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On a New Idiom in the Study of Entailment

R. E. Jennings, Y. Chen and J. Sahasrabudhe

Abstract. This paper is an experiment in Leibnizian analysis. The reader will recall that Leibniz considered all true sentences to be analytically so. The difference, on his account, between necessary and contingent truths is that sentences reporting the former are finitely analytic; those reporting the latter require infinite analysis of which God alone is capable. On such a view at least two competing conceptions of entailment emerge. According to one, a sentence entails another when the set of atomic requirements for the first is included in the corresponding set for the other; according to the other conception, every atomic requirement of the entailed sentence is underwritten by an atomic constituent of the entailing one. The former conception is classical on the twentieth century understanding of the term; the latter is the one we explore here. Now if we restrict ourselves to the formal language of the propositional calculus, every sentence has a finite analysis into its conjunctive normal form. Semantically, then, every sentence of that language can be represented as a simple hypergraph, H, on the powerset of a universe of states. Entailment of the sort we wish to study can be represented as a known relation, subsumption between hypergraphs. Since the lattice of hypergraphs thus ordered is a DeMorgan lattice, the logic of entailment thus understood is the familiar system, FDE of first-degree entailment. We observe that, extensionalized, the relation of subsumption is itself a DeMorgan Lattice ordered by higher-order subsumption. Thus the semantic idiom that hypergraph-theory affords reveals a hierarchy of lattices capable of representing entailments of every finite degree.

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1. Leibnizian Analysis

The work of this paper is prompted by a thought experiment in Leibnizian analysis. Leibniz wrote, though seldom in categorical sentences, as though all significant sentences are categorical. He also supposed that all sentences, whether necessary or contingent, are analytic, necessary sentences finitely so, contingent sentences infinitely so. 'When the analysis of a necessary proposition is continued far enough it arrives at an identical equation; that's what it is to demonstrate a truth with geometrical rigour. But the analysis of a contingent proposition continues to infinity, giving reasons (and reasons for the reasons (and reasons for those reasons...)), so that one never has a complete demonstration. There is always an underlying complete and final reason for the truth of the proposition, but only God completely grasps it, he being the only one who can whip through the infinite series in one stroke of the mind.' [1]

By drawing the distinction between necessary and contingent truths, Leibniz suggested a deeper division between mathematical and logical truths on the one hand, and those comes to what we now regard as the whole body of scientific truths on the other. According to Leibniz, all truths are analytic in the sense that we are to understand them as sequences of reductions to identities. And in the duration, so to speak, of this process of reduction lie all the differences between the necessary and the contingent. The former enjoys a finite demonstration which the latter can never hope to achieve. And we know, at the end of the finite sequence, that we have accomplished our goal, i.e. that we are in possession of a set of identities, by the law of non-contradiction, which guarantees that the negation of any identity leads to absurdity. This law does not apply to scientific truths, as the principle of sufficient reason enjoys no privileged position in logic or mathematics. A meaningful reduction such as can be achieved by human beings, therefore, is confined to logic,[1] or 'reasoning from absolute necessities' as Leibniz called it. The law of non-contradiction, according to Leibniz, is far from being universally applicable to all reasonings as claimed by some practitioners of classical logic. On the other hand, the kind of analysis that suits the contingent, so-called *infinite analysis*, exists only in an ideal world. For, limited as we humans are, it is only imaginable in the metaphorical sense that if analysis were to be done for the other kind of truths in a similar way to that for logical truths, it would be done by a being superior to us, and in an ideal world. *There* the reduction can be seen as the extension to infinity of the finite process of reduction with which we are familiar. But it is not an idea that would give us, here on earth, any practical notion of what analysis may be; for we as human beings would never, by following this path, come to a point when the denial of what we obtain leads to a contradiction. This point is not a theoretical novelty to anyone in the know about science however; for as we are well aware, one can never succeed in *proving* hypotheses of science. As far as Leibniz' idea of analysis is concerned, we shall focus on the finite, viz. the idea that the concept of the predicate is implicitly contained

[1] It is to be understood in a broad sense that includes all branches of mathematics.

in the concept of the subject can be demonstrated in finitely many steps by reduction. At any rate, Leibniz seems not to have contemplated countably infinite analyses. But if we hypothesize an analysis of the subject term of a categorical proposition into denumerable atomic constituents, and of its predicate term likewise, then it would seem that God would find the sentence true provided that every atomic constituent of the predicate term is contained in a single atomic constituent of the subject term. Else, some further analysis of the predicate term must be possible.

