Algorithms and Complexity Results for Persuasive Argumentation *†

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Abstract

The study of arguments as abstract entities and their interaction as introduced by Dung (*Artificial Intelligence* 177, 1995) has become one of the most active research branches within Artificial Intelligence and Reasoning. A main issue for abstract argumentation systems is the selection of acceptable sets of arguments. Value-based argumentation, as introduced by Bench-Capon (*J. Logic Comput.* 13, 2003), extends Dung's framework. It takes into account the relative strength of arguments with respect to some ranking representing an audience: an argument is subjectively accepted if it is accepted with respect to some audience, it is objectively accepted if it is accepted with respect to all audiences.

Deciding whether an argument is subjectively or objectively accepted, respectively, are computationally intractable problems. In fact, the problems remain intractable under structural restrictions that render the main computational problems for non-value-based argumentation systems tractable. In this paper we identify nontrivial classes of value-based argumentation systems for which the acceptance problems are polynomial-time tractable. The classes are defined by means of structural restrictions in terms of the underlying graphical structure of the value-based system. Furthermore we show that the acceptance problems are intractable for two classes of value-based systems that where conjectured to be tractable by Dunne (*Artificial Intelligence* 171, 2007).

1 Introduction

The study of arguments as abstract entities and their interaction as introduced by Dung [12] has become one of the most active research branches within Artificial Intelligence and Reasoning, see, e.g., [3, 6, 24]. Argumentation handles possible conflicts between arguments in form of attacks. The arguments may either originate from a dialogue between several agents or from the pieces of information at the disposal of a single agent, this information may even contain contradictions. A main issue for any argumentation system is the selection of acceptable sets of arguments, where an acceptable set of arguments must be in some sense coherent and be able to defend itself against all attacking arguments. Abstract argumentation provides suitable concepts and formalisms to study, represent, and process various reasoning problems most prominently in defeasible reasoning (see, e.g., [23], [8]) and agent interaction (see, e.g., [22]).

Extending Dung's concept, Bench-Capon [4] introduced *value-based argumentation* systems that allow to compare arguments with respect to their relative strength such that an argument cannot successfully attack another argument that is considered of a higher rank. The ranking is specified by the combination of an assignment of *values* to arguments and an ordering of the values; the latter is called an *audience* [5]. As laid out by Bench-Capon, the role of arguments in this setting is to persuade rather than to prove, demonstrate or refute. Whether an argument can be accepted with respect to *all possible* or *at least one* audience allows to formalize the notions of *objective acceptance* and *subjective acceptance*, respectively.

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An important limitation for using value-based argumentation systems in real-world applications is the *computational intractability* of the two basic acceptance problems: deciding whether a given argument is subjectively accepted is NP-hard, deciding whether it is objectively accepted is co-NP-hard [15]. Therefore it is important to identify classes of value-based systems that are still useful and expressible but allow a polynomial-time tractable acceptance decision. However, no non-trivial tractable classes of value-based systems have been identified so far, except for systems with a tree structure where the degree of nodes and the number of nodes of degree exceeding 2 are bounded [14]. In fact, as pointed out by Dunne [13], the acceptance problems remain intractable for value-based systems whose graphical structures form trees, in strong contrast to the main computational problems for non-value-based argumentation that are linear-time tractable for tree systems, or more generally, for systems of bounded treewidth [13] .

Our Contribution In this paper we introduce nontrivial classes of value-based systems for which the acceptance problems are tractable. The classes are defined in terms of the following notions:

- The *value-width* of a value-based system is the largest number of arguments of the same value.
- The *extended graph structure* of a value-based system has as nodes the arguments of the value-based system, two arguments are joined by an edge if either one attacks the other or both share the same value.
- The *value graph* of a value-based system has as vertices the values of the system, two values v_1 and v_2 are joined by a directed edge if some argument of value v_1 attacks an argument of value v_2 [14].

We show that the acceptance problems are tractable for the following classes of value-based systems:

(P1) value-based systems with a bipartite graph structure where at most two arguments share the same value (i.e., systems of value-width 2);

(P2) value-based systems whose extended graph structure has bounded treewidth; and

(P3) value-based systems of bounded value-width whose value graphs have bounded treewidth.

In fact, we show that both acceptance problems are *linear time tractable* for the classes (P2) and (P3), the latter being a subclass of the former. Our results suggest that the extended graph structure is a suitable structural representation of value-based argumentation systems. The positive results (P1)–(P3) hold for systems with unbounded number of arguments, attacks and values.

We contrast our positive results with negative results that rule out classes conjectured to be tractable. We show that the acceptance problems are (co)-NP-hard for the following classes:

(N1) value-based systems of value-width 2;

(N2) value-based systems where the number of attacks between arguments of the same value is bounded (systems of *bounded attack-width*);

(N3) value-based systems with bipartite value graphs.

In fact, we show that both acceptance problems are intractable for value-based systems of value-width 2 and attack-width 1. Classes (N1) and (N2) were conjectured to be tractable [13], the complexity of (N3) was stated as an open problem [14].

The reminder of the paper is organized as follows. In Section 2 we provide basic definitions and preliminaries. In Section 3 we define the parameters value-width and attack-width and establish the results involving systems of value-width 2, we also discuss the relationship between systems of value-width 2 and dialogues [5]. In Section 4 we consider value-based systems with an extended graph structure of bounded treewidth and show linear time tractability. We close in Section 5 with concluding remarks. Some proofs of technical lemmas are given in an appendix.

The main results of this paper have been presented in preliminary and shortened form at COMMA'10 [21]. Here we provide full proofs, examples, and additional discussions. Further new additions are the results (P3) and (N3) involving value graphs, and the discussion of the relationship between systems of value-width 2 and dialogues.

2 Arguments, attacks, values, and audiences

In this section we introduce the objects of our study more formally.

2.1 Abstract argumentation system

Definition 1. *An* abstract argumentation system *or* argumentation framework *is a pair* (X, A) *where* X *is a finite set of elements called* arguments *and* $A \subseteq X \times X$ *is a binary relation called the* attack relation. *If* $(x, y) \in A$ *we say that* x attacks y.

An abstract argumentation system $F = (X, A)$ can be considered as a directed graph, and therefore it is convenient to borrow notions and notation from the theory of directed graphs [1]. For example we say that a system $F = (X, A)$ is *acyclic* if (X, A) is a DAG (a directed acyclic graph).

Example 1. An abstract argumentation system $F_0 = (X, A)$ with arguments $X = \{a, b, c, d, e, f\}$ and attacks $A = \{(a, d), (a, e), (b, a), (c, d), (d, b), (f, c)\}$ is displayed in Figure 1.

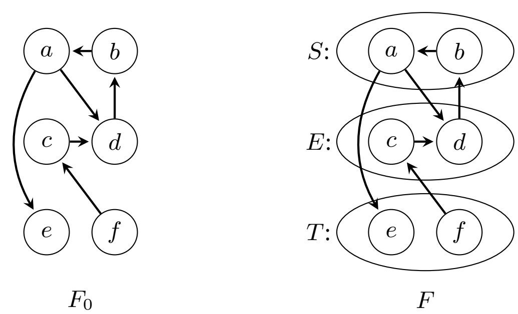

Figure 1: The abstract argumentation system F_0 and value-based system F of Examples 1 and 2, respectively.

Next we define commonly used semantics of abstract argumentation systems as introduced by Dung [12]. For the discussion of other semantics and variants, see, e.g., Baroni and Giacomin's survey [2]. Let $F = (X, A)$ be an abstract argumentation system and $S \subseteq X$.

1. S is *conflict-free* in F if there is no $(x, y) \in A$ with $x, y \in S$.
2. S is *acceptable* in F if for each $x \in S$ and each $y \in X$ with $(y, x) \in A$ there is some $x' \in S$ with $(x', y) \in A$.
3. S is *admissible* in F if it is conflict-free and acceptable.
4. S is a *preferred extension* of F if S is admissible in F and there is no admissible set S' of F that properly contains S.

For instance, the admissible sets of the abstract argumentation system F_0 of Example 1 are the sets $\emptyset$ and $\{f\}$, hence $\{f\}$ is its only preferred extension.

Let $F = (X, A)$ be an abstract argumentation system and $x_1 \in X$. The argument x_1 is *credulously accepted* in F if x_1 is contained in some preferred extension of F, and x_1 is *skeptically accepted* in F if x_1 is contained in all preferred extensions of F.

