline{\mathbf{\Lambda}} &\triangleq \bar{\gamma}(\mathbf{\Sigma} + \bar{\mathbf{X}}\bar{\mathbf{X}}^H) & \underline{\bar{\mathbf{h}}} &\triangleq \bar{\gamma}\bar{\mathbf{X}}^H \mathbf{y}. \end{aligned}$$

This follows from

$$E_{\{b_i\}} \mathrm{E}_{b_\gamma}\left(\gamma\|\mathbf{y}-\mathbf{X}\mathbf{h}\|^2 - \gamma\|\mathbf{y}\|^2\right) = \mathbf{h}^H \underline{\mathbf{\Lambda}}\mathbf{h} - 2\Re(\mathbf{h}^H \underline{\bar{\mathbf{h}}}).$$

Update for x_i $(i = 1, \ldots, n)$*:*

$$\begin{aligned} n^{\mathrm{MF}}_{i\to N(i)} &= m^{\mathrm{MF}}_{p(\mathbf{y}|\mathbf{x},\mathbf{h},\gamma)\to i} \\ &\propto \exp(E_{\{b_{j\neq i}\}} \mathrm{E}_{b_\gamma} \mathrm{E}_{b_{\mathbf{h}}}(\ln p(\mathbf{y} \mid \mathbf{x}, \mathbf{h}, \gamma))) \\ &\propto \exp(-\bar{\gamma}(|x_i|^2[\mathbf{R}_{\mathbf{h}} + \bar{\mathbf{h}}\bar{\mathbf{h}}^H]_{ii} - 2\Re(y_i^*[\bar{\mathbf{h}}]_i x_i)). \end{aligned}$$

Fig. 1 depicts the average BER versus E_b/N_0 of three algorithms. The blue curve denotes the performance of a scheme performing separate decoding and LMMSE channel estimation based on pilot symbols, while knowing the noise precision. The green curve represents the performance of the combined BP/MF approach after convergence is reached. The former "separate" receiver is used to compute the initial values of the channel coefficients and symbol estimates. The red curve depicts the performance of a decoder having perfect knowledge of the channel coefficients and noise precision.

It can be seen that the performance of the BP/MF algorithm is close to that of the scheme having perfect channel knowledge. Moreover, the BP/MF algorithm significantly outperforms the scheme performing separate channel estimation and decoding.

VII. Conclusion

Using the region-based free energy approximation method proposed in [9], we derived message passing update equations for a factor graph where BP is applied to one part of the factor nodes and the MF approximation is implemented on the remaining factor nodes. The proposed theoretical framework provides a mean to determine the way messages computed on the same factor graph using BP and the MF approximation are to be combined.

A simple example confirmed the validity of the BP/MF method. This example shows that the method allows to combine the estimation of densities of continuous parameters with BP processing of discrete variables, unlike methods using the EM algorithm to compute point estimates of these parameters [14].

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