Since Leibniz wished to conceive of all propositions as terms and all hypotheticals as categoricals, it is reasonable to suppose that this understanding could, with his blessing, be extended to conditional sentences. That understanding, in effect, is what we attempt to realize in this paper. Our subject matter is formal, however, and our methods of analysis, are those of classical propositional logic. In this setting, all analyses are *per force* finite, and are represented in normal forms, particularly in conjunctive normal forms, which can be conceived of as conjunctions of (disjunctive) atomic constituents. We can, however, preserve something of the romance of Leibniz's conjectures, if we bear in mind that each of those disjunctions can be thought of as a simple representation of and interval within which one of a set of measurements must lie, namely that set of measurements of magnitudes whose ranges of values determine, for practical purposes, the truth or falsity of the sentence. Leibniz might insist that God and God alone can know their precise values. This, of course, suggests a topological semantic idiom rather than the hypergraph-theoretic idiom that we adopt here. The topological generalization must await a later experiment.

For Leibniz the hypothetical represented a relation between propositions as the copula represented a relation between terms. This must eventually create problems for a representation of propositions more generally as terms, which would require higher degree hypotheticals having hypotheticals as terms. Generally speaking this leads to representations of hypotheticals as binary operations rather than as assertions of relations. We resist that temptation and reserve for later work consideration of the problems attending the Leibnizian approach. With this in mind, we must remind ourselves of the distinction between two kinds of logic:

Definition 1. A unary logic is a set L of formulae closed under the application of certain inferential rules to its members. The members of L are called L-theorems. We write $\vdash_L \alpha$ if $\alpha \in L$; whereas a binary logic is a collection S of ordered pairs $\langle \alpha,\ \beta \rangle$, where β is derivable from α in S satisfying certain closure conditions. We write $\alpha \vdash_S \beta$ if $\langle \alpha,\ \beta \rangle \in S$.

We restrict ourselves in this paper to binary logics that satisfy the following finiteness condition:

Definition 2. Φ is a set of (well-formed) formulae.

Definition 3. In a binary logic S, $\forall \Gamma \subseteq \Phi$, and $\alpha \in \Phi$, if $\Gamma \vdash_S \alpha$, then $\exists \beta_1, \beta_2, \ldots, \beta_n \in \Gamma$ such that $\bigwedge_{i=1}^{n} \beta_i \vdash_S \alpha$.

2. Introducing Analytic Semantics

A traditional principle of entailment asserts that for α to entail β, β must be 'contained' in α. The notion of 'containment' can be interpreted in various ways. Since the relation of containment of the concept of the predicate in the subject is the groundwork on which all truths are built, one might as well understand the relation in a mathematically well-defined manner. Let us first appeal to Leibniz' own description:

> "All this is easily proved from the one assumption that the subject is as it were a container, and the predicate the simultaneous or conjunctive content; or conversely, that the subject is as it were a content, and the predicate an alternative or disjunctive container."
>
> —*Leibniz*, 1679–1686 [2]

Hence the two ways of mapping the analysis of the predicate term into the analysis of the subject term. Both the subject and the predicate concepts can be taken either as the conjunction of their necessary constituents; or as the disjunction of their sufficient conditions. In the first case, the leibnizian[2] notion of 'containment' consists of the underwriting of every necessary constituent of the predicate concept by some such of the subject concept; and the second case naturally describes its dual statement that every sufficient condition of the subject concept is underwritten by some sufficient condition of the predicate concept. To deliver the pair of dual statements in the language of formal semantics, we need to introduce the notion of a hypergraph, which can be understood classically, in the context of a (full) propositional model, that is, an ordered pair

$$\mathscr{M} = \langle\, U,\ V\, \rangle$$

where U is a nonempty set of 'possible worlds' and V

$$V : U \times At \to \{0, 1\}$$

assigns a truth value to every atom at every element (point) of U. A leibnizian analysis of an entailment $\alpha \vdash \beta$ (read α entails β) would invoke some notion of the analyses of α and β, here written as $\mathrm{C}(\alpha)$ and $\mathrm{C}(\beta)$ and would be expressed as follows:

$\forall \alpha, \beta \in \Phi,$

$$\alpha \vdash \beta \Leftrightarrow \forall B \in C(\beta), \exists A \in C(\alpha) : \forall a \in A, \exists b \in B : a \vdash_L b$$

$a \vdash_L b$ indicates that in a binary logic L, b can be inferred from a. It is the underwriting of every analytic condition of β by an analytic condition of α that constitutes the notion of 'containment'. In the context of classical propositional logic where every formula α can be syntactically represented by its CNF, i.e. the conjunction of its *analytic conditions*, each of which is in the form of a disjunction of atoms. The entailment relation in [3] is thereby reduced to a relation between two sets of sets of literals.

