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Brief paper

Autoregressive models of singular spectral matrices[☆]

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ABSTRACT

This paper deals with autoregressive (AR) models of singular spectra, whose corresponding transfer function matrices can be expressed in a stable AR matrix fraction description $D^{-1}(q)B$ with B a tall constant matrix of full column rank and with the determinantal zeros of $D(q)$ all stable, i.e. in $|q| > 1$, $q \in C$. To obtain a parsimonious AR model, a canonical form is derived and a number of advantageous properties are demonstrated. First, the maximum lag of the canonical AR model is shown to be minimal in the equivalence class of AR models of the same transfer function matrix. Second, the canonical form model is shown to display a nesting property under natural conditions. Finally, an upper bound is provided for the total number of real parameters in the obtained canonical AR model, which demonstrates that the total number of real parameters grows linearly with the number of rows in $W(q)$.

1. Introduction

In this paper, we are concerned with singular AR models (see Section 2 for a definition). Our research on singular AR models is strongly motivated by our recent interest in generalized linear dynamic factor models (GDFM's), which are used to model and forecast high-dimensional macroeconomic and financial time series (Banbura, Giannone, & Reichlin, 2010; Forni, Hallin, Lippi, & Reichlin, 2000; Forni & Lippi, 2001; Giannone, Reichlin, & Sala, 2004; Stock & Watson, 2002a,b). In GDFMs, the latent variables are assumed to be stationary and are described as outputs of rational dynamic systems with tall matrix transfer functions (with more rows than columns). Thus the latent variables have singular rational spectra. Tall transfer functions have been shown to be generically zeroless and zeroless transfer functions can be represented by a singular AR model (Anderson & Deistler, 2008a,b; Deistler, Anderson, Filler, Zinner, & Chen, 2010; Filler, 2010) . *In almost all empirical econometric modeling and forecasting exercises using GDFMs, the latent variables are modeled as an singular AR model, i.e. an AR model with singular spectral density*. For example, in Banbura et al. (2010), the latent variables are clearly modeled as a singular AR model with 53 outputs and 3 inputs (factors).

Singular AR models are significantly different from regular AR models because many results for regular AR models are no longer true. For example, the block Toeplitz matrices corresponding to covariances of a stable singular AR model with dimensions high enough will be singular and thus only positive semidefinite. The associated Yule–Walker equations, which are used to construct an AR model given covariance data, may have an infinite number of solutions. Since this non-uniqueness of solutions creates difficulties in AR model identification using Yule–Walker equations, results were derived in Deistler et al. (2010) on how to estimate the real-valued parameters (as opposed to the integer-valued parameters such as the degrees of various polynomials appearing in the system representation).

In this paper, we provide answers to the following three questions: (1) How are different AR models of the same spectrum related? (2) How can we construct a canonical model? (3) What are the properties of the canonical model?

These types of questions have been studied extensively in the literature for ARMA models (see e.g. Deistler & Gevers, 1989; Gevers, 1986 and Hannan & Deistler, 1988). However, it is not immediately clear how the ARMA results should carry over to the AR case; further, our particular interest is with a subclass of AR systems, viz singular AR systems. As already mentioned earlier, singular AR models have led to new challenges in econometric time series modeling, and they deserve and require a special treatment, which is the main task of this paper.

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The paper is organized as follows. In Section 2, we introduce singular AR models. In Section 3, we provide a characterization of the equivalence class of all stable AR systems associated with a given singular spectrum. For every such equivalence class a canonical representative is described in Section 4, and properties of such canonical forms are provided in Sections 5–7. Section 8 provides an example, and conclusions are made in the last section.

2. Model class

We consider vector autoregressive models

$$D(q)y_t = \nu_t. \tag{1}$$

Here q is a complex variable as well as the backward shift on the integers,

$$D(q) = \sum_{j=0}^{k} D_j q^j, \quad D_j \in R^{p\times p},\ p > 1,\ D_k \neq 0, \tag{2}$$

the determinant of $D(q)$ satisfies the stability condition $|D(q)| \neq 0,\ |q| \leq 1$, and (ν_t) is p-dimensional white noise.

Definition 1. An autoregressive model (1) is called singular (regular) if the variance matrix $\mathcal{E}(\nu_t\nu_t')$ is singular (nonsingular).

Denote the rank of $\mathcal{E}(\nu_t\nu_t')$ as m. Clearly, ν_t may be written as $\nu_t = B\varepsilon_t, B \in R^{p\times m}$, where (ε_t) is white noise with covariance matrix $\mathcal{E}(\varepsilon_t\varepsilon_t') = 2\pi I_m$.[2] The spectral density of the stationary solution of (1) (which is the only solution considered in this paper), f_y, is of the form $f_y(\lambda) = W(e^{-i\lambda})W^*(e^{-i\lambda})$, where $W(q) = D^{-1}(q)B$ and * denotes the conjugate transpose. Clearly f_y is rational in $e^{-i\lambda}$ and is singular for $m < p$.

In this paper we are concerned with singular vector AR models. The importance of singular AR models comes from the fact that the latent variables of GDFMs can be described that way. Let us commence from a rational and singular spectral density of rank $m < p$, $\forall\lambda \in [-\pi, \pi]$ for such latent variables, and consider its $p \times m$ stable and mini-phase spectral factors $W(q)$, which are unique up to postmultiplication by constant orthogonal matrices. Then generically (Anderson & Deistler, 2008a,b; Filler, 2010) such a spectral factor corresponds to a singular AR model. Accordingly, they can always be written in the form $D^{-1}(q)B$ for some $D(q), B$ with the properties noted above.