In this paper we are especially interested in finding preferred extensions in *acyclic* abstract argumentation systems. It is well known that every acyclic system $F = (X, A)$ has a unique preferred extension GE(F), and that GE(F) can be found in polynomial time (GE(F) coincides with the "grounded extension" [12]). In fact, GE(F) can be found via a simple labeling procedure that repeatedly applies the following two rules to the arguments in X until each of them is either labeled IN or OUT:

1. An argument x is labeled IN if all arguments that attack x are labeled OUT (in particular, if x is not attacked by any argument).
2. An argument x is labeled OUT if it is attacked by some argument that is labeled IN.

The unique preferred extension $\mathrm{GE}(F)$ is then the set of all arguments that are labeled IN.

2.2 Value-based systems

Definition 2. *A* value-based argumentation framework *or* value-based system *is a tuple $F = (X, A, V, \eta)$ where (X, A) is an argumentation framework, V is a set of* values, *and η is a mapping $X \to V$ such that the abstract argumentation system $F_v = (\eta^{-1}(v), \{ (x, y) \in A \mid x, y \in \eta^{-1}(v) \})$ is acyclic for all $v \in V$.*

We call two arguments $x, y \in X$ to be equivalued *(in F) if $\eta(x) = \eta(y)$.*

The requirement for F_v to be acyclic is also known as the *Multivalued Cycles Assumption*, as it implies that any set of arguments that form a directed cycle in $F = (X, A)$ will contain at least two arguments that are not equivalued [4].

Definition 3. *An* audience *for a value-based system F is a partial ordering $\leq$ on the set V of values of F. An audience $\leq$ is* specific *if it is a total ordering on V.*

For an audience $\leq$ we also define $<$ in the obvious way, i.e., $x < y$ if and only if $x \leq y$ and $x \neq y$.

Definition 4. *Given a value-based system $F = (X, A, V, \eta)$ and an audience $\leq$ for F, we define the abstract argumentation system* induced by $\leq$ from F *as $F_{\leq} = (X, A_{\leq})$ with $A_{\leq} = \{ (x, y) \in A \mid \neg(\eta(x) < \eta(y)) \}$.*

Note that if $\leq$ is a specific audience, then $F_{\leq} = (X, A_{\leq})$ is an acyclic system and thus, as discussed above, has a unique preferred extension $\mathrm{GE}(F_{\leq})$.

Example 2. Consider the value-based system $F = (X, A, V, \eta)$ obtained from the abstract argumentation framework F_0 of Example 1 by adding the set of values $V = \{S, E, T\}$ and the mapping η with $\eta(a) = \eta(b) = S$, $\eta(c) = \eta(d) = E$, $\eta(e) = \eta(f) = T$. The value-based system F is depicted in Figure 1 where the three ellipses indicate arguments that share the same value.

Definition 5. *Let $F = (X, A, V, \eta)$ be a value-based system. We say that an argument $x_1 \in X$ is* subjectively accepted in F *if there exists a specific audience $\leq$ such that x_1 is in the unique preferred extension of $F_{\leq}$. Similarly, we say that an argument $x_1 \in X$ is* objectively accepted in F *if x_1 is contained in the unique preferred extension of $F_{\leq}$ for every specific audience $\leq$.*

Example 3. Consider our running example, the value-based system F given in Example 2. Suppose F represents the interaction of arguments regarding a city development project, and assume the arguments a, b are related to sustainability issues (S), the arguments c, d are related to economics (E), and the arguments e, f are related to traffic issues (T).

Now, consider the specific audience $\leq$ that gives highest priority to sustainability, medium priority to economics, and lowest priority to traffic ($S > E > T$). This audience gives rise to the acyclic abstract argumentation system $F_{\leq}$ obtained from F by deleting the attack (d, b) (as $\eta(b) = S > E = \eta(d)$, d cannot attack b with respect to the audience) and deleting the attack (f, c) (as $\eta(c) = E > T = \eta(f)$, f cannot attack c with respect to the audience).

Figure 2 exhibits the acyclic abstract argumentation systems induced by the six possible specific audiences. The unique preferred extension for each of the six systems is indicated by shaded nodes. We conclude that all arguments of F are subjectively accepted, and e, f are the arguments that are objectively accepted.

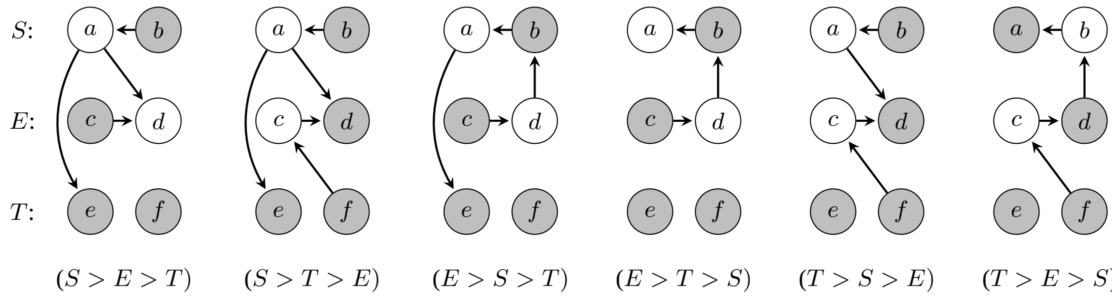

Figure 2: Acyclic abstract argumentation system relative to the six specific audiences on values T, S, E.

2.3 Computational problems for value-based systems

We consider the following decision problems.

SUBJECTIVE ACCEPTANCE

Instance: A value-based system $F = (X, A, V, \eta)$ and a query argument $x_1 \in X$.

Question: Is x_1 subjectively accepted in F?

OBJECTIVE ACCEPTANCE

Instance: A value-based system $F = (X, A, V, \eta)$ and a query argument $x_1 \in X$.

Question: Is x_1 objectively accepted in F?

As shown by Dunne and Bench-Capon [15], SUBJECTIVE ACCEPTANCE is NP-complete and OBJECTIVE ACCEPTANCE is co-NP-complete. Indeed, there are $k!$ possible specific audiences for a value-based system with k values. Hence, even if k is moderately small, say $k = 10$, checking all $k!$ induced abstract argumentation system becomes impractical. Dunne [14] studied properties of value-based systems that allow to reduce the number of audiences to consider.

2.4 Graphical models of value-based systems

In view of the general intractability of SUBJECTIVE ACCEPTANCE and OBJECTIVE ACCEPTANCE, the main decision problems for value-based systems, it is natural to ask which restrictions on shape and structure of value-based systems allow tractability.

A natural approach is to impose structural restrictions in terms of certain graphical models associated with value-based systems. We present three graphical models: the *graph structure* (an undirected graph on the arguments of the value-based system under consideration, edges represent attacks) the *value graph* (a directed graph on the values of the value-based system under consideration, edges represent attacks) and the *extended graph structure* (an undirected graph on the arguments of the value-based system under consideration, edges represent attacks and "equivaluedness"). The concept of value graphs was recently introduced and studied by Dunne [14]. The concept of *extended graph structures* is our new contribution.

Definition 6. *Let $F = (X, A, V, \eta)$ be a value-based system.*

The graph structure *of F is the (undirected) graph $G_F = (X, E)$ whose vertices are the arguments of F and where two arguments x, y are joined by an edge (in symbols $xy \in E$) if and only if X contains the attack (x, y) or the attack (y, x).*

The value graph *of F is the directed graph $G_F^{\text{val}} = (V, E)$ whose vertices are the values of F and where two values u, v are joined by a directed edge from u to v (in symbols $(u, v) \in E$) if and only if there exist some argument $x \in X$ with $\eta(x) = u$, some argument $y \in X$ with $\eta(y) = v$, and $(x, y) \in A$.*

The extended graph structure *of F is the (undirected) graph $G_F^{\text{ext}} = (X, E)$ whose vertices are the arguments of F and where two arguments x, y are joined by an edge if and only if $(x, y) \in A$ or $\eta(x) = \eta(y)$.*

Figure 3 shows the value-based system of Example 2 and the three associated graphical models.

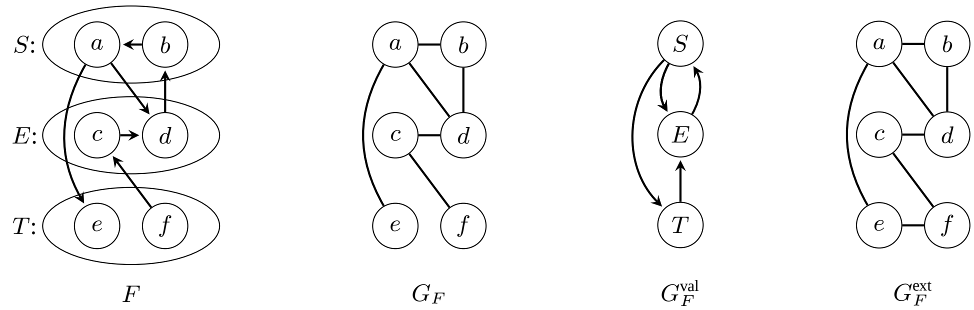

Figure 3: A value-based system F with its graph structure G_F, value graph G_F^{val}, and extended graph structure G_F^{ext}.