[2] The lower case letter 'l' is chosen to avert supposition of attribution, since Leibniz himself did not directly suggest the understanding of containment alluded to.

The resulting formal system is called *first degree* fragment of E, abbreviated as *FDE* in [4] (pp. 158-9).

1. $\vdash \neg\neg p \leftrightarrow p$;
2. $\vdash p \wedge (q \vee r) \to (p \wedge q) \vee r$;
3. $\vdash p \to p \vee q$;
4. $\vdash q \to p \vee q$;
5. $\vdash p \wedge q \to p$;
6. $\vdash p \wedge q \to q$

together with three rules

$$\textit{Transitivity}: \frac{\vdash p \to q \qquad \vdash q \to r}{\vdash p \to r}$$

$$\textit{Left disjunctivity}: \frac{\vdash p \to r \qquad \vdash q \to r}{\vdash p \vee q \to r}$$

$$\textit{Right conjunctivity}: \frac{\vdash p \to q \qquad \vdash p \to r}{\vdash p \to q \wedge r}$$

$$\textit{Contraposition}: \frac{\vdash p \to q}{\vdash \neg q \to \neg p}$$

The decades following the first introduction of *FDE* witnessed the emergence of various semantics to which it is amenable. Among these are the situation semantics given by Dunn [5], possible world semantics with the 'star operation' of Routley and Routley [6], and intensional algebraic semantics given by Belnap [7], and so on. We are naturally inclined to interpret the literals as truth-sets, i.e. as members of $\wp(U)$ where U is the universe in a full propositional model. Accordingly the articular representation of a sentence as a set of sets of literals under this interpretation yields a collection of collections of subsets of U, i.e. a *hypergraph* on $\wp(U)$.

Definition 4. A hypergraph H is a pair $H = (X, E)$ where X is a set of elements, called nodes or vertices, and E is a non-empty set of non-empty subsets of X, each of which is called a (hyper-) edge.

In subsequent discussions, we refer to H so defined as a hypergraph on X. Conventionally, we can write H as a collection of edges, i.e. $H = \{E_1, E_2, \ldots, E_n\}$ where $\forall i, 1 \leq i \leq n, E_i \in X$.

Hypergraph semantics is most closely linked to the Jeffry–Dunn coupled-tree semantics [5]. In a standard model,[3] every sentence is represented by the collection of states that verify it, and every state in that collection is a sufficient condition for the verification of the sentence. Therefore, if a sentence is to be represented by its proposition, its representation is a collection of states, each of which is a set of literals. Since every literal is again represented by a collection of states, a sentence in a full propositional model is to be represented, as Leibniz might have suggested, by a collection of collections of collections of states! Such a representation readily brings to mind a tree-like structure,

[3]By which we mean a (full) propositional model defined on page 4-5 of this paper.

whose branches are all the pathways that make the sentence true, corresponding to the states in the verification collection. Each of the states, that is, every branch of the tree, consists of a collection of literals (a collection of collections of states). Let us now take the tree as a whole to be the representation of the sentence, and regard every branch as a set, then the representation is no other than a hypergraph on a collection of subsets of the universe, i.e. a hypergraph on $\wp(U)$. Given an argument, $\alpha \vdash \beta$ and its truth-tree representation, a test for the validity of the argument requires, according to classical semantics, simply to check, for each branch of the α-tree, whether there exists a branch in the β-tree that is completely covered by it. Such is the basic idea of the coupled tree method introduced by Richard Jeffrey [8]. The test of validity was thus given the name *covering criterion*. However, Jeffrey realized that the covering criterion as it is cannot cover all cases of classical validity. Accordingly he allowed for the two exceptional cases that escape the rule, and added them to the class of valid inferences. What makes the case interesting is that the two exceptional cases are none other than those that make classical logic fail the criterion of paraconsistency. It was Michael Dunn [5] who first realized that the covering criterion itself suffices to give us a paraconsistent logic that is non-classical only in the sense of being paraconsistent, i.e. FDE. Had Jeffrey not added the two exceptional cases that violated the covering criterion in order that his coupled tree method validates all classical inference, he would have arrived at the same system as Anderson and Belnap.