3. Fraction descriptions of transfer functions

For an analysis of observational equivalence, in a first step we commence from the spectral density f_y. As has been noted above, there is an associated transfer function $W(q)$, having no zeros and poles within and on the unit circle, which is unique up to postmultiplication by $m \times m$ constant orthogonal matrices.

As $W(q)$ has no zeros at $q = 0$, $W(0) \in R^{p\times m}$ has full column rank m. The proposition below is straightforward and shows how a unique transfer function can be chosen through the choice of an orthogonal matrix by requiring $W(0)$ to be quasi-lower triangular. (Identify $W(0)$ with R in the proposition.)

Proposition 1. *Let $A \in R^{p\times m}$ with $p \geq m$ and $rk(A) = m$; then there exists a unique factorization of $A = RQ$ where $Q \in R^{m\times m}$ is orthogonal and $R \in R^{p\times m}$ is a quasi-lower triangular matrix, i.e. if the first row of A is not zero then $r_{11} \neq 0$, and $r_{1j} = 0, j > 1$, where r_{ij} is the (i, j) element of R, otherwise $r_{1j} = 0, j \geq 1$. If the second row of A is linearly independent of the first, then $r_{22} \neq 0$ and $r_{2j} = 0, j > 2$ and otherwise $r_{2j} = 0, j \geq 2$, etc.*

[2] Here the appearance of 2π is because $\gamma(s) = \int_{-\pi}^{\pi} f_y(\lambda)e^{i\lambda s}d\lambda$ and $\gamma(0) = \mathcal{E}(y_t y_t')$, where $f_y(\lambda)$ is the spectral density of y_t.

Henceforth, we work with a unique transfer function $W(q)$, which has no poles within and on the unit circle and constant rank for all q. Note that in the definition of a singular AR model, there is no requirement that the polynomial matrix fraction description $A^{-1}(q)B$ be coprime. (A polynomial matrix fraction representation $A^{-1}(q)B$ of a transfer function matrix is said to be coprime Kailath, 1980; Wolovich, 1974 if $[A(q)\ B(q)]$ has full rank for all $q \in \mathbb{C}$.) Note also that even though $W(q)$ has an AR realization, it may have realizations, including coprime realizations. which are not AR. Furthermore, any two coprime AR realizations $A_1^{-1}(q)B_1 = A_2^{-1}(q)B_2$ are not necessarily related by a constant matrix, i.e. a constant X for which $X[A_1(q)\ B_1] = [A_2(q)\ B_2]$, in contrast to the situation where $W(q)$ square or fat and of full row rank almost everywhere.

Consider an $r \times s$ polynomial matrix $D(q)$ with $r \leq s$ and suppose that k_i is the degree of the i-th row of $D(q)$, i.e. the maximum degree of any entry of the row. The value $k_i = 0$ means that the i-th row is independent of q, and is nonzero. By convention, the value $k_i = -\infty$ is used for a row with all zero entries. If there exists a square $r\times r$ submatrix of $D(q)$ whose determinantal degree is $\sum k_i$, the matrix is said to be *row reduced*. Similarly, a *column reduced* matrix can be defined.

We now start to pin down some properties of AR descriptions of $W(q)$.

Theorem 1. *Let a $p \times m$ transfer function $W(q) = D^{-1}(q)B$, with $D_1(q)$ and B not necessarily coprime, correspond to a singular AR model with $p > m$ and $|D(q)| \neq 0,\ |q| \leq 1$. Then there exists a coprime fraction description of the form*

$$W(q) = \bar{D}^{-1}(q)\begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{3}$$

where $|\bar{D}(q)|$ has all zeros in $|q| > 1$, and the matrix $\bar{D}_2(q)$ consisting of the last $p - m$ rows of $\bar{D}(q)$ is row reduced, and $\bar{D}(q)$ has a degree at most equal to that of $D(q)$.

Proof. Noting B is full column rank, a $p \times p$ constant nonsingular matrix Z can be found such that $ZB = \begin{bmatrix} I_m & 0 \end{bmatrix}'$. Set $L(q) = ZD(q)$ (note that this means the degrees of $L(q)$ and $D(q)$ are the same). Then we have $W(q) = L^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$. Define an $m\times p$ polynomial matrix L_1 and a $(p-m)\times p$ polynomial matrix L_2 such that $L(q) = [L_1'(q)\ L_2'(q)]'$. It is evident that $L(q)$, $\begin{bmatrix} I_m & 0 \end{bmatrix}'$ is not coprime if and only if for some $q_0 \neq 0$ the matrix $L_2(q_0)$ has less than full row rank. (That $q_0 \neq 0$ follows from the fact that $|D(0)| \neq 0$.) By making use of the Smith canonical form, there exists a square non-unimodular $E(q)$ with $E(0)$ nonsingular and a matrix $\bar{D}_2(q)$ with full rank for all q such that $L_2(q) = E(q)\bar{D}_2(q)$ and the matrix $\bar{D}_2(q)$ can be assumed to be row reduced without loss of generality.

Let $\bar{D}_1(q) = L_1(q)$ and set $\bar{D}(q) = \begin{bmatrix} \bar{D}_1'(q) & \bar{D}_2'(q) \end{bmatrix}'$, then the pair $\bar{D}(q)$, $\begin{bmatrix} I_m & 0 \end{bmatrix}'$ is coprime, and

$$W(q) = \begin{bmatrix} L_1(q) \\ E(q)\bar{D}_2(q) \end{bmatrix}^{-1}\begin{bmatrix} I_m \\ 0 \end{bmatrix} = \begin{bmatrix} L_1(q) \\ \bar{D}_2(q) \end{bmatrix}^{-1}\begin{bmatrix} I_m \\ 0 \end{bmatrix} = \bar{D}^{-1}(q)\begin{bmatrix} I_m \\ 0 \end{bmatrix}. \tag{4}$$

Using the definition of $L(q)$, it is easy to conclude that $|\bar{D}(q)|$ also has all its zeros in $|q| > 1$.