Definition 7. *A value-based system $F = (X, A, V, \eta)$ is called* bipartite *if its graph structure is a bipartite graph, i.e., if X can be partitioned into two conflict-free sets.*

3 Value-width and attack-width

Dunne [13] suggested to consider restrictions on the number of arguments that share the same value, and the number of attacks between equivalued arguments. We state these restrictions in terms of the following notions.

Definition 8. *Let $F = (X, A, V, \eta)$ be a value-based system. The* value-width *of F is the largest number of arguments that share the same value, i.e., $\max_{v \in V} |\eta^{-1}(v)|$. The* attack-width *of F is the cardinality of the set $\{ (x, y) \in A \mid \eta(x) = \eta(y) \}$.*

For instance, the value-based system of Example 2 has value-width 2 and attack-width 2.

Value-based systems of value-width 1 are not very interesting: Every argument x in such a value-based system is subjectively accepted (x is accepted with respect to any specific audience where $\eta(x)$ is largest), and objectively accepted if and only if x is not attacked by any argument y (if y attacks x then x is not accepted with respect to any specific audience where $\eta(y)$ is largest). Thus, for value-based systems of value-width 1 the problems SUBJECTIVE and OBJECTIVE ACCEPTANCE are trivial, and the expressive power of such value-based systems is very limited.

On the other hand, *value-based systems of value-width 3* are already too expressive to allow a tractable acceptance decision: Dunne [13] showed that the problems SUBJECTIVE and OBJECTIVE ACCEPTANCE are intractable (NP-complete and co-NP-complete, respectively) for value-based systems of value-width 3, even if their graph structure is a tree.

This leaves the intermediate class of *value-based systems of value-width 2* as an interesting candidate for a tractable class. In fact, Dunne [13] conjectured that both acceptance problems are polynomial-time decidable for value-based systems of value-width 2. He also conjectured that the problems are polynomial for value-based systems with an attack-width that is bounded by a constant. We disprove both conjectures and show that the problems remain intractable for value-based systems of value-width 2 and (simultaneously) of attack-width 1.

On the positive side, we show that under the additional assumption that the value-based system is bipartite (that entails value-based systems whose graph structures are trees) both acceptance problems can be decided in polynomial time for value-based systems of value-width 2.

Theorem 1. (A) SUBJECTIVE ACCEPTANCE remains NP-hard for value-based systems of value-width 2 and attack-width 1. (B) OBJECTIVE ACCEPTANCE remains co-NP-hard for value-based systems of value-width 2 and attack-width 1.

Theorem 2. (A) SUBJECTIVE ACCEPTANCE can be decided in polynomial time for bipartite value-based systems of value-width 2. (B) OBJECTIVE ACCEPTANCE can be decided in polynomial time for bipartite value-based systems of value-width 2.

In the remainder of this section we will demonstrate the two theorems.

3.1 Certifying paths

The key to the proofs of Theorems 1 and 2 is the notion of a "certifying path" which defines a certain path-like substructure within a value-based system. We show that in value-based systems of value-width 2, the problems of SUBJECTIVE and OBJECTIVE ACCEPTANCE can be expressed in terms of certifying paths. We then show that in general finding a certifying path in a value-based system of value-width 2 is NP-hard (3SAT can be expressed in terms of certifying paths) but is easy if the system is bipartite.

Definition 9. *Let $F = (X, A, V, \eta)$ be a value-based system of value-width 2. We call an odd-length sequence $C = (x_1, z_1, \ldots, x_k, z_k, t)$, $k \geq 0$, of distinct arguments a* certifying path for $x_1 \in X$ in F *if it satisfies the following conditions:*

- *C1 For every $1 \leq i \leq k$ it holds that $\eta(z_i) = \eta(x_i)$.*
- *C2 For every $1 \leq i \leq k$ there exists a $1 \leq j \leq i$ such that z_i attacks x_j.*
- *C3 For every $2 \leq i \leq k$ it holds that x_i attacks z_{i-1} but x_i does not attack any argument in $\{z_i, x_1, \ldots, x_{i-1}\}$.*
- *C4 Argument t attacks z_k but it does not attack any argument in $\{x_1, \ldots, x_k\}$.*
- *C5 If there exists an argument $z \in X \setminus \{t\}$ with $\eta(z) = \eta(t)$ then either t attacks z or z does not attack any argument in $\{x_1, \ldots, x_k, t\}$.*

Lemma 1. *Let $F = (X, A, V, \eta)$ be a value-based system of value-width 2 and $x_1 \in X$. Then x_1 is subjectively accepted in F if and only if there exists a certifying path for x_1 in F.*

The rather technical proof of this lemma is given in the appendix. We discuss the intuition behind the concept of certifying paths by means of an example.

Example 4. Consider the value-based system F of Example 2. We want to check whether argument a is subjectively accepted, i.e., to identify a specific audience $\leq$ such that a is in the unique preferred extension $\mathrm{GE}(F_{\leq})$ of $F_{\leq}$. Since a is attacked by b and we cannot eliminate this attack (a and b are equivalued), we need to defend a by attacking b. The only possibility for that is to attack b by d. Hence we need to put $S < E$ in our audience. However, since d is attacked by the equivalued argument c, we need to defend it by attacking c by f, hence we need to put $S < E < T$. Since f is not attacked by any other argument we can stop. Via this process we have produced a certifying path $C_a = (a, b, d, c, f)$, and we can check that C_a indeed satisfies Definition 9. For the other subjectively accepted arguments of F we have the certifying paths $C_b = (b)$, $C_c = (c)$, $C_d = (d, c, f)$, $C_e = (e)$ and $C_f = (f)$.

In order to use the concept of certifying paths for objective acceptance, we need the following definition.

Definition 10. *Let $F = (X, A, V, \eta)$ be a value-based system and $v \in V$ a value. We denote by $F - v$ the value-based system obtained from F by deleting all arguments with value v and all attacks involving these arguments.*

Lemma 2. *Let $F = (X, A, V, \eta)$ be a value-based system of value-width 2 and $x_1 \in X$. Then x_1 is objectively accepted in F if and only if for every argument $p \in X$ that attacks x_1 it holds that $\eta(p) \neq \eta(x_1)$ and p is* not *subjectively accepted in $F - \eta(x_1)$.*

Again, the technical proof is moved to the appendix.

Example 5. In our example, consider the argument e. We want to check whether e is objectively accepted. Since e is only attacked by a, and since $\eta(a) \neq \eta(e)$, it remains to check whether a is not subjectively accepted in $F - \eta(e)$. In fact, $F - \eta(e)$ contains no certifying path for a. Hence e is objectively accepted in F.

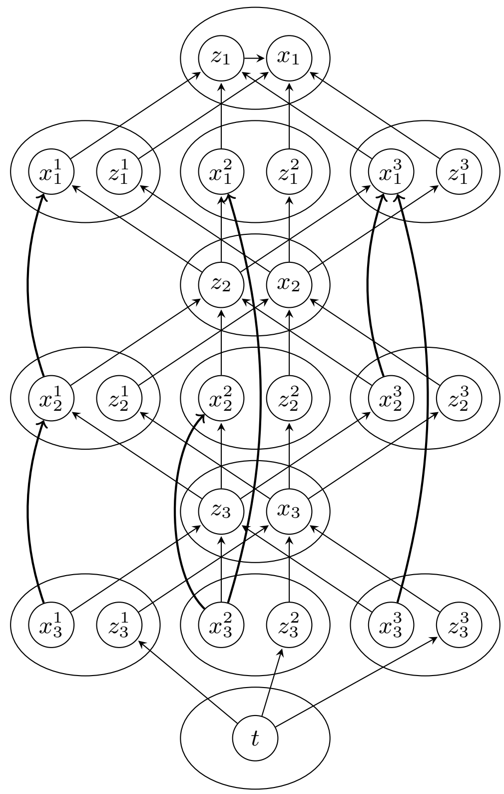

Figure 4: The value-based system F in the proof of Theorem 1 for the 3CNF Formula $(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$.