3. Hypergraph Semantics

The theory of hypergraphs lies at the heart of our semantic analysis of entailment. Its usefulness as an instrument of analysis lies in its capacity to represent a wide range of formulae while preserving an independently stable structure of its own. Those natural properties of hypergraphs that ground the analysis of entailment are seen to best advantage in models that assign propositional formulae to hypergraphs. In [9], Jennings and Chen gave a hypergraph semantics using the *simple*, that is subset-disordered, hypergraphs on $\wp(U)$. When both vertices and edges are sets, the '$\subseteq$' ordering provides the natural candidate for the 'underwriting' relation involved in the analysis. In this section we explore some basic implications of a semantic approach that has as its goal the interpretation of *all* formulae as hypergraphs, so that we can grasp the fundamental theorem laid down in the next section with a more general understanding.

An articular model is an ordered quadruple $\mathscr{M} = \langle U, \leq, \mathbb{H}, \mathbf{H} \rangle$ where

1. $U \neq \varnothing$ is a partially ordered set;
2. $\leq$ is the partial order on U;
3. $\mathbb{H} \subseteq \wp\wp(U)$ such that every member of $\mathbb{H}$ is a simple hypergraph.
4. $\mathbf{H} : At \rightarrow \mathbb{H}$.

That is, $\mathbb{H}$ is a set of *simple* hypergraphs, and to each p_i, $\mathbf{H}$ assigns a simple hypergraph on U, $\mathbf{H}(p_i)$.

Definition 5. H is a simple hypergraph if and only if $\forall E, E' \in H,\ E \nprec E'$

where $E \prec E'$ abbreviates $\forall e \in E, \exists e' \in E' : e \leq e'$. Since not all set-theoretic operations naturally preserve the simplicity of hypergraphs, for present purposes, we obtain from every non-simple hypergraph H a simple hypergraph by a star operation that casts out super-edges of H.

Definition 6. $\star H = H$ - $\{E \in H \mid \exists E' \in H : E' \prec E\}$.

3.1. Extending H to Φ

The account of $\mathbf{H}(\cdot)$ which extends $\mathbf{H}$ to Φ, requires some preliminary definitions. For some set S,

Definition 7. If $A \subseteq \wp(S)$, then b is an intersector of A iff $\forall a \in A, b \cap a \neq \phi$.

Definition 8. If $A \subseteq \wp(S)$, then $\tau(A) = \{b \mid b$ is a minimal intersector of A$\}$.

Definition 9. If S is a set and f an operation, then $f[S] = \{f(a) \mid a \in S\}$.

Definition 10. If H is a hypergraph, then $\tau(H)$ is the blocker of H.

A hypergraph understood in the sense of definition 4 and its blocker are dual to each other. The function τ is also called a blocker function.

Definition 11. A DeMorgan lattice is a distributive lattice L with a negation operation $\neg : L \longrightarrow L$ that satisfies

1. $x \geq y \Leftrightarrow \neg y \geq \neg x$
2. $\neg\neg x = x$

Theorem. *Every hypergraph lattice, with its blocker playing the role of negation, is a DeMorgan lattice.*

The proof is trivial. As we have pointed out, a simple hypergraph and its blocker are duals.

$\mathbf{H}(\cdot)$ thus extends $\mathbf{H}$ to Φ:

1. $H_{P_i} = \mathbf{H}(P_i)$;
2. $H_{\neg\alpha} = \{f[B_i] \mid B_i \in \tau(H_\alpha)\}$;
3. $H_{\alpha\vee\beta} = \star\{a \cup b \mid a \in H_\alpha,\ b \in H_\beta\}$;
4. $H_{\alpha\wedge\beta} = \star(H_\alpha\ \cup\ H_\beta)$.

f in 2 is an operation such that $\forall a, b \in B_i,\ f^2(a) = a$ and $a \leq b \Rightarrow f(a) \geq f(b)$, i.e. $f[B_i]$ forms a DeMorgan lattice. Semantic entailment is therefore defined as a relation between two simple hypergraphs on $\wp(U)$, which we now state.

Definition 12. $\forall H, H' \in \mathbb{H}, H \sqsubseteq H'$, ($H'$ subsumes H) if and only if $\forall E' \in H'$, $\exists E \in H$ such that $\forall e \in E,\ \exists e' \in E' :\ e \leq e'$.

The proposition of a sentence is its corresponding hypergraph. An algebraic structure naturally emerges from the definition of subsumption relation, i.e. that $\langle \mathbb{H}, \sqsubseteq \rangle$ is a lattice. It is easily seen that '$\sqsubseteq$' is a partial ordering; moreover,

$$\forall \alpha, \beta \in \Phi, sup(H_\alpha, H_\beta) = H_{\alpha\vee\beta} \tag{1}$$

$$\forall \alpha, \beta \in \Phi, inf(H_\alpha, H_\beta) = H_{\alpha\wedge\beta} \tag{2}$$

As it happens, any composition of the language, be it an atom or an entailment of degree n, that is interpreted as a hypergraph of some sort, is an element in the lattice. Thus by representing formulae as hypergraphs, binary articular logic with subsumption serves as a base system for the interpretation of entailments.