Consider now the degree of $\bar{D}_2(q)$. Denote the i-th row of $L_2(q)$ by $L_{2i}(q)$, and the ij element of $E(q)$ by $e_{ij}(q)$. Let the i-th row degree of $\bar{D}_2(q)$ be k_i. Note that $\bar{D}_2(q)$ is row reduced, then by the predictable degree property (Kailath, 1980, p. 387), there holds $\deg L_{2i}(q) = \max_j[\deg e_{ij}(q) + k_j]$. The degree of a zero element is set equal to $-\infty$. Let $k_{\max}$ be the largest of the k_j. Consider the corresponding column of $E(q)$. Since this column cannot contain all zero elements, it follows that for some i, we have $\deg L_{2i}(q) \geq k_{\max} = \deg \bar{D}_2(q)$ from which the claim is immediate noting that $L_1(q) = \bar{D}_1(q)$ and that the degree of $L(q)$ is the same as that of $D(q)$. □

For a $W(q)$ with a coprime matrix fraction description of the form $\bar{D}^{-1}(q)\begin{bmatrix} I_m & 0\end{bmatrix}'$, note that $W(q)$ may have a pole at infinity (i.e. for some i, j $W_{ij}(\infty)$ is unbounded) or it may have a zero at $q = \infty$ (i.e. the rank of $W(\infty)$ is less than m), even though it has no finite zero. Examples are provided by

$$W_1(q) = \begin{bmatrix} 1 & 1 \\ q+1 & q \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -q \\ q+1 \end{bmatrix},$$

$$W_2(q) = \begin{bmatrix} q & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{q-4} \\ \dfrac{-2}{q-4} \end{bmatrix}$$

where $W_1(q)$ has a pole at $q = \infty$, and $W_2(q)$ has a zero at $q = \infty$.

Using Theorem 2.2.1 in Hannan and Deistler (1988), it is easy to obtain the following corollary of Theorem 1, treating the nonuniqueness of $\bar{D}(q)$.

Corollary 1. *Given a coprime matrix fraction description* $W(q) = \bar{D}^{-1}(q)\begin{bmatrix} I_m & 0\end{bmatrix}'$ *with* $|\bar{D}(q)|$ *having all zeros in* $|q| > 1$, *then there exists another matrix fraction description* $W(q) = \Delta^{-1}(q)\begin{bmatrix} I_m & 0\end{bmatrix}'$ *with* $|\Delta(q)|$ *having all zeros in* $|q| > 1$, *if and only if there exists a nonsingular polynomial matrix* $V(q)$ *with all zeros of* $|V(q)|$ *in* $|q| > 1$, *satisfying* $V(q)\begin{bmatrix} I_m & 0\end{bmatrix}' = \begin{bmatrix} I_m & 0\end{bmatrix}'$ *and* $\Delta(q) = V(q)\bar{D}(q)$. *Moreover, this second fraction description is coprime if and only if* $V(q)$ *is unimodular.*

Partition $V(q)$ as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \tag{5}$$

where V_{11} is $m \times m$. Then it follows from Corollary 1 that $\begin{bmatrix} V_{11}' & V_{21}' \end{bmatrix} = \begin{bmatrix} I_m & 0\end{bmatrix}'$ and $|V_{22}(q)|$ has all zeros in $|q| > 1$. Further, $V_{22}(q)$ is unimodular if and only if $V(q)$ is unimodular. The matrix $V_{12}(q)$ is free.

For a given AR model with a transfer function $W(q)$, which has a coprime fraction description $\bar{D}^{-1}(q)\begin{bmatrix} I_m & 0\end{bmatrix}'$with a stable $\bar{D}(q)$, the equivalence class of all stable singular AR models can be described by the following set

$$S_{\text{eq}} = \left\{ \left(\Delta(q), \begin{bmatrix} I_m \\ 0 \end{bmatrix} \right) \middle| \Delta(q) = V(q)\bar{D}(q), \right.$$
$$\left. V = \begin{bmatrix} I_m & V_{12}(q) \\ 0 & V_{22}(q) \end{bmatrix}, |V_{22}(q)| \neq 0, V_{22}(q) \text{ is stable} \right\}. \tag{6}$$

Here, no degree restriction is imposed on the set of $\Delta(q)$.

4. Canonical singular AR model of equivalence class

In this section, our interest is in finding a canonical form, i.e. a uniquely defined representative of the equivalence class S_{eq} as described above. We shall show how we can obtain the canonical representative of S_{eq} in two steps. In the first step, we will choose the matrix V_{22} to ensure that the transformed matrix, $\tilde{D}_2$ is canonical. Then, in the second step, we shall focus on how to choose V_{12} to secure a canonical representative including some control over column degrees.

4.1. Choice of V_{22} to secure canonical $\tilde{D}_2$ with minimum row degrees

We shall first consider the choice of V_{22} with a view to getting a canonical form for $V_{22}\bar{D}_2$. This is actually standard.

We now recall some material from Kailath (1980) and Wolovich (1974). An $r \times s$ polynomial matrix $X(q)$ of normal rank r is said to be in *Popov form* or *row polynomial–echelon form* if the following properties hold:

(1) It is row reduced and the row degrees are in descending order, say $k_1 \geq k_2 \geq \cdots \geq k_r$. (It is actually conventional to assume the row degrees are in ascending order; however, the difference is immaterial, and the choice of descending order facilitates the statement of certain results later.)
(2) For row i with $1 \leq i \leq r$, there is a pivot index p_i such that d_{ip_i} is monic and has degree k_i, and degree $d_{ij} < k_i$ for all $j > p_i$. Here d_{ij} denotes the degree of the i, j entry of the matrix $X(q)$.
(3) If $k_i = k_j$ and $i < j$, then $p_i < p_j$, i.e. the pivot indices corresponding to the same row degree are increasing.
(4) d_{ip_j} has degree less than k_j if $i \neq j$.