3.2 Hardness for value-based systems of value-width 2

This subsection is devoted to prove Theorem 1. We devise a polynomial reduction from 3SAT. Let Φ be a 3CNF formula with clauses $C_1, \dots, C_m$ and $C_j = x_{j,1} \vee x_{j,2} \vee x_{j,3}$ for $1 \leq j \leq m$. In the following we construct a value-based system $F = (X, A, V, \eta)$ of value-width 2 and attack-width 1 such that the query argument $x_1 \in X$ is subjectively accepted in F if and only if Φ is satisfiable. See Figure 4 for an example.

The set X contains the following arguments:

1. a pair of arguments x_j, z_j for $1 \leq j \leq m$;
2. a pair of arguments x_j^i, z_j^i for $1 \leq j \leq m$ and $1 \leq i \leq 3$;
3. an argument t.

The set A contains the following attacks:

1. (z_1, x_1);
2. (x_j^i, z_j) and (z_j^i, x_j) for $1 \leq j \leq m$ and $1 \leq i \leq 3$;
3. (x_{j+1}, z_j^i) and (z_{j+1}, x_j^i) for $1 \leq j \leq m-1$ and $1 \leq i \leq 3$;
4. (t, z_m^i) for $1 \leq i \leq 3$;
5. $(x_j^i, x_{j'}^{i'})$ for $1 \leq j' < j \leq m$ and $1 \leq i, i' \leq 3$ whenever $x_{j,i}$ and $x_{j',i'}$ are complementary literals.

The set V contains one value for each x, z pair, and one value for argument t, i.e., $|V| = 4m + 1$. Consequently, the mapping η is defined such that $\eta(x_j) = \eta(z_j) = v_j$, $\eta(x_j^i) = \eta(z_j^i) = v_j^i$ for $1 \leq j \leq m$, $1 \leq i \leq 3$, and

$\eta(t) = v_t$. Evidently F has attack-width 1 and value-width 2, and it is clear that F can be constructed from Φ in polynomial time.

We establish part (A) of Theorem 1 by showing the following claim.

Claim 1. *Φ is satisfiable if and only if x_1 is subjectively accepted in F.*

Proof. First we note that every certifying path for x_1 in F must have the form $(x_1, z_1, x_1^{i_1}, z_1^{i_1}, x_2, z_2, x_2^{i_2}, z_2^{i_2}, x_3, z_3, \dots, x_m, z_m, x_m^{i_m}, z_m^{i_m}, t)$ where $i_j \in \{1, 2, 3\}$ for every $1 \leq j \leq m$ and for every pair $1 \leq j < j' \leq m$ there is no attack $(x_{j'}^{i_{j'}}, x_j^{i_j}) \in A$. Hence there exists a certifying path for x_1 in F if and only if there exists a set L of literals that corresponds to a satisfying truth assignment of Φ (i.e., L contains a literal of each clause of Φ but does not contain a complementary pair of literals). □

In order to show part (B) of Theorem 1, let F be the value-based system as constructed above and define $F' = (X', A', V', \eta')$ to be the value-based system with

1. $X' := X \cup \{x_0\}$,
2. $A' := A \cup \{(x_1, x_0)\}$,
3. $V' := V \cup \{v_0\}$,
4. $\eta'(x_0) = v_0$ and $\eta'(x) = \eta(x)$ for every $x \in X$.

Part (B) of Theorem 1 follows from the following claim which follows from Claim 1 and Lemma 2.

Claim 2. *Φ is satisfiable if and only if x_0 is not objectively accepted in F'.*

By a slight modification of the above reduction we can also show the following, answering a research question recently posed by Dunne [14]. The detailed argument is given in the appendix.

Corollary 1. SUBJECTIVE *and* OBJECTIVE ACCEPTANCE *remain* NP-*hard and* co-NP-*hard, respectively, for value-based systems whose value graphs are bipartite.*

3.3 Certifying paths and dialogues

Bench-Capon, Doutre, and Dunne [5] developed a general *dialogue framework* that allows to describe the acceptance of arguments in a value-based system in terms of a game, played by two players, the proponent and the opponent. The proponent tries to prove that a certain argument (or a set of arguments) is accepted, the opponent tries to circumvent the proof. An argument is subjectively accepted if the proponent has a winning strategy, that is, she is able to prove the acceptance regardless of her opponent's moves.

In the following we outline a simplified version of the dialogue framework that applies to value-based systems of value-width 2. We will see that certifying paths correspond to winning strategies for the proponent.

Let $F = (X, A, V, \eta)$ be a value-based system of value-width 2. We have two players, the proponent and the opponent, who make moves in turn, at each move asserting a new argument. This produces a sequence $(x_1, y_1, x_2, y_2 \dots)$ of arguments and a set of audiences $\leq$ with $\eta(x_1) = \eta(y_2) < \eta(x_2) = \eta(y_2) < \dots$. The proponent has the first move, where she asserts the query argument x_1 whose subjective acceptance is under consideration. After each move of the proponent, asserting argument x_i, the opponent asserts a new argument $y_i \in X \setminus \{x_1, y_1, \dots, x_{i-1}, y_{i-1}, x_i\}$ which has the same value as x_i but is not attacked by x_i, and attacks some argument asserted by the proponent. If no such argument y_i exists, the proponent has won the game. After each move of the opponent asserting an argument y_i, it is again the proponent's turn to assert a new argument $x_{i+1} \in X \setminus \{x_1, y_1, \dots, x_i, y_i\}$. This argument x_{i+1} must attack the opponent's last argument y_i, but must not attack any argument asserted by the proponent. If no such argument x_{i+1} exists, the proponent has lost the game. Because the value-width of F is assumed to be 2, the opponent has at most one choice for each move. Therefore, the proponent's wining strategy does not need to consider several possibilities for the opponent's counter move. Hence, a winning strategy is not a tree but just a path and can be identified with a sequence $(x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n)$ that corresponds to a play won by the proponent. It is easy to verify that such a sequence is exactly a certifying path.

Example 6. Consider again the value-based system F of Example 2. The proponent wants to prove that argument a is subjectively accepted in F and asserts a with her first move. Now, it is the opponent's turn. He has no other choice but to assert b (the only argument different from a with the same value as a). Now, it is again the proponent's turn. She must assert an argument $x \notin \{a, b\}$ that attacks b but does not attack a. Argument d satisfies this property (it happens that this is the only choice). Next, the opponent asserts c, and the proponent asserts f, and it is again the opponent's turn. The only argument with the same value as f is argument e, but e does not attack any of the arguments in $\{a, d, f\}$. Hence, the proponent wins. The sequence of arguments (a, b, d, c, f) produced by this play is indeed a certifying path for a in F. Hence a is subjectively accepted.

3.4 Polynomial-time algorithm for bipartite value-based systems of value-width 2

In this subsection we prove Theorem 2. Throughout this section, we assume that we are given a bipartite value-based system $F = (X, A, V, \eta)$ together with a query argument x_1. Furthermore, let X_{even} and X_{odd} be the subsets of X containing all arguments x such that the length of a shortest directed path in F from x to x_1 is even and odd, respectively.

Lemma 3. *Let $C = (x_1, z_1, \ldots, x_k, z_k, t)$ be a certifying path for x_1 in F. Then $\{\, x_i \mid 1 \leq i \leq k \,\} \cup \{t\} \subseteq X_{\text{even}}$ and $\{\, z_i \mid 1 \leq i \leq k \,\} \subseteq X_{\text{odd}}$.*

Proof. The claim follows easily via induction on k by using the properties of a certifying path and the fact that F is bipartite. □

Based on the observation of Lemma 3, we construct an auxiliary directed graph $H_F := (V, E)$ as follows. The vertex set of H_F is the set V of values of F. There is a directed edge from u to v if and only if there is an argument $x \in X_{\text{even}}$ with $\eta(x) = u$ and an argument z with $\eta(z) = v$ such that $(x, z) \in A$. Note that $z \in X_{\text{odd}}$ since F is bipartite.

Lemma 4. *If $C = (x_1, z_1, \ldots, x_k, z_k, t)$ is a certifying path for x_1 in F, then $(\eta(t), \eta(x_k), \ldots, \eta(x_1))$ is a directed path from $\eta(t)$ to $\eta(x_1)$ in H_F.*

Proof. By the definition of a certifying path, we have $(t, z_k) \in A$ and for every $2 \leq i \leq k$ it holds that $(x_i, z_{i-1}) \in A$. Lemma 3 implies that for t and x_i are contained in X_{even} for every $1 \leq i \leq k$, and hence $(\eta(t), \eta(x_k)), (\eta(x_i), \eta(x_{i-1})) \in E$ for every $1 < i \leq k$. □

Lemma 4 tells us that each certifying path in F gives rise to a directed path in H_F.