The idea of subsumption also sheds light on familiar concepts. We have introduced the star function as an operation applicable to all hypergraphs, simplifying a hypergraph by casting out its super-edges. It is obvious that

$$\star : H \rightarrow \tau\tau(H)$$

The set of all hypergraphs on $\wp(U)$ therefore is the union of a set of equivalence classes, each of which is a set of hypergraphs that get mapped to by the above function to the same simple hypergraph. Let H be the set of all hypergraphs on U, the lattice $\langle[\mathsf{H}], \leq\rangle$ where

1. $[\mathsf{H}] = \{[\mathsf{H}]_x \mid x \in \mathbb{H}\}$ is the set of equivalence classes; there is a one-to-one correspondence between $[\mathsf{H}]$ and $\mathbb{H}$.
2. $\forall x \in \mathbb{H}$, and $\forall H, H' \in [\mathsf{H}]_x$, $\star(H) = \star(H') \in \mathbb{H}$.
3. $[\mathsf{H}]_x \leq [\mathsf{H}]_y$ if and only if $x \sqsubseteq y$.

is isomorphic with $\langle \mathbb{H}, \sqsubseteq\rangle$. Simplification operation, so to speak, is wired in the hypergraph lattice $\langle \mathbb{H}, \sqsubseteq\rangle$, which we shall refer to in the subsequent passages of this paper as the hypergraph lattice, unless otherwise specified.

Definition 13. $\forall\alpha, \beta \in \Phi$, $\alpha \models \beta$ (α semantically entails β) if and only if $\forall \mathscr{M} = \langle U, \mathbf{H}\rangle$, $H_\alpha \sqsubseteq H_\beta$. Alternatively, we say that $\alpha \vdash \beta$ is valid. So, *mutatis mutandis*, for $\Gamma \models \alpha$ (for Γ an arbitrary set of formulae).

4. Algebraic Properties of Hypergraph Lattices

Definition 14. In an arbitrary lattice L, if for some y, z, $x = y \wedge z$ if and only if $x = y$ or $x = z$, then x is a meet-irreducible element in L.

It is obvious that every element of a finite lattice is the meet of some meet-irreducible elements.

Theorem. *Every finite distributive lattice is isomorphic to a hypergraph lattice.*

Proof. Let D be a distributive lattice. Let $M(D)$ denote the set of meet irreducible elements of D. Let η be a function that assigns each element of D a (simple) hypergraph on the ground set $M(D)$. We first define η for the set of meet-irreducible elements in D.

$$\eta(x) = \{\{y : y \in M(D), y \ngeq x\}\}$$

We extend the definition in a practical manner.

$$\eta(x) \quad = \prod_{\alpha \geq x, \alpha \in M(D)} \eta(\alpha)$$

We claim η is an isomorphism $\eta : D \longrightarrow \eta(D)$. We prove that η is both a homomorphism and a bijection. First we prove that it is a homomorphism, viz. $x \geq y \Leftrightarrow \eta(x) \sqsupseteq \eta(y)$.

The proof is divided into two cases. For the first case, assume that x, y are meet-irreducible, then $x \geq y \Leftrightarrow \{z : z \ngeq x\} \supseteq \{z : z \ngeq y\}$. Therefore $\eta(x) = \{\{z : z \ngeq x\}\} \sqsupseteq \{\{z : z \ngeq y\}\}$, i.e. $\eta(x) \sqsupseteq \eta(y)$.

In the more general case, we prove the necessity and sufficiency conditions separately.

($\Rightarrow$) Let $x \geq y$ for $x, y \in D$, then $x = \bigwedge_i^n x_i,\ y = \bigwedge_j^m y_j$ with x_i, y_j as meet-irreducible elements. So $\forall 1 \leq i \leq n,\ x_i \geq x$; and $\forall 1 \leq j \leq m,\ y_j \geq y$. Now we prove that $\eta(x) \sqsupseteq \eta(y)$. $\forall i, x_i \geq x_1 \wedge \ldots \wedge x_n = x \geq y = y_1 \wedge \ldots \wedge y_m$. Therefore, $x_i \vee (y_1 \wedge \ldots \wedge y_m) = x_i$. Since D is a distributive lattice, $(x_i \vee y_1) \wedge \ldots \wedge (x_i \vee y_m) = x_i$. Because x_i is meet-irreducible, $x_i \vee y_j = x_i$ for some $1 \leq j \leq m$. That is, $\forall 1 \leq i \leq n,\ x_i \geq y_j$ for some $1 \leq j \leq m$. Since x_i and y_j are meet-irreducible, for each $\eta(x_i)$ there exists $\eta(y_j)$ such that $\eta(x_i) \sqsupseteq \eta(y_j)$, given the result we proved for the first case. But $\eta(x) = \sqcap_i^n \eta(x_i)$ and $\eta(y) = \sqcap_j^m \eta(y_j)$. Therefore, $\eta(x) \sqsupseteq \eta(y)$.