The major results are summarized in the following.

Lemma 1. *Given an arbitrary* $r \times s$ *polynomial matrix* $D(q)$ *with* $r \leq s$ *and of full row rank for almost all* q, *there exists a unimodular left multiplying matrix such that the product is row reduced. Further, the vector of row degrees modulo reordering is unique, i.e. if a different unimodular left multiplier produces a row reduced product, the row degrees must be the same (though not necessarily their order). In addition, every polynomial matrix of full row rank for almost all* q *can be reduced through pre-multiplication by a unimodular matrix to row polynomial–echelon form, and the form is canonical, i.e. any two such full row rank polynomial matrices which are related through pre-multiplication by a unimodular matrix have the same row polynomial–echelon form.*

This result indicates how the matrix V_{22} should be chosen in (6): it should bring $\bar{D}_2$ to row polynomial–echelon form.

4.2. Choice of V_{12} to secure canonical $\tilde{D}_1$

Observe first that the task of choosing V_{12} in (6) is not affected by the choice of V_{22}. This is because $V_{12}\bar{D}_2 = [V_{12}V_{22}^{-1}]V_{22}\bar{D}_2$ and with V_{22} unimodular, it is evident that $V_{12}V_{22}^{-1}$ will be polynomial if and only if V_{12} is polynomial.

So we suppose that $\tilde{D}_2$ is in row polynomial–echelon form, and we seek to exploit the freedom in V_{12} to minimize the column degrees of a certain submatrix of $\bar{D}_1 + V_{12}\bar{D}_2$ to select a canonical representative for $\bar{D}_1 + V_{12}\bar{D}_2$.

To obtain a canonical member of the associated equivalence class, proceed as follows. Let E_2 be the square submatrix determined by deleting the non-pivot-index columns of $\tilde{D}_2$, then let E_1 denote the submatrix of $\bar{D}_1$ comprising the columns with the same indices. Thus $[E_1' \; E_2']'$ will be obtained from $[\bar{D}_1' \; \tilde{D}_2']'$ by deleting columns not containing pivot indices of $\tilde{D}_2$. As noted above, E_2 is column reduced. Form the matrix $E_1E_2^{-1}$ and express it as the sum of a polynomial term $-V_{12}$ and strictly proper remainder. This additive decomposition is of course unique. Then evidently, for some polynomial F_1, there holds

$$E_1E_2^{-1} = -V_{12} + F_1E_2^{-1} \tag{7}$$

or equivalently

$$F_1 = E_1 + V_{12}E_2. \tag{8}$$

Further, because E_2 is column reduced and $F_1E_2^{-1}$ is strictly proper, the column degrees of F_1 will be less than those of E_2. Now take $\tilde{D}_1$ to be $\bar{D}_1 + V_{12}\tilde{D}_2$, and it is evident that in those columns of $\tilde{D}_1$ below which $\tilde{D}_2$ has a pivot index, the degree of the entries will be less than the degree of the pivot index element.

It is not hard to argue that the transformed $\tilde{D}_1$ is unique, i.e. we have obtained a canonical representative, because the additive decomposition referred to above is unique.

We sum up the construction as follows.

Theorem 2. *Consider a tall transfer function matrix* $W(q) = \bar{D}^{-1}\begin{bmatrix} I_m & 0 \end{bmatrix}'$ *where* $\bar{D}(q)$ *has all determinantal zeros in* $|q| > 1$ *and* $(\bar{D}(q), (I_m, 0)')$ *is coprime. Consider other singular AR descriptions of* $W(q)$ *obtained as* $[V(q)\bar{D}(q)]^{-1}\begin{bmatrix} I_m & 0 \end{bmatrix}'$ *via a unimodular matrix satisfying*

$$V(q) = \begin{bmatrix} I_m & V_{12} \\ 0 & V_{22} \end{bmatrix}. \tag{9}$$

Then a canonical representative in the set of coprime factorizations with numerator $\begin{bmatrix} I_m & 0 \end{bmatrix}'$ *is obtainable by the following two-step procedure:*

(1) *Choose a unimodular* V_{22} *such that* $\tilde{D}_2 = V_{22}\bar{D}_2$ *is in row reduced echelon form.*
(2) *Let* E_2 *be the submatrix of* $\tilde{D}_2$ *comprising those columns defined by the pivot indices of* $\tilde{D}_2$ *and let* E_1 *be the submatrix of* $\bar{D}_1$ *defined by selecting the same columns. Choose* V_{12} *(which is actually unique) such that the column degrees of* $F_1 = E_1 + V_{12}E_2$ *are less than the corresponding column degrees of* E_2.

It should be noted that once we have obtained the canonical form for coprime matrix fraction descriptions in S_{eq}, *ipso facto* we have found a canonical form for all matrix fraction descriptions in S_{eq}. Therefore, it follows from Theorem 2 that the procedure above leads to a canonical form, i.e. a unique representative of the equivalent class S_{eq}.

5. Maximum lag of the canonical AR model

Given a transfer function $W(q) = D^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$ with $D(q) = \sum_{j=0}^{k} D_j q^j$, we shall establish the relationship between the maximum lag of the AR realization $D^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$ (i.e. k) and the maximum lag of the AR realization $\tilde{D}^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$ (i.e. the degree of the canonical $\tilde{D}(q)$), and show that the maximum lag of the AR realization $\tilde{D}^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$ is minimal among all equivalent stable matrix fraction descriptions of $W(q)$ with the same numerator $\begin{bmatrix} I_m & 0 \end{bmatrix}'$. The upshot is that the introduction of a canonical form is costless as far as the maximum lag is concerned.

