Practical Estimation of Rapidly Varying Channels for OFDM Systems

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***Abstract*—We propose a novel pilot-aided algorithm for estimation of rapidly varying wireless channels in OFDM systems. Our approach is specifically designed for channels varying on the scale of a single OFDM symbol duration, which occur, for example, in mobile WiMAX, WAVE, and DVB-T. From the pilot information, we recover information about the channel taps in the framework of the Basis Expansion Model (BEM). We derive explicit formulas for the BEM coefficients in terms of the receive signal. Algebraically, the algorithm is FFT-based, and can be easily implemented in hardware. For a system with L channels taps, our method uses $\mathcal{O}(L \log L)$ operations and $\mathcal{O}(L)$ memory per OFDM symbol. This complexity is the best possible up to the order of magnitude. Previously published methods require $\mathcal{O}(L^2)$ operations and $\mathcal{O}(L^2)$ memory.**

Numerical simulations illustrate performance gains achieved by our estimator at sufficiently high Doppler frequencies. Our approach does not assume any prior statistical information.

***Index Terms*—Channel estimation, OFDM, time-varying channel, basis expansion model, mobile WiMAX.**

I. Introduction

A. Motivation and Previous Work

ORTHOGONAL frequency-division multiplexing (OFDM) is a popular multicarrier modulation technique with several desirable features, e.g. robustness against multipath propagation and high spectral efficiency. OFDM is increasingly used in high-mobility wireless communication systems, e.g. mobile WiMAX (IEEE 802.16e), WAVE (IEEE 802.11p), and DVB-T (ETSI EN 300 744). Usually, OFDM systems are designed so that no channel variations occur within an individual OFDM symbol duration. Recently, however, there has been an increasing interest in rapidly varying channels, with the channel coherence time less than the OFDM symbol duration. Rapid channel variations are caused by, for example, user mobility, or carrier frequency offsets. Moreover, OFDM is a likely candidate for future aeronautical communication systems, see [1]. In such applications, substantial relative Doppler shifts are possible.

Rapidly varying channels act like time-varying filters with a finite impulse response (FIR). In the case of frequency-selective channels in OFDM systems, estimation in the frequency-domain is unmatched in simplicity and accuracy, see [2].

In the case of doubly-selective channels, the channel taps, change with time. The Basis Expansion Model (BEM) is commonly used to model doubly selective channels, see [3], [4], [5], [6].

The BEM approximates the channel taps by combinations of prescribed basis functions in the time domain. In this context, channel estimation amounts to approximate computation of the basis coefficients. Several bases have been proposed for modeling doubly-selective channel taps. The BEM with complex exponential (CE-BEM) [7], [8] uses a truncated Fourier series, and is remarkable because the resulting frequency-domain channel matrix is banded. However, this method has a limited accuracy due to a large modeling error. Specifically, [4], [9] observe that the reconstruction with a truncated Fourier series introduces significant distortions at the boundaries of the data block. The errors are due to the Gibbs phenomenon, and manifest themselves as a spectral leakage, especially in the presence of significant Doppler spreads. A more suitable exponential basis is provided by the Generalized CE-BEM (GCE-BEM) [10], which employs complex exponentials oversampled in the frequency domain. A basis of discrete prolate spheroidal wave functions is discussed in [4], [11]. Finally, the polynomial BEM (P-BEM) is presented in [12]. Definitive references on pilot-aided transmission in doubly-selective channels are [13], [14].

For channels varying at the scale of one OFDM symbol duration, a general framework for estimation of the BEM coefficients is developed in [3]. With L channel taps, the method requires $\mathcal{O}(L^2)$ operations and $\mathcal{O}(L^2)$ memory.

Contemporary broadband outdoor applications use scalable OFDM, in which the number of subcarriers increases with the available bandwidth, while keeping the symbol duration and the intercarrier frequency spacing fixed. This is accomplished by increasing the sampling rate, which in turn increases the number of discrete multipaths L, see [15, p. 370], [16, p. 54] for the relation between the number of resolvable multipaths and the bandwidth. For example, mobile WiMAX (IEEE 802.16e) with K subcarriers typically exhibits a discrete path delay of ca. $\frac{K}{8}$, see [17]. The algorithm developed in [3] requires the solution of a linear system of size $\mathcal{O}(L)$, which is equivalent to $\mathcal{O}(K)$ for a broadband communication channel. Since the coefficient matrix in the linear system is independent of the receive signal, the inverse can be precomputed off-line. Therefore for a broadband application,

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the algorithm developed in [3] has a complexity of $\mathcal{O}(K^2)$.

Therefore it is important to develop estimation algorithms, whose computational and memory requirements scale with the number of OFDM subcarriers. Furthermore, such new methods can be combined with recent low-complexity equalization algorithms, see [18]–[20].

B. Contributions

We develop a systematic approach to wireless channel estimation, which is aimed at channels varying on the scale of a single OFDM symbol duration. From the pilot information, we compute the Fourier coefficients of the channel taps, and then the BEM coefficients of the taps. We use a frequency-domain Kronecker delta (FDKD) pilot arrangement, see [13]. The proposed estimation algorithm is FFT-based, and therefore fairly easy to implement in hardware. We use the basis of Legendre polynomials, but the method can be applied with arbitrary bases. The BEM with the Legendre polynomials falls within the framework of the polynomial BEM (P-BEM). The P-BEM with the Legendre polynomials is, however, more numerically stable than the P-BEM using monomials introduced in [12].

The main contributions of this work can be summarized as follows.