($\Leftarrow$) Now suppose $\eta(x) \sqsupseteq \eta(y)$ for $x, y \in D$. By definition of η, $\eta(x) = \eta(x_1) \sqcap \ldots \sqcap \eta(x_n)$, $\eta(y) = \eta(y_1) \sqcap \ldots \sqcap \eta(y_m)$ where x_i and y_j are meet irreducibles. Therefore for each $\eta(x_i)$ there exists $\eta(y_j)$ such that $\eta(x_i) \sqsupseteq \eta(y_j)$. Since x_i and y_j are meet irreducibles, given the result proved for the first case, we have for every x_i there exists y_j such that $x_i \geq y_j$. Thus, $x = x_1 \wedge \ldots \wedge x_n \geq y_1 \wedge \ldots \wedge y_m = y$. This proves the result. $\square$

Conjecture. *Every finite DeMorgan lattice is isomorphic to a hypergraph lattice with the blocker operation realizing negation.*

By saying that the blocker *realizes* negation in a DeMorgan lattice, we mean that if L is a DeMorgan lattice there exists a hypergraph lattice H that is isomorphic to L, and if $\phi : L \longrightarrow H$ gives our isomorphism, we require that, in addition to being a lattice isomorphism, ϕ satisfies $\phi(\neg x) = \tau(\phi(x))$. The following is an instance that confirms the conjecture.

Example. We give an explicit construction of a hypergraph representation of the familiar lattice of the set of subsets of $\{1, 2, 3\}$ with complementation serving as negation. We define a class of hypergraphs over the base set $X = \{a, b, c, d\}$. We associate $\{1, 3\} \rightsquigarrow \{\{a, b\}, \{c, d\}\}$ and $\{1, 2\} \rightsquigarrow \{\{a, c\}, \{b, d\}\}$ and $\{2, 3\} \rightsquigarrow \{\{a, d\}, \{b, c\}\}$. This defines the whole lattice, by taking joins and meets. Checking that negation corresponds to blocker, we calculate. $\{1, 3\}^c \rightsquigarrow b(\{\{a, b\}, \{c, d\}\}) = \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$, which is the hypergraph meet of the hypergraphs assigned to $\{2, 3\}, \{2, 1\}$. One may check that all other relations hold.

So far we have not happened upon any general techniques for the construction of hypergraph lattices. The above example was constructed using parallel classes of the two dimensional finite affine geometry over the field GF(2). This trick does not extend naturally. We believe that the difficulty of these constructions perhaps hints at a intimate link between these classes of objects. We also note that these constructions are of interest from the point of view of combinatorial optimization. The blocker function on hypergraphs indeed models a duality relationship that is found throughout combinatorial optimization.

5. Sufficiency of Hypergraph Semantics

Let the canonical model $\mathscr{M}^*$ be the ordered pair $\langle U^*, V^*\rangle$ where

1. U^*: A set of maximal FDE-consistent sets.
2. V^*: $V^*(P_i) = \{\{|P_i|\}\}$.[4]

We define the CNF of a formula α to be a set $\mathbf{CNF}_\alpha$ of equivalence classes CNF_α modulo permutation,[5] where each member of an equivalent class CNF_α is a conjunction of disjunctions of literals provably equivalent to α such that $\forall CNF_\alpha^m, CNF_\alpha^n \in CNF_\alpha$, then $CNF_\alpha^m \dashv\vdash CNF_\alpha^n$. Every class differs from other classes by the set of literals having occurrences in the CNF of the class. For any two CNFs within an equivalence class, CNF_α^i and CNF_α^j,

$$At\,(CNF_\alpha^i) = At\,(CNF_\alpha^j)$$

Among the equivalent classes in $\mathbf{CNF}_\alpha$, we define one particular class as the *standard CNF class* of α.