Theorem 3. *Given a transfer function* $W(q)$ *with a matrix fraction description* $D^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$ *with maximum lag* k*, suppose that* $\tilde{D}^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$ *is the canonical AR description of* $W(q)$ *as described in Section 4 with maximum lag* $\tilde{k}$*. Then* $\tilde{k} \le k$ *holds.*

Proof. According to Theorem 1, without loss of generality we can always commence with a coprime matrix fraction $W(q) = \bar{D}^{-1}(q)\begin{bmatrix} I_m & 0 \end{bmatrix}'$ with the degree of $\bar{D}(q)$ less or equal to k. Then, we can obtain $\tilde{D}(q)$ using the procedure presented in Theorem 2.

Define E_1, E_2, and F_1 as in the statement of Theorem 2. Let J_1 be the submatrix of $\bar{D}_1$ obtained by deleting the columns of E_1 and J_2 the submatrix of $\tilde{D}_2$ obtained by deleting the columns of E_2.

Note that there exists a unimodular matrix $V_{22}(q)$ such that $\tilde{D}_2 = V_{22}\bar{D}_2$, we have $\bar{D}_2 = V_{22}^{-1}\tilde{D}_2$. Since V_{22}^{-1} is a polynomial matrix and $\tilde{D}_2$ is row reduced, similarly to the proof of Theorem 1 it can be shown that the degree of $\tilde{D}_2$ is at most equal to that of $\bar{D}_2$ and thus at most k. This proves that the degrees of E_2 and J_2 are at most k. The degrees of E_1 and J_1 are at most k because of the fact that the degree of $\bar{D}_1$ is at most k. Since the degree of F_1 is less than that of E_2, the degree of F_1 is less than k as well.

We have proved that the degree of $F_1 - E_1$ is at most k. Note that $F_1(q) - E_1(q) = V_{12}(q)E_2(q)$ and $E_2(q)$ is row reduced, by the predictable degree property, we have for any $1 \le i \le p - m$

$$\max_j(\deg(v_{ij}) + k_j) \le k \tag{10}$$

where $\deg(\cdot)$ is the degree of a polynomial, v_{ij} is the ij element of V_{12} and k_j, $j = 1, \ldots, p - m$ are the row degrees of E_2.

Let $J_2(q) = (h_{jr})$ and $\Psi(q) = (\psi_{ir}) = V_{12}(q)J_2(q)$, then we have $\psi_{ir} = \sum_{j=1}^{p-m} V_{ij}h_{jr}$. Denote the row degrees of $J_2(q)$ as $l_1, \ldots, l_{p-m}$, it is easy to see that $\deg(\psi_{ir}) \le \max_j(\deg(V_{ij}) + l_j)$ for all r. Note that the construction of the canonical form ensures that $l_j \le k_j$, it follows from (10) that $\deg(\psi_{ir}) \le \max_j(\deg(V_{ij}) + l_j) \le \max_j(\deg(V_{ij}) + k_j) \le k$ for all i and r, which means the degree of $\Psi(q) = V_{12}(q)J_2(q)$ is at most k. This together with the fact that the degree of J_1 is at most k proves that the degree of $J_1(q) + V_{12}(q)J_2(q)$ is at most k. Observing that the columns of $\tilde{D}_1(q)$ consist of the columns of $J_1(q) + V_{12}(q)J_2(q)$ and $F_1(q)$, we have proved that the degree of $\tilde{D}_1(q)$ is at most k, which completes the proof. □

Theorem 3 proves that the maximum lag of the canonical AR model constructed in Section 4 will not increase above the maximum lag of the original AR model that is used to derive the canonical AR model. This implies that the maximum lag of the canonical AR model is minimal among all the matrix fraction descriptions in S_{eq}.

6. A nesting property of the canonical AR model

Recall that the McMillan degree of a transfer function $W(q)$ is defined as the total number of its finite and infinite poles with appropriate allowance for multiplicity and appearance of the pole in possibly more than one entry of $W(q)$ (see Kailath, 1980 for more details on definitions and discussions on finite and infinite poles of a transfer function based on the Smith–McMillan form).

In this section, we make some new assumptions that are typical in the GDFM literature cited in the introduction and which reflect the tendency to consider output processes of varying dimensions.

Assumptions. For some fixed integer m, a set of transfer functions $W_p(q)$ with $p = m + 1, m + 2, \ldots$ is specified where (1) $W_p(q)$ is $p \times m$; (2) the $W_p(q)$ are nested, in the sense that for all p the first p rows of $W_{p+1}(q)$ are $W_p(q)$; and (3) there exists some p_0 such that for all $p \ge p_0$, (a) the normal rank of $W_p(q)$ is m, (b) the McMillan degree of $W_p(q)$ is the same as that of $W_{p_0}(q)$ and (c) $W_{p_0}(q)$ has no zeros (apart possibly from $q = \infty$), and therefore the same is true of $W_p(q)$.

6.1. Kronecker indices and the defect of a rational transfer function matrix

The *Kronecker indices* of a rational transfer function matrix $W(q)$ are closely related to its left or right null spaces. The left (right) null space $\mathcal{N}_l$ ($\mathcal{N}_r$) is a vector space of rational vectors in q such that $f(q)W(q) = 0$ ($W(q)f(q) = 0$) for any vector $f(q) \in \mathcal{N}_l$ ($f(q) \in \mathcal{N}_r$). They are defined in Kailath (1980) as follows.