- We propose a pilot-aided method for channel estimation in OFDM systems, which explicitly separates the computation of the Fourier coefficients of the channel taps, and a subsequent computation of the BEM coefficients of the channel taps.
- We formulate a fast and accurate algorithm for approximate computation of the Fourier coefficients of the channel taps from the receive signal using an FDKD-type pilot placement.
- We derive explicit formulas for the BEM coefficients in terms of the Fourier coefficients. We illustrate the proposed method using the basis of the Legendre polynomials.
- With L channel taps, the proposed method requires overall $\mathcal{O}(L \log L)$ operations and $\mathcal{O}(L)$ memory per OFDM symbol, and this complexity is the best possible up to the order of magnitude.

Previously published methods, e.g. [3], require $\mathcal{O}(L^2)$ operations and memory. The improvement in operation count from $\mathcal{O}(L^2)$ to $\mathcal{O}(L \log L)$ is remarkable. For example, mobile WiMAX with $K = 2048$ subcarriers exhibits a typical discrete path delay of $L = \frac{K}{8} = 256$, see [17]. In this case, the operation count of the proposed algorithm is lower by a factor of approximately $\frac{L}{\log L} = 32$, see Section IV-E for details. Moreover, reducing the memory use from $\mathcal{O}(L^2)$ to $\mathcal{O}(L)$ further accelerates the execution, since the computations better utilize a fast cache.

We emphasize, that the proposed algorithm does not reconstruct the full channel matrix, but only estimates the BEM coefficients of the channel taps. It can be combined with a recent equalization algorithm, which only uses the BEM coefficients, see [20]. In this way, the whole transmission is efficient in terms of both computational complexity and memory. If needed, the channel matrix can be easily reconstructed from the BEM coefficients. However, reconstruction of the full channel matrix with K subcarriers requires $\mathcal{O}(LK)$ operations and memory, and dramatically increases the use of resources.

Extensive computer simulations show that our scheme is superior to the estimation method presented in [3]. At higher mobile velocities, our method is better than conventional time-invariant least squares (LS) estimation [2]. Our transmission is typically simulated for a user velocity of 300 km/h, and energy per data bit to noise spectral density (E_b/N_0) of 20 dB.

The paper is organized in the following way. In Section II, we discuss theoretical foundations of the proposed estimation algorithm. In Section III, we introduce the system model, and then the proposed channel estimator in Section IV. We present simulation results in Section V, and our conclusions in Section VI.

II. Theoretical Foundations of the Estimation Algorithm

A. Overview

We develop a systematic framework for channel estimation in OFDM systems with significant channel variations within one OFDM symbol duration. We divide this task into two separate steps,

- pilot-aided estimation of the Fourier coefficients of the channel taps.
- estimation of the BEM coefficients of the channel taps.

B. Fourier Coefficients of the Channel Taps

We use pilot symbol assisted modulation (PSAM), with uniformly distributed blocks of pilot sub-carriers, each block having the FDKD pilot arrangement [13], [14]. Pilots are inserted in every OFDM symbol in order to capture rapid variations of path gains. The first few Fourier coefficients of the channel taps are computed for each individual OFDM symbol. In Section IV-C, we derive an efficient and accurate method for estimation of the Fourier coefficients of the channel taps from the receive signal. The Fourier coefficients are computed using FDKD-type pilot carriers, see Sections IV-A and IV-C. Our algorithm is FFT-based, and fairly easy to implement.

Since the channel taps are in general non-periodic, a straightforward reconstruction of the channel taps as truncated Fourier series from the estimated Fourier coefficients is inaccurate. This problem is well known, and is commonly referred to as the Gibbs phenomenon. In the context of wireless channels, the failure of reconstruction with the Fourier basis is discussed in detail in [9]. However, it turns out that the information content of the Fourier coefficients can be used more effectively than in the straightforward approach, as we explain in the next subsection.

C. BEM Coefficients of the Channels Taps

The second stage is to estimate BEM coefficients of the channel taps from their Fourier coefficients in a way which remedies the Gibbs phenomenon. Several accurate algorithms have been proposed for overcoming the Gibbs phenomenon, see [21], [22] or [23]. A theoretical analysis of the resolution

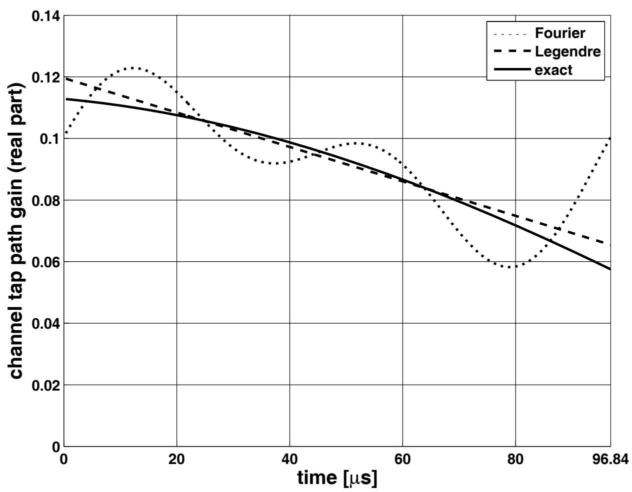


Fig. 1. A typical channel tap (real part) across one OFDM symbol, the normalized Doppler equals 0.2.

of the Gibbs phenomenon is beyond the scope of this paper, see a review article [21], and also [23], and the references therein for further details.