Example 7. Figure 5 shows a bipartite value-based system F and the associated auxiliary graph H_F. The query argument is x_1. Hence $X_{\text{even}} = \{x_1, \ldots, x_5\}$ and $X_{\text{odd}} = \{z_1, \ldots, z_5\}$. The query argument x_1 is subjectively accepted in F as $C = (x_1, z_1, x_2, z_2, x_4, z_4, x_5)$ is a certifying path for x_1 in F. Indeed, C gives rise to the directed path v_5, v_4, v_2, v_1 (i.e., $\eta(x_5), \eta(x_4), \eta(x_2), \eta(x_1)$) in H_F, as promised by Lemma 4.

It would be desirable if we could find certifying paths by searching for directed paths in H_F. However, not every directed path in H_F gives rise to a certifying path in F. To overcome this obstacle, we consider for each value $v \in V$ the subgraph H_F^{-v} of H_F which is obtained as follows:

If there is an argument $z \in X_{\text{odd}} \cap \eta^{-1}(v)$ that is not attacked by some equivalued argument, then for every argument $y \in X_{\text{even}}$ that is attacked by z we remove the vertex $\eta(y)$ from H_F.

Figure 5 shows the graphs H_F^{-v} for the value-based system F of Example 7.

Lemma 5. *Consider an odd-length sequence $C = (x_1, z_1, \ldots, x_k, z_k, t)$ of distinct arguments of a bipartite value-based system F of value width 2. Then C is a certifying path for x_1 in F if and only if the following conditions hold:*

(1) $\eta(x_i) = \eta(z_i)$ *for* $1 \leq i \leq k$.

(2) $(\eta(t), \eta(x_k), \ldots, \eta(x_1))$ *is a directed path from* $\eta(t)$ *to* $\eta(x_1)$ *in* $H_F^{-\eta(t)}$.

(3) None of the sub-sequences $\eta(x_i), \ldots, \eta(x_1)$ *is a directed path from* $\eta(x_i)$ *to* $\eta(x_1)$ *in* $H_F^{-\eta(x_i)}$ *for* $1 \leq i \leq k$.

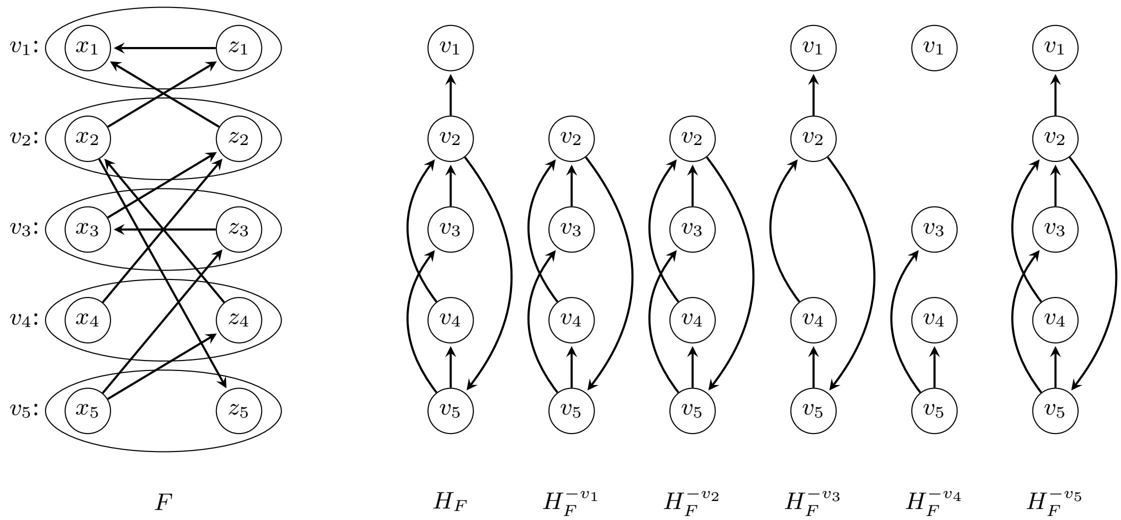

Figure 5: A bipartite value-based system F, the associated axillary graph H_F, and various subgraphs obtained by deleting a value.

Proof. Assume $C = (x_1, z_1, \ldots, x_k, z_k, t)$ is a certifying path for x_1 in F. Property (1) follows from condition C1 of a certifying path, property (2) follows from condition C5 and Lemma 4. Property (3) follows from conditions C2 and C3.

To see the reverse assume that C satisfies properties (1)–(3). Condition C1 follows from property (1). Conditions C3, C4 and C5 follow from property (2) and the assumption that F is bipartite. Condition C2 follows from property (3). Hence C is a certifying path for x_1 in F. □

Indeed, consider the certifying path C of Example 7 which gives rise to the sequence of values v_5, v_4, v_2, v_1. This sequence is a directed path in $H_F^{-v_5}$, however v_4, v_2, v_1 is not a directed path in $H_F^{-v_4}$, v_2, v_1 is not a directed path in $H_F^{-v_2}$, and v_1 is not a directed path in $H_F^{-v_1}$.

Lemma 5 suggests a simple strategy for finding a certifying path for x_1 in F, if one exists. For each value v we search for a directed path $v_k, \ldots, v_1$ from $v = v_k$ to $v_1 = \eta(x_1)$ in H_F^{-v}. If we find such a path $v_k, \ldots, v_1$, we check for each subsequence $v_i, \ldots, v_1$, $1 \leq i < k$, whether it is a directed path in $H_F^{-v_i}$. If the answer is NO for all i, then $v_k, \ldots, v_1$ satisfies the conditions of Lemma 5. Hence the sequence of arguments in X whose values form P is a certifying path for x_1 in F. If, however, the answer is YES for some $i < k$, we take the smallest i for which the answer is YES. Now the sequence $v_i, \ldots, v_1$ satisfies the conditions of Lemma 5 and so gives rise to a certifying path for x_1 in F. On the other hand, if there is no value v such that H_F^{-v} contains a directed path from v to v_1, then there is no certifying path for x_1 in F. The pseudo code for this algorithm is given in Figure 6.

Proposition 1. *The algorithm* DETECT CERTIFYING PATH *correctly returns a certifying path for x_1 in $F = (X, A, V, \eta)$ if one exists and returns NO otherwise in time $O(|V|^2 \cdot (|X| + |A| + |V|))$.*

Proof. The correctness of DETECT CERTIFYING PATH follows from Lemma 5. For $v \in V$, building H_F^{-v} and finding a shortest directed path from v to $v_1 = \eta(x_1)$, if one exists, takes linear time in the input size of F (which we estimate by the term $O(|X| + |A| + |V|)$). As we iterate over all vertices of V, and we check for at most $|V|$ subsequences $v_i, \ldots, v_1$ whether it is a directed path in $H_F^{-v_i}$, the claimed running time follows. □

We are now ready to combine the above results to a proof of Theorem 2. Statement (A) of the theorem follows from Lemma 1 and Proposition 1. Statement (B) follows from Statement (A) and Lemma 2.

Algorithm DETECT CERTIFYING PATH

Input: value-based system $F = (X, A, V, \eta)$, query argument $x_1 \in X$

Output: a directed path in H_F that corresponds to a certifying path for x_1 in F, or NO if there is no certifying path for x_1 in F

```
for all v ∈ V do
    check if H_F^{-v} contains a directed path from v to v_1
    if yes do
        find such a path, v_k, ..., v_1, v = v_k
        for i = 1, ..., k do
          check if v_i, ..., v_1 is a directed path in H_F^{-v_i}
          if yes, output v_i, ..., v_1 and terminate
return NO and terminate
```

Figure 6: Polynomial-time algorithm for the detection of a certifying path in a bipartite value-based system of value-width 2.

4 Linear-time algorithm for value-based systems of bounded treewidth

As mentioned above, it is known that both acceptance problems remain intractable for value-based systems whose graph structure is a tree. This is perhaps not surprising since two arguments can be considered as linked to each other if they share the same value. In fact, such links may form cycles in an otherwise tree-shaped value-based system. Therefore we propose to consider the *extended graph structure* of the value-based system (recall Definition 6 in Section 2.4) that takes such links into account. We show that the problems SUBJECTIVE and OBJECTIVE ACCEPTANCE are easy for value-based systems whose extended graph structure is a tree, and more generally, the problems can be solved in *linear-time* for value-based systems with an extended graph structure of *bounded treewidth*.