Definition 2. Suppose that the nontrivial left (right) null space of a rational matrix $W(q)$ exists and has a minimal polynomial basis, $f_1(q), \ldots, f_\gamma(q)$, whose degrees are $\mu_1, \ldots, \mu_\gamma$ with $\mu_1 \ge \cdots \ge \mu_\gamma$. Then $\mu_1, \ldots, \mu_\gamma$ are called the *left (right) Kronecker indices* of $W(q)$.

The next concept is that of the *defect* of a rational matrix $W(q)$. This can be defined using properties of the Smith–McMillan form of $W(q)$. However, we shall take as our starting point the following:

Definition 3. Let n and n_z be the McMillan degree and the number of zeros (allowing for zeros at $q = \infty$) of $W(q)$. Then the *defect* of $W(q)$ is defined as def $W(q) = n - n_z$.

The defect is related to the Kronecker indices, Theorem 6.5–11 of Kailath (1980). The result is as follows:

Theorem 4. *Let $W(q)$ be a rational transfer function matrix. Then def $W(q)$ is the sum of the left and right Kronecker indices of $W(q)$.*

6.2. Structure of the canonical denominator matrices

We will now use the result on the defect of a rational matrix to determine more aspects of the structure of a canonical singular AR description.

Adopt the assumptions at the start of the section, and fix some $p \geq p_0$. Let

$$W_p(q) = \tilde{D}_p^{-1}\begin{bmatrix} I_m \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{D}_{1p} \\ \tilde{D}_{2p} \end{bmatrix}^{-1}\begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{11}$$

where the denominator matrix is canonical, $\tilde{D}_{1p}$ is $m \times p$, $\tilde{D}_{2p}$ is $(p-m) \times p$, and is in row polynomial–echelon form. Using the fact that $\tilde{D}_{2p}W(q) = 0$ with $\tilde{D}_{2p}$ row-reduced, one can show that:

Lemma 2. *The left Kronecker indices of $W_p(q)$ are precisely the row degrees of $\tilde{D}_{2p}$.*

With this lemma in hand, we can state the main result of this section.

Theorem 5. *Adopt the assumptions listed at the start of the section, and let n denote the McMillan degree of $W_{p_0}(q)$. Then, for $p > \max\{m+n, p_0\}$, there holds*

$$\tilde{D}_p = \begin{bmatrix} \tilde{D}_{1p} \\ \tilde{D}_{2p} \end{bmatrix} = \begin{bmatrix} F_{11}(q) & 0 \\ F_{21}(q) & 0 \\ F_{31} & I \end{bmatrix} \tag{12}$$

where $\tilde{D}_{1p}$ is $m \times p$, $F_{11}(q)$ is a polynomial matrix in q and $m \times (m+n)$, $F_{21}(q)$ is a polynomial matrix in q, and $n \times (m+n)$, F_{31} is a constant matrix and $(p-(m+n)) \times (m+n)$ and the identity matrix is $(p-(m+n)) \times (p-(m+n))$. Moreover, for $p > \max\{m+n, p_0\}$, the $\tilde{D}_p$ are nested, in that $\tilde{D}_{p-1}$ is obtainable from $\tilde{D}_p$ by deleting the last row and column.

Proof. We shall first identify the structure of $\tilde{D}_{2p}$. Since $\tilde{D}_{2p}$ is in row polynomial–echelon form and has $p-m > n$ rows, we can denote its row degrees as $k_1, k_2, \ldots, k_n, \ldots, k_{p-m}$, which satisfy $k_1 \geq k_2 \geq \cdots \geq k_n \geq \cdots \geq k_{p-m}$. Recall that the assumptions on $W_p(q)$ ensure that it has no zeros other than possibly at $q = \infty$ (and suppose there are $n_{p\infty}$ such zeros), and its McMillan degree is the same for all $p \geq p_0$, say n. Further, it has full column rank, and thus no right Kronecker indices. Then by Theorem 4 and the preceding lemma the sum of the left Kronecker indices, which is the sum of the row degrees of $\tilde{D}_{2p}$, satisfies $k_1 + k_2 + \cdots + k_n + \cdots + k_{p-m} = n - n_{p\infty} \leq n$. Using the fact that $k_1 \geq k_2 \geq \cdots \geq k_n \geq \cdots \geq k_{p-m}$, it follows that $k_j = 0$, $n < j \leq p-m$ (notice that no row degree can be $-\infty$, since $\tilde{D}_p$ is necessarily a nonsingular matrix). This proves that $\tilde{D}_p$ is in the form of (12). Recall further that all entries to the right of, above and below a pivot element equal to 1 must be zero. All this means that, for $p > \max\{m+n, p_0\}$, the matrix $\tilde{D}_{2p}$ has the following structure:

$$\tilde{D}_{2p} = \begin{bmatrix} F_{21}(q) & 0 \\ F_{31} & I \end{bmatrix} \tag{13}$$

where $F_{21}(q)$ is $n \times (m+n)$ and is in row polynomial–echelon form, and F_{31} is $(p-(m+n)) \times (m+n)$, this following from the fact that the last $p-(m+n)$ rows of $\tilde{D}_{2p}$ have row degree zero, and are part of a matrix in row polynomial–echelon form. The identity matrix is of course $(p-(m+n)) \times (p-(m+n))$.

The structure of $\tilde{D}_{1p} = [F_{11}(q)\ 0]$, and in particular the fact that the last $(p-(m+n))$ columns are zero, is a consequence of the canonical form construction, which ensures that in those columns of $\tilde{D}_{1p}$ which correspond to pivot indices appearing in $\tilde{D}_{2p}$, the degree of each entry is less than that of the corresponding pivot entry. The nonzero entries of the identity matrix in the bottom right corner of $\tilde{D}_{2p}$ are all pivot entries, and this gives rise to the zeros in the last $(p-(m+n))$ columns of $\tilde{D}_{1p}$.