In this paper, for illustration, we implement our method using the basis of Legendre polynomials adapted to individual OFDM symbols, however our approach can accommodate arbitrary bases(see Section IV-D for details). We make an a priori assumption that the channel taps are analytic, although not necessarily periodic. Such functions can be represented by a rapidly converging expansion known as the Legendre series. Specifically, the Legendre series of an analytic function converges exponentially fast, see [24], p. 52, Theorem 10. A truncated Fourier series is converted into a truncated Legendre series by orthogonal projection. We emphasize, that no truncated Fourier series is ever formed. Instead, the Legendre coefficients are computed from the Fourier coefficients by applying a matrix, whose entries are derived in Section IV-D. Specifically, the matrix entries are the Legendre coefficients of complex exponentials, and have explicit expressions in terms of the spherical Bessel functions of the first kind [25]. A corresponding matrix for another bases in place of the Legendre polynomials can be readily obtained using a method described in Sec. IV-D.

Although the Legendre coefficients are computed from the Fourier coefficients, a truncated Legendre series is in fact more accurate than a truncated Fourier series with a similar number of terms. The quality of the reconstruction with the truncated Legendre series is illustrated in Fig. 1, where the real part of a typical channel tap is plotted along with its approximation by a truncated Fourier series and a truncated Legendre series. In Fig. 1, we use a three-term Fourier series, and then a two-term Legendre series, see Section IV for details. The observed improvement can be explained by a dramatic difference in approximation properties of the two bases considered. In the case of a non-periodic function, the approximation error of the truncated Fourier series cannot be made arbitrarily small. On the other hand, the Legendre series converges exponentially fast. Other bases which well approximate channel taps can be used in place of the Legendre polynomials.

Our numerical simulations confirm that for doubly-selective channels estimation with a truncated Legendre series is dramatically more accurate than the reconstruction with a truncated Fourier series.

III. System Model

A. Transmitter-Receiver Model

We consider an equivalent baseband representation of a single-antenna OFDM system with K subcarriers. We assume a sampling period of $T_{\mathrm{s}} = 1/B$, where B denotes the transmit bandwidth. A cyclic prefix of length L_{cp} is used in every OFDM symbol. We choose L_{cp} so large that $L_{\mathrm{cp}}T_{\mathrm{s}}$ exceeds the channel's maximum delay, in order to avoid inter symbol interference (ISI). Consequently, throughout this paper, we deal with one OFDM symbol at a time.

Each subcarrier is used to transmit a symbol $A[k]$ ($k = 0, \dots, K-1$) from a finite symbol constellation. A subset of these symbols serves as pilots for channel estimation (cf. Section IV-C). The OFDM modulator uses the inverse discrete Fourier transform (IDFT) of size K to map the frequency-domain transmit symbols $A[k]$ to the time-domain transmit signal $x[n]$

$$x[n] = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} A[k]\, e^{j2\pi \frac{nk}{K}}, \tag{1}$$

$n = -L_{\mathrm{cp}}, \dots, K-1$.

After discarding the cyclic prefix, the receive signal satisfies

$$y[n] = \sum_{l=0}^{L-1} h_l[n]\, x[n-l] + w[n], \qquad n = 0, \dots, K-1, \tag{2}$$

where, $w[n]$ denotes circularly complex additive noise of variance N_0, $h_l[n]$ is the complex channel tap associated with the delay l, and L is the channel length (maximum discrete-time delay). Consequently, the channel's maximum delay equals $(L-1)T_s$. For simplicity, we make the worst-case assumption $L = L_{\mathrm{cp}}$. The OFDM demodulator performs a discrete Fourier transform (DFT) of size K to obtain the frequency-domain receive signal $Y[k]$,

$$Y[k] = \frac{1}{\sqrt{K}} \sum_{n=0}^{K-1} y[n]\, e^{-j2\pi \frac{nk}{K}}. \tag{3}$$

Combining equation (2) with equation (3) we get

$$Y[k] = \sum_{l=0}^{L-1} (H_l * X_l)[k] + W[k], \tag{4}$$

where $*$ denotes the cyclic convolution of length K, and $k = 0, \dots, K-1$. In this expression, the quantities $Y[k]$, $H_l[k]$, $X_l[k]$, and $W[k]$ denote the DFTs of $y[n]$, $h_l[n]$, $x[n-l]$, and $w[n]$, respectively. Specifically,

$$H_l[k] = \frac{1}{\sqrt{K}} \sum_{n=0}^{K-1} h_l[n]\, e^{-j2\pi \frac{nk}{K}} \tag{5}$$

are the Fourier coefficients of the individual channel taps, and

$$X_l[k] = e^{-j2\pi \frac{lk}{K}} A[k] \tag{6}$$

for $k = 0, \dots, K-1$ and $l = 0, \dots, L-1$.

B. BEM With the Legendre Polynomials

In this subsection, we discuss a BEM with the Legendre polynomials. However, other bases which well approximate channel taps can be used in place of the Legendre polynomials.

Each channel tap $h_l[n]$ is modeled as a linear combination of the first M Legendre polynomials rescaled to a single OFDM symbol duration (without the cyclic prefix)

$$h_l[n] = \sum_{m=0}^{M-1} b_{lm}\, p_m[n]\,, \qquad l = 0, \ldots, L-1, \qquad (7)$$

where b_{lm} is the mth Legendre coefficient of the lth channel tap, and M is the BEM model order. Furthermore,

$$p_m[n] = p_m(nT_s), \qquad (8)$$

and,

$$p_m(t) = P_m\left(\frac{2t}{KT_s} - 1\right), \qquad 0 \le t \le KT_s, \qquad (9)$$

where P_m is the Legendre polynomial of degree m, as defined in Appendix.