Definition 15. CNF_α is the standard CNF class of α if and only if

1. $CNF_\alpha \in \mathbf{CNF}_\alpha$;
2. $At\,(\alpha) = At\,(CNF_\alpha)$;
3. At least one of each literal pair based on the language of α is in every conjunct.

We then define $CNF(\alpha)$ to be the standard CNF of α, which is the member of the standard CNF class with the least number of conjuncts and the least number of literals in every conjunct.

Definition 16. $\forall\alpha \in \Phi$, suppose an arbitrary CNF in the equivalence class $CNF(\alpha)$ is in the form of a conjunction of Δ_i $(1 \le i \le n)$, which are disjunctions of literals δ_j $(1 \le j \le m_i)$,[6] then the deformulated set corresponding to $CNF(\alpha)$, denoted by $\mathbb{CNF}\,(\alpha)$, is a collection of Δ_i $(1 \le i \le n)$ such that $\Delta_i = \{\delta_j \mid 1 \le j \le m_i\}$.

Fundamental Theorem: $\forall\alpha \in \Phi, H_\alpha^* = \{|[\Delta_i]|$ [7] $\mid \Delta_i \in \mathbb{CNF}\,(\alpha), 1 \le i \le n\}$.

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[4] $|P_i|$ is a maximal FDE-consistent set that contains P_i.

[5] Informally it is understood in the sense of rearrangement, e.g. there are six permutations of the set $\{1, 2, 3\}$, namely $[1, 2, 3]$, $[1, 3, 2]$, $[2, 1, 3]$, $[2, 3, 1]$, $[3, 1, 2]$, and $[3, 2, 1]$. We here take the formal definition that corresponds to this meaning in group theory and algebra. A permutation of a set S is a bijection from S to itself (i.e., a map $S \to S$ for which every element s of S occurs exactly once as image value). To such a map f is associated the rearrangement of S in which each element s takes the place of its image $f(s)$.

[6] This notation m_i suggests that the number of literals in the ith disjunction is the output of a function m taking i as input.

[7] For any set S and any operation $\dagger$, we use $\dagger[S]$ to denote the set $\{\dagger s \mid s \in S\}$. Given that $\Delta_i = \{\delta_j \mid 1 \le j \le m_i\}$, $|[\Delta_i]|$ denotes the set of proof sets of δ_j $(1 \le j \le m_i)$.

The actual proof is notationally cumbrous, but the idea is simple. So we will here present the proof strategy. The deformulated set $\mathbb{CNF}\ (\alpha)$ corresponding to the standard CNF of α is a hypergraph on the union[8] of $Lit(\alpha)$, the set of literal pairs in the language of α, what we want to demonstrate here is that the hypergraph H^*_α, i.e. the representation of α in the canonical model, is structurally the same as $\mathbb{CNF}\ (\alpha)$, in fact it is the same as $\mathbb{CNF}\ (\alpha)$ except that in place of each literal p_i, we put its proof set $|p_i|$ instead. Therefore H^*_α is a hypergraph on the set of proof sets of literals based on the language of α, where the vertices are all proof sets of literals, and edges collections of proof sets of literals.

6. Analysis of Entailment

FDE as the first degree fragment of E admits of various semantic modelings. These semantics, in contrast with hypergraph semantics, have a trait in common: with each newly introduced element in the language, the interpretive expansions depend upon the introduction of new truth-conditions. The validity of $p \vdash p \vee (q \rightarrow r)$ cannot be determined by the model that validates such first-degree formula as $p \vdash p \vee q$, unless the valuation function is extended recursively to interpret $q \rightarrow r$ along with $\neg q$, $p \wedge q$ and $p \vee q$. The satisfaction of such a condition would qualify the model with the expanded valuation as a semantic model for some higher degree entailments. Hence, the validity of higher degree substitutional instances of first degree entailments can only be determined in a new semantic model for higher degree entailment. This, however, is not the case for hypergraph semantics.

Entailment in the binary articular logics was represented as a relation between simple hypergraphs. It can be summed up in a simple statement to the effect that α entails β if and only if every edge of H_β stands in an ordering relation to some edge of H_α. Adding entailment to our language, denoted by '$\rightarrow$', each necessary component of the entailment of β from α can be represented as a collection of pairs of edges $\langle E^i_\alpha, E^j_\beta\rangle$ with a fixed j to signify a particular β-edge whereas i ranges over the entire set of α-edges. Collections as such with E^j_β ranging over the entire set of β-edges constitute the representation of $\alpha \rightarrow \beta$. Suppose there are m edges of H_α and n edges of H_β, then the representation of $\alpha \rightarrow \beta$ can be written as the set