Finally, to establish the nesting property, recognize that $W_{p-1}(q) = [I_{p-1}\ 0]W_p(q)$, where the submatrix of zeros is in fact a $(p-1)$-dimensional vector of zeros. Denote the matrix obtained from $\tilde{D}_p$ by deleting the last row and column as $\hat{D}_{p-1}$. It is easily verified that

$$\tilde{D}_p^{-1} = \begin{bmatrix} \hat{D}_{p-1}^{-1} & 0 \\ X & 1 \end{bmatrix} \tag{14}$$

where X is a row vector whose entries are inessential. It follows that

$$W_{p-1}(q) = [I_{p-1}\ 0]\begin{bmatrix} \hat{D}_{p-1}^{-1} & 0 \\ X & 1 \end{bmatrix}\begin{bmatrix} I_m \\ 0 \end{bmatrix} = \hat{D}_{p-1}^{-1}\begin{bmatrix} I_m \\ 0 \end{bmatrix}. \tag{15}$$

It is trivial to check that $\hat{D}_{p-1}$ is canonical because $\tilde{D}_p$ has this property. Hence there necessarily holds $\hat{D}_{p-1} = \tilde{D}_{p-1}$, as required. □

6.3. Another way to reveal nesting: linking AR models of static factors to AR models of outputs of a GDFM

Static factors are sometimes used in describing GDFMs. Given the latent variables y_k of a GDFM, a static factor z_k is defined as a process which has a dimension at most equal to that of the latent variables and satisfies $y_k = Lz_k$ with L being a constant matrix, and a minimal static factor is a static factor of the least dimension (Deistler et al., 2010). It has been shown in Deistler et al. (2010) that (1) the dimension of a minimal static factor z_k is the rank of the zero-lag covariance matrix of y_k; and (2) there is an AR model linking the dynamic factor process[3] to the latent variables if and only if there is an AR model linking the dynamic factor process to any minimal static factor process.

Suppose that such AR models exist, as is generically the case. Denote the dynamic factor process driving the GDFM as u_k and suppose u_k, z_k, y_k are of dimensions m, l, p respectively. Suppose also that the singular AR model linking the dynamic factor process to the minimal static factor is described as $D_z(q)z_k = \begin{bmatrix} I_m & 0 \end{bmatrix}' u_k$ with $\mathcal{E}(u_k u_j^T) = I_m\delta_{kj}$.

Also, without significant loss of generality, we shall assume that the minimal static factor comprises the first l entries of the output, so that $y_k = \begin{bmatrix} I_l & H_2' \end{bmatrix}' z_k$.

Then it is easy to show the following:

Proposition 2. *Define*

$$D_y(q) = \begin{bmatrix} D_z(q) & 0 \\ -H_2 & I_{p-l} \end{bmatrix}. \tag{16}$$

Then a singular AR model linking the dynamic factor process to the latent variables is provided by $D_y(q)y_k = \begin{bmatrix} I_m & 0 \end{bmatrix}' u_k$.

The above proposition reveals an important fact that the latent variables of a GDFM is a singular AR process if the static factor process is a singular AR process. If we denote by $D_y^p(q)$ the denominator matrix when y has dimension p, then another immediate consequence of the proposition is that for $p > l$,

[3] The dynamic factor process is the driving white noise process in the AR model.

the $D_y^p(q)$ are nested, in that $D_y^{p-1}(q)$ is obtainable from $D_y^p(q)$ by deleting the last row and column.

In the following, we argue that the dimension of the minimal static factor, l, will be bounded by the sum of the McMillan degree and the number of a minimal dynamic factor of the GDFM. Denote the transfer function from u_k to y_k as $W_p(q) = (D_y^p)^{-1}\begin{bmatrix} I_m & 0\end{bmatrix}'$. Assume that there exists p_0 such that the McMillan degree of $W_p(q)$ is the same as that of $W_{p_0}(q)$ (denoted as n) for $p \geq p_0$. Then, it follows from Theorem 5 that for $p > \max\{m+n, p_0\}$, $W_p(q)$ can be written as

$$W_p(q) = \begin{bmatrix} F(q) & 0 \\ G & I \end{bmatrix}^{-1} \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{17}$$

where $F(q)$ is a $(m+n)\times(m+n)$ polynomial matrix in q, G is a $(p-(m+n))\times(m+n)$ constant matrix and the identity matrix is $(p-(m+n))\times(p-(m+n))$.

Denote $y_k = \begin{bmatrix} y_{k,1}' & y_{k,2}' \end{bmatrix}'$ with $y_{k,1}$ being $(m+n)$-dimensional. Then an immediate consequence of (17) is that $y_{k,2} = -Gy_{k,1}$. This leads to $y_k = \begin{bmatrix} I_{m+n} & G' \end{bmatrix}' y_{k,1}$.

The above equation implies that the rank of the zero-lag covariance matrix of y_k is bounded by $m+n$ for $p > \max\{m+n, p_0\}$. This in turn shows that the dimension of a minimal static factor z_k, i.e. l, is bounded by $m+n$ for $p > \max\{m+n, p_0\}$.

7. Counting the number of real parameters in the canonical form

In this section, we shall derive an upper bound for the number of real parameters in a canonical AR model.

Based on the procedure of Theorem 2, it should be easy to obtain the following result by parameter counting.