IV. Proposed Channel Estimator

A. Analysis of Intercarrier Interactions

In our system model, channel estimation amounts to computing the LM BEM coefficients $\{b_{lm}\}$ from the receive signal $Y[k]$ $(y[n])$ and the pilot symbols. We first estimate the Fourier coefficients of the channel taps (cf. (5)), and then we compute approximate BEM coefficients from the Fourier coefficients, as discussed in Section IV-D.

For a fixed positive integer D, we approximate the channel taps with their D-term Fourier series

$$h_l[n] \approx \sum_{d=D^-}^{D^+} H_l[d]\, e^{j2\pi\frac{dn}{K}}, \qquad (10)$$

where $D^- = -\lfloor (D-1)/2 \rfloor$ and $D^+ = \lfloor D/2 \rfloor$ ($\lfloor\cdot\rfloor$ denotes the floor operation). Clearly, $D^- \leqslant 0 \leqslant D^+$, and $D^+ - D^- = D - 1$. For a negative index $-d$, $H_l[-d]$ is set equal to $H_l[K-d]$, which in turn is defined in Equation (5).

The representation of the channel taps described by equation (10) is commonly known as the Basis Expansion Model with complex exponentials (CE-BEM) [7], [8]. We use this model only for computation of the Fourier coefficients of the channel taps, but not for reconstruction of the taps themselves. Combining (2), (4), (6) and (10), we obtain

$$\begin{aligned} Y[k] &= \sum_{l=0}^{L-1}\sum_{d=D^-}^{D^+} H_l[d]\, X_l[k-d] + \widetilde{W}[k] \\ &= \sum_{l=0}^{L-1}\sum_{d=D^-}^{D^+} H_l[d]\, e^{-j2\pi\frac{l(k-d)}{K}} A[k-d] + \widetilde{W}[k], \\ &= \sum_{d=D^-}^{D^+} A[k-d] \sum_{l=0}^{L-1} H_l[d]\, e^{-j2\pi\frac{l(k-d)}{K}} + \widetilde{W}[k], \quad (11) \end{aligned}$$

where $k = 0, \ldots, K-1$, and $\widetilde{W}[k]$ denotes additive noise $W[k]$ combined with the approximation error resulting from (10). From the above equation, we notice that the value $Y[k]$ depends only on the D transmit symbols $A[k-D^+], \ldots, A[k-D^-]$ at the neighboring subcarriers.

The input-output relationship in the frequency domain (11) is visualized in [3, Fig. 3].

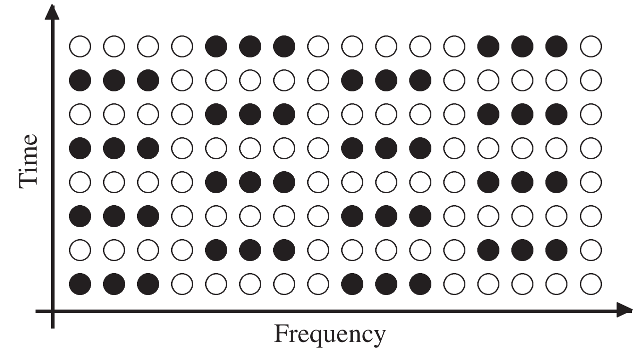


Fig. 2. An illustration of the proposed pilot arrangement with $K = 16$, $L = 2$, and $D = 2$ ('∘' represents data symbols and '•' represents pilot symbols). Only the central pilot in each block is non-zero. The offset k_0 is chosen equal to 0 and 4 in the even and odd symbol periods, respectively.

B. Pilot Arrangement

We assume that $I = \frac{K}{L}$ is an integer, which can always be achieved by an appropriate choice of L. Typically, both K and L are integer powers of 2. We use an FDKD pilot arrangement. Within each OFDM symbol, we distribute pilots in the frequency domain in L blocks of size $2D-1$ each, uniformly spaced every I subcarriers. Of course, this is only possible if $2D-1 \leqslant I$. Denoting the location of the first pilot subcarrier by k_0, $0 \leqslant k_0 \leqslant I-(2D-1)$, the pilot locations have the form

$$k_0 + q + iI, \qquad (12)$$

where $q = 0, \ldots, 2D-2$, and $i = 0, \ldots, L-1$. An example of such an arrangement is shown in Fig. 2. Within each block, all the pilot values are zero, except for the central pilot, which is set to a value a_0 common to all blocks. Thus only the L symbols $A[k_0+D-1+iI]$, $i = 0, \ldots, L-1$, carry non-zero pilots.

C. Estimation of Fourier Coefficients

We create D length-L subsequences of the frequency-domain receive signal $Y[k]$ by uniform subsampling as follows

$$\tilde{Y}_d[i] = Y[k_0 + D^+ + d + iI], \qquad (13)$$

for $i = 0, \ldots, L-1$ and $d = 0, \ldots, D-1$. From (11), we obtain

$$\tilde{Y}_d[i] = \sum_{d'=D^-}^{D^+} A\left[k_0 + D^+ + d + iI - d'\right] \times \sum_{l=0}^{L-1} H_l[d']\, e^{-j2\pi\frac{l(k_0+D^++d+iI-d')}{K}} + \widetilde{W}_d[i]\,, \quad (14)$$

where $\widetilde{W}_d[i] = \widetilde{W}[k_0 + D^+ + d + iI]$. In view of our pilot arrangement (12), it is clear that for any $d = 0, \ldots, D-1$ and $i = 0, \ldots, L-1$, the summation in formula (14) involves the known pilot symbols, but no data symbols. Moreover, if $d' = d + D^-$, then

$$A\big[k_0 + D^+ + d + iI - d'\big] = A\big[k_0 + D - 1 + iI\big] = a_0. \quad (15)$$