Treewidth is a popular graph parameter that indicates in a certain sense how similar a graph is to a tree. Many otherwise intractable graph problems (such as 3-COLORABILITY) become tractable for graphs of bounded treewidth. Bounded treewidth (and related concepts like *induced width* and *d-tree width*) have been successfully applied in many areas of AI, see, e.g., [18, 11, 10, 20]. Deciding acceptance for argumentation frameworks of bounded treewidth has been investigated by Dunne [13] and by Dvorák, Pichler, and Woltran [16]. However, for value-based argumentation, the concept of bounded tree-width has not been applied successfully: the basic decision problems for value-based systems remain intractable for value-based systems of value width 3 whose graph structure has treewidth 1 [13]. Hardness even prevails for value-based systems whose value graph has pathwidth 2 [14]. These negative results are contrasted by our Theorem 3, which indicates that the extended graph structure seems to be a suitable and adequate graphical model for value-based systems.

The treewidth of a graph is defined using the following notion of a tree decomposition (see, e.g., [7]).

Definition 11. *A* tree decomposition *of an (undirected) graph $G = (V, E)$ is a pair (T, χ) where T is a tree and χ is a labeling function that assigns each tree node t a set $\chi(t)$ of vertices of the graph G such that the following conditions hold:*

1. *Every vertex of G occurs in $\chi(t)$ for some tree node t.*
2. *For every edge $\{u, v\}$ of G there is a tree node t such that $u, v \in \chi(t)$.*
3. *For every vertex v of G, the tree nodes t with $v \in \chi(t)$ form a connected subtree of T.*

The width *of a tree decomposition (T, χ) is the size of a largest bag $\chi(t)$ minus 1 among all nodes t of T. A tree decomposition of smallest width is* optimal*. The* treewidth *of a graph G is the width of an optimal tree decomposition of G.*

Example 8. Figure 7 exhibits a graph (the extended graph structure of the value-based system of Example 2) and a tree decomposition of it. The width of the tree decomposition is 2, and it is not difficult to see that this is optimal. Hence the treewidth of the graph in the figure is 2.

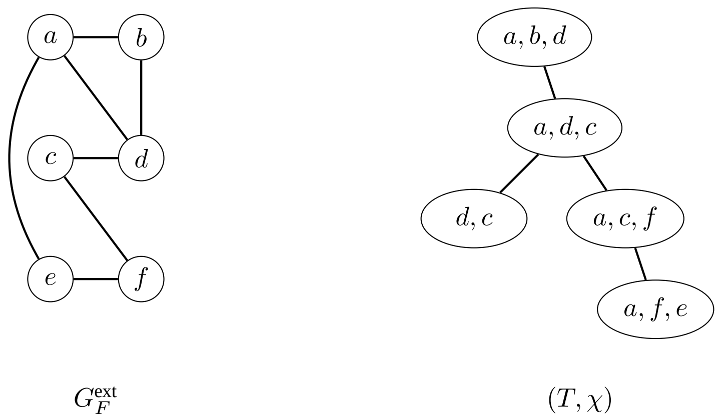

Figure 7: A graph and its tree decomposition.

4.1 Linear-time tractability for value-based systems with extended graph structures of bounded treewidth

We are going to establish the following result.

Theorem 3. *The problems* SUBJECTIVE *and* OBJECTIVE ACCEPTANCE *can be decided in linear time for value-based systems whose extended graph structure has bounded treewidth.*

To achieve tractability we have to pay a price in generality: The mentioned hardness results of [14, 13] imply that if SUBJECTIVE ACCEPTANCE is fixed-parameter tractable for any parameter p, then, unless P = NP, parameter p cannot be bounded by a function of any of the following three parameters: the treewidth of the graph structure, the treewidth of the value graph, and the value-width. This even holds if the bounding function is exponential. Indeed, the treewidth of the extended graph structure can be arbitrarily large for value-based systems where one of these three parameters is bounded by a constant.

The reminder of this section is devoted to a proof of Theorem 3. We shall take a logic approach and use the celebrated result of Courcelle [9], which states that all properties that can be expressed in a certain formalism (Monadic Second-Order logic, MSO) can be checked in linear time for graphs (or more generally, for finite structures) of bounded treewidth. Courcelle's Theorem is constructive in the sense that it not only promises the existence of an algorithm for the particular problem under consideration, but it provides the means for actually producing such an algorithm. The algorithm produced in this general and generic way leaves much room for improvement and provides the basis for the development of problem-specific and more practical algorithms.

In the following we use Courcelle's result as laid out by Flume and Grohe [17]. Let S denote a finite relational structure and φ a sentence in monadic second-order logic (MSO logic) on S. That is, φ may contain quantification over atoms (elements of the universe) and over sets of atoms. Furthermore, we associate with the structure S its *Gaifman graph* $G(S)$, whose vertices are the atoms of S, and where two distinct vertices are joined by an edge if and only if they occur together in some tuple of a relation of S. We define the *treewidth of structure* S as the treewidth of its Gaifman graph $G(S)$. Now Courcelle's theorem states that for a fixed MSO sentence φ and a fixed integer k, one can check in linear time whether φ holds for a given relational structure of treewidth at most k. The proof of Theorem 3 boils down to the following two tasks:

Task A. To represent a value-based system F and a query argument x_1 by a relational structure $S[F, x_1]$ such that bounded treewidth of the extended graph structure of F implies bounded treewidth of $S[F, x_1]$.

Task B. To construct formulas φ_s and φ_o in MSO logic such that for every value-based system F and every argument x_1 of F it holds that φ_s is true for $S[F, x_1]$ if and only if x_1 is subjectively accepted in F, and φ_o is true for $S[F, x_1]$ if and only if x_1 is objectively accepted in F.

4.2 Reference graphs

For many problems it is rather straight-forward to find an MSO formulation so that Courcelle's Theorem can be applied. In our case, however, we have to face the difficulty that we have to express that "a certain property holds for some total ordering" (subjective acceptance) and "a certain property holds for all total orderings" (objective acceptance), which cannot be directly expressed in MSO. Our solution to this problem lies in the introduction of an auxiliary directed graph R, the *reference graph*, which will allow us to quantify over total orderings of V. The relational structure $S[F, x_1]$ will then be defined to represent F together with R.

Definition 12. *Let $F = (X, A, V, \eta)$ be a value-based system and let $\prec$ be an arbitrary but fixed total ordering of V. The* reference graph *$R = (V, E_R)$ is the directed graph where V is the set of values of F and E_R consists of all directed edges (u, v) for which*

1. *$u \prec v$ in the fixed ordering, and*
2. *A contains an attack (x, x') with $\eta(x) = u$ and $\eta(x') = v$ or $\eta(x) = v$ and $\eta(x') = u$.*

For a subset $Q \subseteq E_R$ let $R[Q] = (V, E_R[Q])$ be the directed graph obtained from the reference graph R by reversing all edges in Q, i.e., $E_R[Q] := \{ (u, v) \mid (u, v) \in E_R \setminus Q) \} \cup \{ (v, u) \mid (u, v) \in E_R \cap Q \}$.

We also define the abstract argumentation system $F[Q] := (X, A[Q])$ as the system obtained from F with $A[Q] := \{ (u, v) \in A \mid (\eta(u), \eta(v)) \notin E_R[Q]) \}$.

Note that the reference graph R is by definition acyclic (in contrast to the value graph G_F^{val} whose definition is similar but distinct).

Every specific audience $\leq$ of F can now be represented by some subset $Q \subseteq E_R$ for which the directed graph $R[Q]$ is acyclic, and conversely, every set $Q \subseteq E_R$ such that $R[Q]$ is acyclic represents a specific audience $\leq$. These observations are made precise in the following lemma whose easy proof is omitted.

Lemma 6. *An argument x_1 is subjectively accepted in F if and only if there exists a set $Q \subseteq E_R$ such that $R[Q]$ is acyclic and x_1 is in the unique preferred extension of $F[Q]$. An argument x_1 is objectively accepted in F if and only if for every set $Q \subseteq E_R$ such that $R[Q]$ is acyclic it holds that x_1 is in the unique preferred extension of $F[Q]$.*

Since we can test for acyclicity with MSO logic (see the next subsection), we can now express subjective and objective acceptance in MSO logic as "a certain property holds for some subset Q of E_R for which $R[Q]$ is acyclic" and "a certain property holds for all subsets Q of E_R for which $R[Q]$ is acyclic", respectively. Next we give a more detailed description of how to accomplish the two tasks for our proof.

4.3 Task A: representing the value-based system

We define a relational structure $S[F, x_1]$ that represents the value-based system F together with the reference graph $R = (V, E_R)$. The universe of $S[F, x_1]$ is the union of the sets X, V, and E_R. $S[F, x_1]$ has one unary relation U_a^* and four binary relations H, T, B_a and B_η that are defined as follows:

1. $\mathrm{U}_a^*(x)$ if and only if $x = x_1$ (used to "mark" the query argument).
2. $\mathrm{T}(t, (u, v))$ if and only if $t = u$ (used to represent the "tail relation" of E_R)
3. $\mathrm{H}(h, (u, v))$ if and only if $h = v$ (used to represent the "head relation" of E_R)
4. $\mathrm{B}_a(x, y)$ if and only if $(x, y) \in A$ (used to represent the attack relation).
5. $\mathrm{B}_\eta(x, v)$ if and only if $\eta(x) = v$ (used to represent the mapping η).