$$\Big\{ \big\{ \langle E^1_\alpha, E^1_\beta\rangle, \ldots, \langle E^m_\alpha, E^1_\beta\rangle \big\}, \ldots \Big\{ \langle E^1_\alpha, E^j_\beta\rangle, \ldots, \langle E^i_\alpha, E^n_\beta\rangle \Big\} \Big\}$$

which is a hypergraph $H = (R, E)$ built on a certain relation R between H_α and H_β. We call it a hypergraph on R, according to definition 3. Each edge of the hypergraph is a sub-relation of the relation R. The dual of this construction is the set of projections from H_β to H_α:

$$\tau(H_{\alpha\rightarrow\beta}) = \Big\{ f \mid H_\beta \xrightarrow{f} H_\alpha \Big\}.$$

8 S is a set of sets s, the union of the set S, $\cup S$, is the union of its element sets, i.e. $\cup S = \{s' | s' \in s \text{ for some } s\}$.

$H_\alpha \sqsubseteq H_\beta$ as such is a property of $H_{\alpha\to\beta}$ given that

$$H_\alpha \sqsubseteq H_\beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B \textit{ such that } a \leq b \qquad \text{(Sub)}$$

where '$\leq$' is a partial ordering. With ordered pairs being added as vertices of some hypergraph, we give a specific parsing of (Sub) where the partial ordering is subsumption:

$$\models \alpha \to \beta \Leftrightarrow H_\alpha \sqsubseteq H_\beta.$$

We define subsumption in an articular model $\mathscr{M} = \langle U, \leq, \mathbb{H}, \ \mathbf{H}\rangle$ inductively:

Definition 17.

$$a \sqsubseteq b = \begin{cases} a \leq b & a, b \in U; \\ \forall x \in b, \exists y \in a : y \leq x & if a, b \in \wp(U); \\ \forall x \in b, \exists y \in a : \forall y' \in y, \exists x' \in x \textit{ such that } y' \sqsubseteq x' & otherwise. \end{cases}$$

The reasons that motivate this specification are largely due to the understanding of ordered pairs as sets with inner structures. For example, $\langle a, \ b\rangle$ is understood as the higher order set $\{\{a, b\}, a\}$. As we know, subsumption satisfies the three structural rules. For two sets S and x not necessarily of the same order,

Ref $x \in S \Rightarrow S \sqsubseteq \{x\}$;
Mon $S \leq S', \ S \sqsubseteq x \Rightarrow S' \sqsubseteq x$;
Cut $S \cup S' \leq x, \ S \sqsubseteq S' \Rightarrow S \sqsubseteq x$.

Thus we can work out the entailment layer by layer, subsumption by subsumption, until we arrive at a clear set-theoretic inequality. The definition is incomplete without the qualification that in the extreme case, when a and b are sets, $a \sqsubseteq b$ amounts to '$a \supseteq b$'. This is so because we can see that the inner clause of (Sub): $\forall a \in A, \exists b \in B$ *such that* $a \leq b$ represents a partial ordering, which in the extreme case is the identity relation. It is routine work to verify that the following binary formulae of higher-degree entailments are valid under the interpretation of hypergraph on relation.

1. $p \to q \vdash p \wedge r \to q$;
2. $(p \to q) \wedge (p \to r) \dashv\vdash (p \to q \wedge r)$;
3. $(p \vee q \to r) \vdash (p \to r) \wedge (q \to r)$.

Applying the same technique as many times as necessary, we will find that the generalized subsumption comes finally down to an ordinary subsumption of the representations of some zero-degree formulae, as in FDE. Here again it reinforces the idea that FDE serves as a free-standing base system for entailment. However, the same technique may be employed to falsify many such 'unbalanced'[9] higher-degree entailments as *contraction*: $p \to (p \to q) \vdash p \to q$, whose absence from the validity set, we believe, would not have escaped the notice of an acute reader. There are other representations of the arrow that generate systems where all the theorems of the paradigm relevance logic R are

[9] It vaguely refers to the entailments whose antecedent and consequent are of different degrees.

valid,[10] but this is beyond the scope of our current discussion. We here concentrate more on the method of representation than its syntactic consequences.

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R. E. Jennings, Y. Chen and J. Sahasrabudhe
Laboratory for Logic and Experimental Philosophy
Simon Fraser University
8888 University Drive
Burnaby, BC V5A 1S6
Canada
URL: http://www.sfu.ca/llep/
e-mail: jennings@sfu.ca;
yechennings@gmail.com; nek@sfu.ca;
jds16@sfu.ca

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[10] A detailed discussion of it can be found in [10].