Corollary 2. *Consider an AR description $D_p^{-1}(q)\begin{bmatrix} I_m & 0\end{bmatrix}'$ of $W_p(q)$ with $D_p(q)$ of degree k in q and denote the McMillan degree of $W_p(q)$ as n. Suppose the canonical form constructed using $D_p(q)$ is $\tilde{D}_p(q)$. Then the number of real parameters in $\tilde{D}_p(q)$ is at most $(n+m+1)p + mn + km^2$.*

Since k is independent of p, Corollary 2 demonstrates the linear relationship between p and the number of real parameters if the McMillan degree of $W_p(q)$ is bounded. This implies that the canonical AR model constructed in this paper does not suffer from the "curse of dimensionality". Recall that in the normal course of events, one might expect that a canonical form for the $p\times p$ denominator matrix $\tilde{D}_p(q)$ of a coprime fraction description of a tall $p\times m$ transfer function matrix $W_p(q)$ with numerator matrix $\begin{bmatrix} I_m & 0\end{bmatrix}'$ would be likely to have $O(p^2)$ real parameters.

8. Example

The main application of interest to us for singular AR models is in econometric modeling where the process may have dimension exceeding 200 or even 300. The following example is of much more modest dimensions, but nevertheless illustrates the canonical form. We start with a 3×1 transfer function

$$W(q) = \begin{bmatrix} \dfrac{1-q/2}{(-4+q)(-6+q)} \\ \dfrac{1-q/3}{(-6+q)(-5+q)} \\ \dfrac{1-q/2}{(-4+q)(-6+q)} + \dfrac{1-q/3}{(-6+q)(-5+q)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{(-4+q)(-6+q)} & 0 & 0 \\ 0 & \dfrac{1}{(-6+q)(-5+q)} & 0 \\ \dfrac{1}{(-4+q)(-6+q)} & \dfrac{1}{(-6+q)(-5+q)} & 1 \end{bmatrix} \times \begin{bmatrix} 1-q/2 \\ 1-q/3 \\ 0 \end{bmatrix} \tag{18}$$

such that

$$W(q) = A^{-1}(q)B(q)$$

with

$$A(q) = \begin{bmatrix} (-4+q)(-6+q) & 0 & 0 \\ 0 & (-6+q)(-5+q) & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

$$B(q) = \begin{bmatrix} 1-q/2 \\ 1-q/3 \\ 0 \end{bmatrix}. \tag{19}$$

Extending $B(q)$ to a unimodular matrix gives us

$$G(q) = \begin{bmatrix} 1-q/2 & 3/2 & 0 \\ 1-q/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$G^{-1}(q) = \begin{bmatrix} -2 & 3 & 0 \\ (2-2q/3) & (-2+q) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

which we use to obtain a numerator polynomial matrix of the form $[I\ 0]'$, i.e., $G(q)^{-1}B(q) = [1\ 0\ 0]'$ and

$$D(q) = G(q)^{-1}A(q)$$

$$= \begin{bmatrix} -2(q-4)(q-6) & 3(q-6)(q-5) & 0 \\ \left(2-\dfrac{2q}{3}\right)(q-4)(q-6) & (q-2)(q-6)(q-5) & 0 \\ -1 & -1 & 1 \end{bmatrix}. \tag{21}$$

As $D(q)$ and $[1\ 0\ 0]'$ are not coprime (there is a zero at $q=6$) we obtain

$$\bar{D}(q) = \begin{bmatrix} -2(-4+q)(-6+q) & 3(-6+q)(-5+q) & 0 \\ (2-2q/3)(-4+q) & (-2+q)(-5+q) & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

with $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -6+q & 0 \\ 0 & 0 & 1 \end{bmatrix}\bar{D}(q) = D(q)$ and $W(q) = \bar{D}^{-1}(q)[1\ 0\ 0]'$.

Now we have to choose V_{22} and V_{12} according to (6) to obtain our canonical form. As the last two rows of $\bar{D}(q)$ are already in row polynomial–echelon form (with pivot indexes equal to 2 and 3) $V_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, by using the same notation as in Section 4.2, we have $E_2 = \begin{bmatrix} (-2+q)(-5+q) & 0 \\ -1 & 1 \end{bmatrix}$ and $E_1 = [3(-6+q)(-5+q)\ 0]$. According to Eq. (7) $V_{12} = [-3\ 0]$.

Thus the canonical representative is

$$\tilde{D}(q) = \begin{bmatrix} 6(-4+q) & -12(-5+q) & 0 \\ (2-2q/3)(-4+q) & (-2+q)(-5+q) & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

with

$$W(q) = \tilde{D}^{-1}(q)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

9. Conclusions

In this paper, we have considered tall transfer function matrices of dimension $p \times m$ which have no finite zeros. Such transfer functions can be represented as singular AR models. A complete characterization has been provided on the observational equivalence class of all AR matrix fraction descriptions of the same transfer function. A canonical AR model has been constructed for the observational equivalence class.

The proposed canonical form has been shown to possess several desirable properties. The first property says that the canonical form proposed corresponds to a D of minimal degree k in (2). The second property reveals that, when the sequence $W_p(q)$ is nested and p is big enough, a nesting property for the canonical AR models associated with $W_p(q)$ can be found. The last property shows that the total number of real parameters in the canonical AR model grows linearly with the number of rows of the transfer function, i.e. p, when p varies, with $W_p(q)$ remaining nested and of bounded McMillan degree.

Our result has been motivated by the analysis of Generalized Dynamic Factor Models which are used for modeling and forecasting for high-dimensional time series. The following problems are left for further research:

(1) The integer valued parameters $k_1, \ldots, k_{p-m}$ (row degrees) and $p_1, \ldots, p_{p-m}$ (pivot indices) associated with our canonical form do not directly describe the set of all canonical forms associated with the same integers. This is a consequence of the fact that we have neither been able to give exact bounds of the degree of those columns of $\tilde{D}_1$ not corresponding to pivot indices, nor that the corresponding real valued parameters are free.
(2) The structure theory in this paper has been developed in view of the more general problem of identification in particular of GDFMs. Major open questions are the estimation of integer and real valued parameters.

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