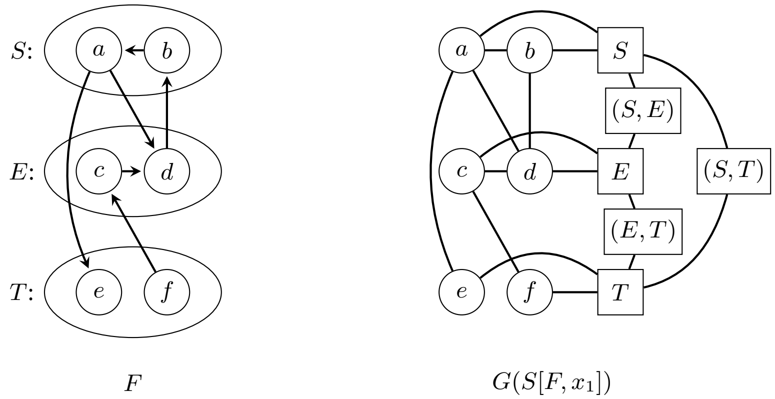

Figure 8: Value-based system F of Example 2 and the corresponding Gaifmann graph $G(S[F, x_1])$.

Consequently, the Gaifman graph of $S[F, x_1]$ is the graph $G(S[F, x_1]) = (V_{S[F,x_1]}, E_{S[F,x_1]})$ with $V_{S_F} = X \cup V \cup E_R$ and $E_{S_F} = \{ \{u, v\} \mid (u, v) \in \mathrm{T} \cup \mathrm{H} \cup \mathrm{B}_a \cup \mathrm{B}_\eta \}$, see Figure 8 for an illustration.

Lemma 7. *The treewidth of $S[F, x_1]$ is at most twice the treewidth of the extended graphs structure of F plus* 1.

The easy proof is given in the appendix.

4.4 Task B: expressing acceptance in MSO

In order to define φ_s and φ_o we introduce the following auxiliary formulas:

A formula $\mathrm{TH}(t, h, a)$ that holds if and only if t is the tail and h is the head of $a \in E_R$:

$$\mathrm{TH}(t, h, a) := \mathrm{T}(t, a) \wedge \mathrm{H}(h, a)$$

A formula $\mathrm{E}(t, h, Q)$ that holds if and only if the directed edge (t, h) is contained in $R[Q]$:

$$\mathrm{E}(t, h, Q) := \exists a\ [(\neg Qa \wedge \mathrm{TH}(t, h, a)) \vee (Qa \wedge \mathrm{TH}(h, t, a))]$$

A formula $\mathrm{ACYC}(Q)$ that checks whether $R[Q]$ is acyclic. We use the well-known fact that a directed graph contains a directed cycle if and only if there is a nonempty set C of vertices each having an out-neighbor in C (see, e.g., [1]).

$$\mathrm{ACYC}(Q) := \neg\exists C\ (\exists x C x \wedge \forall t \exists h [Ct \rightarrow (Ch \wedge \mathrm{E}(t, h, Q))])$$

A formula $\mathrm{B}'_a(t, h, Q)$ that holds if and only if t attacks h in $F[Q]$:

$$\mathrm{B}'_a(t, h, Q) := \mathrm{B}_a(t, h) \wedge \exists v_h \exists v_t\ [\mathrm{B}_\eta(t, v_t) \wedge \mathrm{B}_\eta(h, v_h) \wedge \neg\mathrm{E}(v_h, v_t, Q)]$$

A formula $\mathrm{ADM}(S, Q)$ that checks whether a set $S \subseteq X$ is admissible in $F[Q]$:

$$\mathrm{ADM}(S, Q) := \forall x \forall y\ [(\mathrm{B}'_a(x, y, Q) \wedge Sy) \rightarrow (\neg Sx \wedge \exists z (Sz \wedge \mathrm{B}'_a(z, x, Q)))]$$

Now the formula φ_s can be defined as follows:

$$\varphi_s := \exists Q\ [\mathrm{ACYC}(Q) \wedge (\exists S (\forall x (\mathrm{U}^*_a(x) \rightarrow Sx) \wedge \mathrm{ADM}(S, Q)))]$$

It follows from Lemma 6 that φ_s is true for $S[F, x_1]$ if and only if x_1 is subjectively accepted in F. A trivial modification of φ_s gives us the desired sentence φ_o:

$$\varphi_o := \forall Q\ [\mathrm{ACYC}(Q) \rightarrow (\exists S (\forall x (\mathrm{U}^*_a(x) \rightarrow Sx) \wedge \mathrm{ADM}(S, Q)))]$$

It follows from Lemma 6 that φ_o is true for $S[F, x_1]$ if and only if x_1 is objectively accepted in F.

We summarize the above construction in the next lemma.

Lemma 8. *There exists an MSO sentence φ_s such that φ_s is true for $S[F, x_1]$ if and only if x_1 is subjectively accepted in F. Similarly, there exists an MSO sentence φ_o such that φ_o is true for $S[F, x_1]$ if and only if x_1 is objectively accepted in F.*

In view of Lemmas 7 and 8, Theorem 3 now follows by Courcelle's Theorem.

If both the treewidth of the value graph and the value-width of an value-based system are bounded, then also the extended graph structure has bounded treewidth, hence we have the following corollary.

Corollary 2. *The problems* SUBJECTIVE *and* OBJECTIVE ACCEPTANCE *can be decided in linear time for value-based systems for which both value-width and the treewidth of their value graphs are bounded.*

Proof. Let k and k' be constants. Let (T, χ) be a tree decomposition of the value graph of a value-based system F of width k and assume the value-width of F is k'. Then (T, χ') with $\chi'(t) = \bigcup_{v \in \chi(t)} \eta^{-1}(v)$ is a tree decomposition of the extended graph structure of F. Since $|\chi'(t)| \leq |\chi(t)| \cdot k' \leq (k+1)k'$ holds for all nodes t of T, it follows that the width of (T, χ') is bounded by the constant $k'' = (k+1)k' - 1$. We conclude, in view of Theorem 3, that we can decide both acceptance problems for F in linear time. ☐

5 Conclusion

We have studied the computational complexity of persuasive argumentation for value-based argumentation frameworks under structural restrictions. We have established the intractability of deciding subjective or objective acceptance for value-based systems with value-width 2 and attack-width 1, disproving conjectures stated by Dunne. It might be interesting to note that our reductions show that intractability even holds if the attack relation of the value-based system under consideration forms a directed acyclic graph. On the positive side we have shown that value-based systems with value-width 2 whose graph structure is bipartite are solvable in polynomial time. These results establish a sharp boundary between tractability and intractability of persuasive argumentation for value-based systems with value-width 2. Furthermore we have introduced the notion of the *extended graph structure* of a value-based system and have shown that subjective and objective acceptance can be decided in linear-time if the treewidth of the extended graph structure is bounded (that is, the problems are *fixed-parameter tractable* when parameterized by the treewidth of the extended graph structure). This is in strong contrast to the intractability of the problems for value-based systems where the treewidth of the graph structure or the treewidth of their value graph is bounded. Therefore we conclude that the extended graph structure seems to be an appropriate graphical model for studying the computational complexity of persuasive argumentation. It might be interesting for future work to extend this study to other graph-theoretic properties or parameters of the extended graph structure.

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Appendix: Technical proofs

Proof of Lemma 1. Let $C = (x_1, z_1, \ldots, x_k, z_k, t)$ be a certifying path for x_1 in F. Take a specific audience $\leq$ such that $\eta(x_1) < \cdots < \eta(x_k) < \eta(t)$ and all other values in V are smaller than $\eta(x_1)$. We claim that the unique preferred extension $P = \mathrm{GE}(F_\leq)$ of $F_\leq$ includes $\{x_1, \ldots, x_k, t\}$ and excludes $\{z_1, \ldots, z_k\}$, which means that x_1 is subjectively accepted in F. It follows from C5 that t is not attacked by any other argument in $F_\leq$ and hence $t \in P$ (see also Section 2 for a description of an algorithm to find the unique preferred extension of an acyclic abstract argumentation system). From C4 it follows that $z_k \notin P$. Furthermore, if there exists an argument $z \neq t$, $\eta(t) = \eta(z)$ then either $(t, z) \in A_\leq$ or z does not attack an argument in $\{x_1, \ldots, x_k, t\}$. In the first case $z \notin P$ and does not influence the membership in P for any other arguments in X. In the second case $z \in P$ but it does not attack any argument in $\{x_1, \ldots, x_k, t\}$. In both cases it follows that $x_k \in P$. Using C3 it follows that $z_{k-1} \notin P$ and since we already know that $z_k \notin P$ it follows that $x_{k-1} \in P$. A repeated application of the above arguments establishes the claim, and hence $x_1 \in P$ follows.