By construction, all the other pilot symbols in (14) are zero, and (14) reduces to the following

$$\tilde{Y}_d[i] = a_0 \sum_{l=0}^{L-1} H_l[d + D^-]\, e^{-j2\pi \frac{l(k_0 + D - 1 + iI)}{K}} + \widetilde{W}_d[i]. \quad (16)$$

Performing the length-L IDFT with respect to the variable i, we obtain

$$\begin{aligned} \tilde{y}_d[l] &= \frac{1}{\sqrt{L}} \sum_{i=0}^{L-1} \tilde{Y}_d[i]\, e^{j2\pi \frac{il}{L}} \\ &= a_0 \sqrt{L}\, H_l[d + D^-]\, e^{-j2\pi \frac{l(k_0 + D - 1)}{K}} + \tilde{w}_d[l], \end{aligned} \quad (17)$$

where $\tilde{y}_d[l]$ and $\tilde{w}_d[l]$ denote the IDFTs of $\tilde{Y}_d[i]$ and $\tilde{W}_d[i]$, respectively. Ignoring the noise term $\tilde{w}_d[l]$, the solution of the system of DL equations (17) gives approximate Fourier coefficients of the channel taps

$$\hat{H}_l[d] = \frac{1}{a_0 \sqrt{L}}\, e^{j2\pi \frac{l(k_0 + D - 1)}{K}}\, \tilde{y}_{(d - D^-)}[l], \quad (18)$$

for, $d = D^-, \dots, D^+$, and $l = 0, \dots, L-1$. We observe that the computation of the quantities $\hat{H}_l[d]$ is accomplished using D IDFTs of length L, where typically $D \leq 3$. By way of contrast, previous approaches to the computation of the Fourier (CE-BEM) coefficients from the receive signal over one OFDM symbol require $\mathcal{O}(L^2)$ operations and memory, see Subsection IV-B in [3].

Reconstruction of the channel taps as truncated Fourier series using equation (10) and the estimated Fourier coefficients (18) is inaccurate because of the Gibbs phenomenon, see Fig. 1. In the next subsection, we present a simple method for the mitigation of the Gibbs phenomenon by replacing the complex exponentials with a more suitable basis.

D. Estimation of the BEM Coefficients

We regard the channel taps as analytic functions of time, and represent them by means of a rapidly converging expansion known as the Legendre series [24]. It turns out, that one of the simplest methods to reduce the Gibbs phenomenon is to convert a truncated Fourier series into a truncated Legendre series by orthogonal projection, see [21] for theoretical foundations. We describe how this is accomplished by a linear mapping transforming the Fourier coefficients into approximate Legendre coefficients, without ever creating the truncated Fourier series (10) explicitly. This approach can be used with any other basis in place of the Legendre polynomials.

In order to derive this linear mapping, let us project the truncated Fourier expansion (see equation (10)) onto the rescaled Legendre polynomials p_m (see equation (9)), which form an orthogonal basis on the interval $[0, KT_s]$. Denoting the mth Legendre coefficient of the exponential function $e^{j2\pi \frac{d}{KT_s} t}$ by $e_d(m)$, we have

$$e_d(m) = \frac{\int_0^{KT_s} e^{j2\pi \frac{d}{KT_s} t} p_m(t) dt}{\int_0^{KT_s} p_m^2(t) dt}. \quad (19)$$

In the appendix, we derive an explicit expression for $e_d(m)$

$$e_d(m) = j^m (2m+1)\, (-1)^d\, \mathrm{j}_m(\pi d), \quad (20)$$

where j_m is the spherical Bessel function of the first kind and order m, which is also defined in the appendix. Combining this equation with (7) and (10), we obtain

$$\hat{b}_{lm} = j^m (2m+1) \sum_{d=D^-}^{D^+} (-1)^d\, \mathrm{j}_m(\pi d)\, \hat{H}_l[d], \quad (21)$$

where $\hat{b}_{lm}$ denotes the estimate of b_{lm}. The linear mapping (21) amounts to applying the $M \times D$ matrix $\boldsymbol{J}$ with entries

$$\boldsymbol{J}_{md} = j^{(m-1)} (2m-1)\, (-1)^d\, \mathrm{j}_{m-1}\big(\pi(d - D^+)\big) \quad (22)$$

to the length-D vector $(\hat{H}_l[D^-], \dots, \hat{H}_l[D^+])^T$ of the estimated Fourier coefficients, resulting in the length-M vector $(\hat{b}_{l0}, \dots, \hat{b}_{l(M-1)})^T$ of the Legendre coefficients.

If necessary, the channel taps can be reconstructed as truncated Legendre series using the coefficients $\hat{b}_{lm}$ as in equation (7)

$$\hat{h}_l[n] = \sum_{m=0}^{M-1} \hat{b}_{lm}\, p_m[n], \qquad l = 0, \dots, L-1. \quad (23)$$

However, the estimated BEM coefficients can be directly used for equalization, without actually creating the channel matrix (see [20]).

The proposed estimation of the BEM coefficients is not limited to the Legendre polynomials, but can be used with arbitrary bases. The BEM coefficients are constructed from the Fourier coefficients according to equation (21). This amounts to applying the matrix $\boldsymbol{J}$, whose entries are given by an expression analogous to equation (19). In the general case, the integrals in (19) might not be available analytically, and are computed numerically instead.