Conversely, suppose that there exists a specific audience $\leq$ such that x_1 is contained in the unique preferred extension $P = \mathrm{GE}(F_\leq)$ of $F_\leq$. We will now construct a certifying path C for x_1 in F. Clearly, if there is no $z_1 \in X \setminus \{x_1\}$ with $\eta(z_1) = \eta(x_1)$ and $(z_1, x_1) \in A$, then (x_1) is a certifying path for x_1 in F. Hence, it remains to consider the case where such a z_1 exists. Since $x_1 \in P$ it follows that $z_1 \notin P$. The sequence (x_1, z_1) clearly satisfies properties C1–C3. We now show that we can always extend such a sequence until we have found a certifying path for x_1 in F. Hence, let $S = (x_1, z_1, \ldots, x_l, z_l)$ be such a sequence satisfying conditions C1–C3, and in addition assume S satisfies the following two conditions:

S1 It holds that $\eta(x_1) < \cdots < \eta(x_l)$.

S2 For every $1 \leq i \leq l$ we have $x_i \in P$ and $z_i \notin P$.

Clearly, the sequence (x_1, z_1) satisfies S1 and S2, hence we can include these conditions in our induction hypothesis. It remains to show how to extend S to a certifying path. Let $Y := \{\, y \in P \mid (y, z_l) \in A \wedge \eta(y) > \eta(x_l) = \eta(z_l) \,\}$. Then $Y \neq \emptyset$ because $z_l \notin P$ by condition S2 and the assumption that P is a preferred extension.

For each $y \in Y$ let $C_y = (x_1, z_1, \ldots, x_l, z_l, y)$. If there is an argument $y \in Y$ such that C_y is a certifying path for x_1 in F we are done. Hence assume there is no such $y \in Y$.

We choose $x_{l+1} \in Y$ arbitrarily. Note that $C_{x_{l+1}}$ satisfies the condition C4; $(x_{l+1}, z_l) \in A$ (as $x_{l+1} \in Y$) and $(x_{l+1}, x_i) \notin A$ for $1 \leq i \leq l$ (as $x_{l+1}, x_i \in P$ and P is conflict-free). Since we assume that $C_{x_{l+1}}$ is not a certifying path, $C_{x_{l+1}}$ must violate C5.

It follows that there exists some argument z_{l+1} with $\eta(z_{l+1}) = \eta(x_{l+1})$ such that $(x_{l+1}, z_{l+1}) \notin A$ and $(z_{l+1}, x_i) \in A$ for some $1 \leq i \leq l+1$. We conclude that $S' = (x_1, z_1, \ldots, x_l, z_l, x_{l+1}, z_{l+1})$ satisfies conditions C1–C3 and S1–S2. Hence, we are indeed able to extend S and will eventually obtain a certifying path for x_1 in F. □

Proof of Lemma 2. Assume that x_1 is objectively accepted in F. Suppose there is a $p \in X$ that attacks x_1 and $\eta(p) = \eta(x_1)$. If we take a specific audience $\leq$ where $\eta(x_1)$ is the greatest element, then x_1 is not in the unique preferred extension of $F_\leq$, a contradiction to the assumption that x_1 is objectively accepted. Hence $\eta(p) \neq \eta(x_1)$ for all arguments $p \in X$ that attack x_1. Next suppose there is an argument $p \in X$ that attacks x_1 and is subjectively accepted in $F - \eta(x_1)$. Let $\leq$ be a specific audience such that p is in the unique preferred extension of $(F - \eta(x_1))_\leq$. We extend $\leq$ to a total ordering of V ensuring $\eta(x_1) \leq \eta(p)$. Clearly x_1 is not in the unique preferred extension of $F_\leq$, again a contradiction. Hence indeed for all $p \in X$ that attack x_1 we have $\eta(p) \neq \eta(x_1)$ and p is not subjectively accepted in $F - \eta(x_1)$

We establish the reverse direction by proving its counter positive. Assume that x_1 is not objectively accepted in F. We show that there exists some $p \in X$ that attacks x_1 and where either $\eta(p) = \eta(x_1)$ or p is subjectively accepted in $F - \eta(x_1)$. Let $\leq$ be a specific audience of F such that x_1 is not in the unique preferred extension $P = \mathrm{GE}(F_\leq)$ of $F_\leq$. In view of the labeling procedure for finding P as sketched in Section 2, it follows that there exists some $p \in P$ that attacks x_1 with $\eta(x_1) \leq \eta(p)$. If $\eta(x_1) = \eta(p)$ then we are done. On the other hand, if $\eta(p) \neq \eta(x_1)$, then p is in the unique preferred extension of $(F - \eta(x_1))_\leq$, and so p is subjectively accepted in $F - \eta(x_1)$. □

Proof of Corollary 1. We slightly modify the reduction from 3SAT as given in Section 3.2. Let $C_1, \ldots, C_m$ be the clauses of the 3CNF formula Φ. It is well-known that 3SAT remains NP-hard for formulas where each clause is either positive (all three literals are unnegated variables) or negative (all three literals are negated variables), see [19]. Hence we may assume that for some $2 \leq k \leq m$, $C_1, \ldots, C_k$ are positive clauses and $C_{k+1}, \ldots, C_m$ are negative clauses. Let F and F' be the two value-based systems corresponding to Φ as constructed in Section 3.2. We obtain from F the value-based system F_B by adding a new pair of arguments x_B, y_B with a new value $v_B = \eta(x_B) = \eta(y_B)$ and inserting the pair between the pairs x_k^i, z_k^i and the pair x_{k+1}, z_{k+1}. That is, for $1 \leq i \leq 3$ we replace the attacks (x_{k+1}, z_k^i) and (z_{k+1}, x_k^i) with the attacks (x_B, z_k^i) and (z_B, x_k^i), and we add the attacks (x_{k+1}, z_B), (z_{k+1}, x_B). By the same modification we obtain from F' the value-based system F'_B. Clearly Claims 1 and 2 still hold for the modified value-based systems, i.e., Φ is satisfiable if and only if x_1 is subjectively accepted in F, and Φ is satisfiable if and only if x_1 is not objectively accepted in F'.

In order to establish the corollary it remains to show that the value graphs of F_B and F'_B are bipartite.

We partition the set of arguments into two sets X_0 and X_1. X_0 contains the values v_j for $j \leq k$, the value v_B, and the values v_j^i for $j > k$. X_1 contains the values v_j for $j > k$, the values v_j^i for $j \leq k$, and the value v_t. For F'_B, X_1 contains also the value v_0. It is easy to check that there is no attack (a, b) with $\eta(a), \eta(b) \in X_0$ or $\eta(a), \eta(b) \in X_1$, hence F_B and F'_B have bipartite value graphs. ☐

Proof of Lemma 7. Let G' be the graph obtained from $G(S[F, x_1]) = (V_{S[F,x_1]}, E_{S[F,x_1]})$ by replacing every path of the form $(t, (t, h), h)$ for $t, h \in V$ by an edge $\{t, h\}$; i.e., $G' = (V', E')$ where $V' = X \cup V$ and $E' = (E_{S[F,x_1]} \cap \{ \{u, v\} \mid u, v \in X \cup V \}) \cup \{ \{t, h\} \mid (t, h) \in E_R \}$. Conversely one can obtain $G(S[F, x_1])$ from G' by subdividing all edges of the form $\{t, h\}$ for $t, h \in V$ and $(t, h) \in E_R$ with a vertex (t, h). However, subdividing edges does not change the treewidth of a graph [7], hence it suffices to show that the treewidth of G' is at most twice the treewidth of the extended graph structure of F plus 1. Let $\mathcal{T} = (T, \chi)$ be a tree decomposition of the extended graph structure of F. We observe that $\mathcal{T}' = (T, \chi')$ where $\chi'(t) = \chi(t) \cup \{ \eta(v) \mid v \in X \cap \chi(t) \}$ is a tree decomposition of G' where $|\chi'(t)| \leq 2 \cdot |\chi(t)|$ for all nodes t of T; hence the width of $\mathcal{T}'$ is at most twice the width of $\mathcal{T}$ plus 1. ☐