E. Algorithm Summary and Complexity

We summarize the proposed channel estimation algorithm as applied to one OFDM symbol, assuming that OFDM demodulation according to (4) has already been performed, and that the matrix $\boldsymbol{J}$ in (22) has been precomputed.

Step 1: Apply the size-L IDFT to each of the D subsequences $\tilde{Y}_i[d]$ according to (17).

Step 2: Compute the Fourier coefficient estimates $\hat{H}_l[d]$ according to (18).

Step 3: Calculate the estimates $\hat{b}_{lm}$ of the BEM coefficients via (21).

We note that conventional time-invariant least squares (LS) estimation [2] is a special case of our algorithm with model parameters $D = 1$, $M = 1$. It is essential for practical applications, that the estimated BEM coefficients can be directly used for equalization, without ever creating the channel matrix (see [20]).

In Table I, we report the computational complexity of our scheme in complex floating point operations. Step 1 requires $DL \log L$ complex operations, while Step 2 uses DL complex multiplications. Step 3 requires DM complex operations per tap, with the total of DML complex operations. Altogether,

TABLE I
OPERATION COUNT FOR THE PROPOSED ALGORITHM PER OFDM SYMBOL OBTAINED FOR $K = 256$, $L = 32$, $D = 3$, AND $M = 2$ (AS USED IN THE SIMULATIONS).

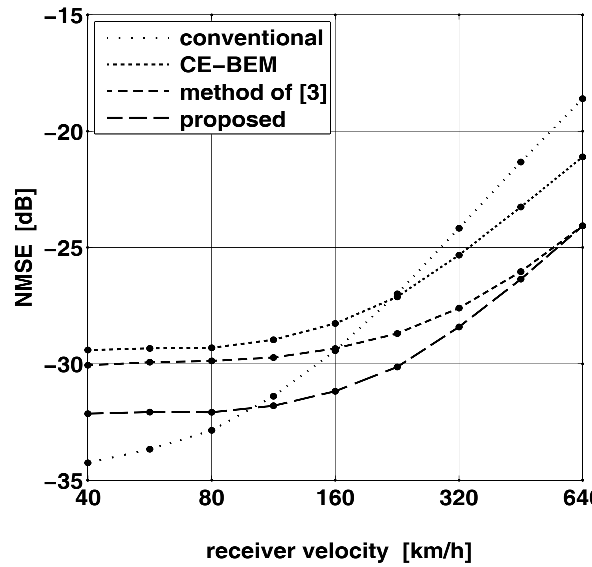
step	*description*	*operations*	*example*
1	L-point IDFTs of $\tilde{Y}_k[i]$	$DL \log L$	480
2	computation of Fourier coefficients	DL	96
3	computation of BEM coefficients	MDL	192

the proposed algorithm performs $D(\log L + M + 1)L$ complex floating point operations. For comparison, the estimation method presented in [3] requires at least DML^2 operations.

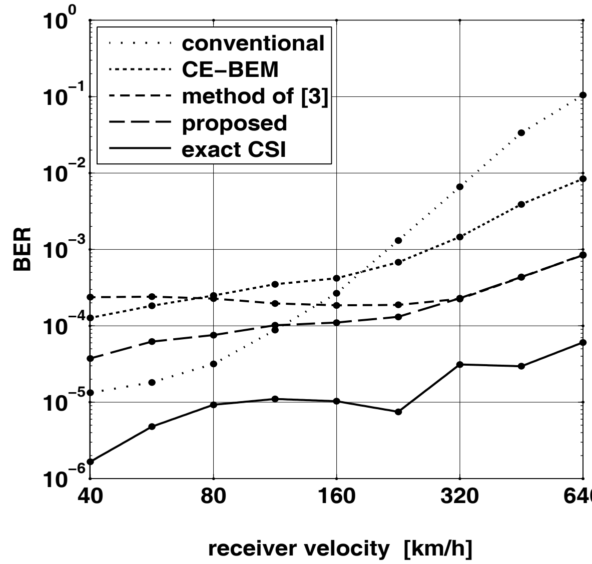
The precomputed matrix $\boldsymbol{J}$ is dimensioned $M \times D$, and its storage is trivial, since typically $M \leqslant D \leqslant 3$. The estimated Fourier coefficients are stored as DL floating point complex numbers, while the estimated BEM coefficients are stored as ML complex numbers. The Fourier coefficients are discarded during the computation of the BEM coefficients. Overall, the proposed algorithm stores approximately DL complex numbers in addition to the receive signal. On the other hand, the method of [3] requires at least DML^2 of storage, so the improvement is extraordinary.

F. Choice of Parameters

The proposed algorithm uses only two parameters: the number of estimated Fourier coefficients per channel tap, denoted by D, and the number of basis functions used to reconstruct the channel taps, denoted by M. For doubly selective channels, D should increase when the Doppler shift increases, since more Fourier coefficients are needed to describe the channel taps in high Doppler regimes. In Section V, we present results obtained with $D = 1$ and $D = 3$. The proposed algorithm requires $(2D - 1)L$ pilot carriers to estimate D Fourier coefficients of L discrete channel taps when the FDKD pilot arrangement is used. Hence the value of the parameter D should be selected considering the Doppler effect in the channel, the required accuracy, and the required bit rate. For example, if an accurate data transmission is required, one should use $D = 3$, although the data rate is lower due to the pilot overhead. This is the case e.g. in transmission of binary executables. On the other hand, for online video streaming, where data rate should be high, and higher BERs are acceptable, one should use $D = 2$. A discussion of the number of samples used for estimation and the accuracy of the estimation can be found in Subsection IV-D of [3]. The number of basis functions M should be large enough to accurately model the channel taps. However, smaller values of M improve conditioning of the matrix transforming the Fourier coefficients to the basis coefficients.



(a)



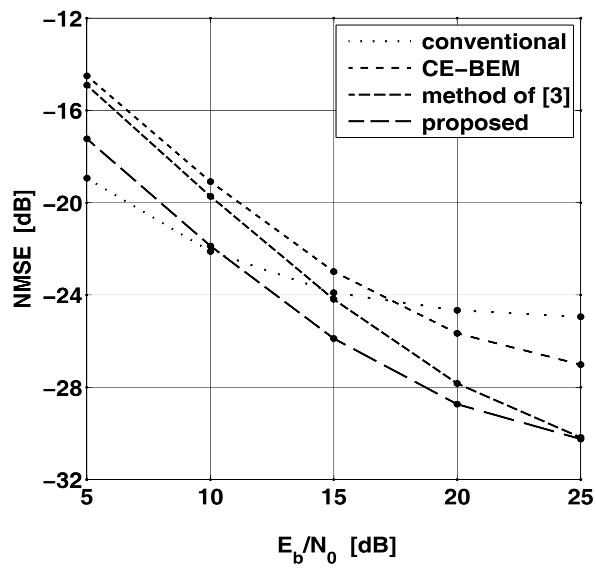
(b)

Fig. 3. (a) NMSE versus receiver velocity and (b) BER versus receiver velocity for a fixed SNR of $E_b/N_0 = 20\,\text{dB}$.

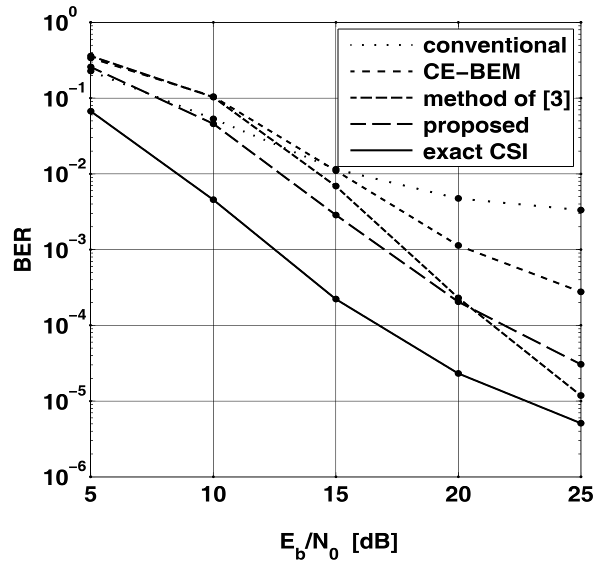
V. NUMERICAL SIMULATIONS

A. Setup

Our transmission simulation setup conforms to mobile WiMAX standards (IEEE 802.16e), but our method is general, and allows a variety of OFDM transmission schemes. We simulate a coded OFDM system with $K = 256$ subcarriers, the transmit bandwidth $B = 2.8\,\text{MHz}$, and the carrier frequency $f_c = 5.8\,\text{GHz}$. The length of the cyclic prefix is $L_{\text{cp}} = 32$, and the total symbol duration is $102.9\,\mu\text{s}$. The information bits are encoded using a convolutional code of rate 1/2, passed through an interleaver, and mapped to 4-QAM symbols. We insert pilots as described in Section IV-B. We use the MATLAB



(a)



(b)

Fig. 4. (a) NMSE versus SNR and (b) BER versus SNR for a fixed receiver velocity of 300 km/h.

Communications Toolbox to create a Rayleigh fading channel with a maximum delay of $11.4\,\mu$s, which corresponds to the worst case of $L = L_{\rm cp} = 32$ taps. All channel taps have an average path gain of -2 dB, and a Jakes Doppler spectrum. This channel model is chosen only for illustration, our method does not make any particular assumptions about the wireless channel. The normalized Doppler frequency ν is related to the receiver velocity v by the formula

$$\nu = \frac{v}{c} f_c T_s K, \tag{24}$$

where f_c is the carrier frequency, $\frac{1}{T_s K}$ is the intercarrier frequency spacing, and c is the speed of light. The receiver performs channel estimation followed by the MMSE equalization [26] and decoding. We use MMSE equalization as our benchmark in order to obtain results independent of any particular equalization method. We compare the results obtained by our estimator (using $D = 3$ Fourier modes and $M = 2$ Legendre polynomials) with those obtained by conventional time-invariant LS estimation for frequency selective channels (see [2]), with those obtained by an estimator based on the CE-BEM with $D = 3$ complex exponentials, and with results obtained using the method of [3]. Each of the schemes uses the same density of pilots. Specifically, the number of pilot carriers equals $(2D - 1)L = 160$. Additionally, we report the bit error rate (BER) obtained using the exact channel state information (CSI). The normalized mean squared error (NMSE) is computed as the expected mean square error between the exact channel tap $h_l(t)$, and the estimated channel tap $\hat{h}_l(t)$, normalized by the power of the exact channel. The BER and the NMSE are computed by averaging over $100,000$ OFDM symbols in order to capture even extremely low BERs.

B. Results of Simulations

Fig. 3 shows the BER and the NMSE as functions of the receiver velocity for a fixed signal-to-noise ratio (SNR) with $E_b/N_0 = 20\,$dB. Here, E_b denotes the energy per data bit, excluding the pilots, and N_0 is the variance of the AWGN. As expected, the performance deteriorates with increasing velocity. For the chosen system parameters, conventional time-invariant LS estimation is the best of all the methods at velocities less than 113 km/h (5.6% normalized Doppler). We note, the LS estimation is a special case of the proposed estimation algorithm with the Fourier model order $D = 1$ and the Legendre model order $M = 1$. For rapidly varying channels occurring at velocities over 113 km/h, our estimator with the Fourier model order $D = 3$ and the Legendre model order $M = 2$ performs best, having approximately one order of magnitude lower a BER than that of the CE-BEM. Consequently, the proposed method allows us to adapt the model order to the severity of the Doppler effect for better estimation. We also notice that the proposed algorithm consistently gives a lower BER than the method of [3], but approximately one order of magnitude greater than the one obtained using the exact CSI.

Fig. 4 shows the BER and the NMSE as functions of the SNR at a fixed receiver velocity of 300 km/h. This velocity corresponds to a maximum Doppler shift of 1.61 kHz, which is about 14.7% of the subcarrier spacing. We note, that from the vantage point of a stationary receiver, the Doppler effect of a moving reflector is twice as large as that of a moving transmitter. Consequently, the same Doppler effect is caused by a reflector moving with velocity 150 km/h, which is common in modern mobile environments.

Our scheme achieves a BER of $2.9 \cdot 10^{-3}$ at $E_b/N_0 = 15$ dB, and a BER of $2.0 \cdot 10^{-4}$ at $E_b/N_0 = 20\,$dB. It consistently outperforms the LS and the CE-BEM-based estimation methods, especially at higher SNRs. At a BER of $1.0 \cdot 10^{-3}$, our estimator outperforms the CE-BEM by about 3 dB. We also notice that the proposed algorithm gives a BER approximately one order of magnitude higher than the one obtained with the exact CSI. At high velocities, the BERs of the proposed algorithm are comparable to those obtained using the method of [3]. However, the computational complexity of the proposed algorithm is only $\mathcal{O}(L \log L)$, in contrast to the complexity of $\mathcal{O}(L^2)$ required by the method of [3].

VI. Conclusions

We develop a novel, low-complexity channel estimator for OFDM systems, which is reliable at high Doppler spreads. The main idea is an FFT-based estimation of the Fourier coefficients of the channel taps, followed by a conversion to BEM coefficients. The BEM coefficients of the channel taps are computed from explicit formulas involving the pilot values and the receive signal. We use the basis of Legendre polynomials, but the approach can be applied with arbitrary bases. Our method is meant to be combined with equalization algorithms, which only use the BEM coefficients, without actually creating the channel matrix.

Conventional time-invariant least-squares (LS) estimation is a method of choice for doubly-selective channels with low Doppler spreads. Our proposed algorithm is aimed at doubly-selective channels with high Doppler spreads, corresponding to reflector velocities in the range of $60 - 200$ km/h and a carrier frequency of 5.8 GHz. The LS estimation is a special case of the proposed method with the Fourier model order $D = 1$ and the BEM model order $M = 1$. At higher Doppler spreads, reliable channel estimates are obtained with higher models orders, at the expense of the transmission capacity.

For a system with L channels taps, our method uses $\mathcal{O}(L \log L)$ operations and $\mathcal{O}(L)$ memory per OFDM symbol. This complexity is the best possible up the order of magnitude. Previously published methods require $\mathcal{O}(L^2)$ operations and $\mathcal{O}(L^2)$ memory.

VII. Appendix

A. Legendre Polynomials

The Legendre polynomial P_n of degree $n = 0, 1, \ldots$, is defined by the formula [27, Sec. 22.11.5],

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n\right]. \qquad (25)$$

For example,

$$P_0(x) = 1, \qquad (26)$$
$$P_1(x) = x, \qquad (27)$$
$$P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2}. \qquad (28)$$

B. Spherical Bessel Functions

The spherical Bessel function j_n of the first kind and order $n = 0, 1, \ldots$, is given by the following formula [27, Sec. 10.1.25],

$$\mathsf{j}_n(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}, \qquad (29)$$

For example,

$$\mathsf{j}_0(x) = \frac{\sin x}{x}, \qquad (30)$$
$$\mathsf{j}_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \qquad (31)$$
$$\mathsf{j}_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x. \qquad (32)$$

The Fourier transform of the Legendre polynomials can be expressed in terms of the spherical Bessel functions j_n [28, Sec. 7.243],

$$\int_{-1}^{1} e^{\jmath d\pi x} P_n(x)\,dx = 2\,\jmath^n\,\mathsf{j}_n(d\pi) \qquad (33)$$

valid for all real numbers d. In order to normalize the Legendre polynomials, one needs the following formula [28, Sec. 7.221],

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{1}{n + \frac{1}{2}}. \qquad (34)$$

C. Derivation of Equation (20)

We evaluate both integrals in (19) using the substitution $x = \frac{2t}{KT_s} - 1$, in combination with (34) and (33) for $n = m$,

$$\int_0^{KT_s} p_m^2(t)dt = \frac{KT_s}{2}\int_{-1}^{1} P_m^2(x)dx = \frac{KT_s}{2m+1}, \qquad (35)$$

$$\begin{aligned}\int_0^{KT_s} e^{j2\pi\frac{d}{KT_s}t} p_m(t)dt &= \frac{KT_s}{2}\int_{-1}^{1} e^{j\pi d(x+1)} P_m(x)dx \\ &= \frac{KT_s}{2}(-1)^d \int_{-1}^{1} e^{j\pi dx} P_m(x)dx \\ &= KT_s\,(-1)^d\, j^m\, \mathsf{j}_m(d\pi). \qquad (36)\end{aligned}$$

Combining (35) and (36), we obtain (20).

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