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Convergence of adaptive FEM-BEM coupling driven by residual-based error estimators

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Why Adaptivity?

A broad range of physical problems can be described as transmission problems. Efficient methods for solving these problems are FEM-BEM couplings together with an adaptive mesh-refinement, because the accuracy of the computed approximation hinges on the singularities of the given data and / or on the singularities of the unknown exact solution. In contrast to a uniform strategy, adaptive mesh-refinement aims to resolve these singularities effectively. For the presentation we use the Johnson-Nédélec coupling and the error estimator from [2] to steer the adaptive mesh-refinement. Finally, the rigorous mathematical theory is underlined by a numerical experiment: We consider the computation of the magnetostatic potential, which is a bottleneck in most micromagnetic simulations.

Model problem

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain with polygonal boundary Γ, diameter $\mathrm{diam}(\Omega) < 1$, and outer normal unit vector ν. The interface problem reads

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ -\Delta u^{\mathrm{ext}} &= 0 && \text{in } \mathbb{R}^2 \backslash \overline{\Omega}, \\ u - u^{\mathrm{ext}} &= u_0 && \text{on } \Gamma, \\ \partial_\nu u - \partial_\nu u^{ext} &= \phi_0 && \text{on } \Gamma, \\ u^{\mathrm{ext}} &= O\left(|x|^{-1}\right) && \text{as } |x| \to \infty. \end{aligned}$$

The problem admits a unique solution if the given data $(f, u_0, \phi_0) \in H^1(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ fulfills $\langle f\,;1\rangle_\Omega + \langle \phi_0\,;1\rangle_\Gamma = 0$.

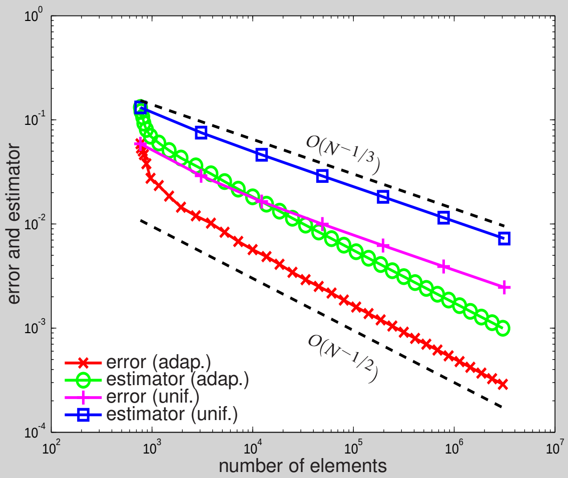
FEM-BEM coupling

The variational form of the Johnson-Nédélec coupling reads: Find $(u, \phi) \in \mathcal{H} = H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \langle \nabla u\,; \nabla v\rangle_\Omega - \langle \phi\,; v\rangle_\Gamma &= \langle f\,; v\rangle_\Omega + \langle \phi_0\,; v\rangle_\Gamma, \\ \langle \psi\,; (\tfrac{1}{2} - \mathfrak{K})u\rangle_\Gamma + \langle \psi\,; \mathfrak{V}\phi\rangle_\Gamma &= \langle \psi\,; (\tfrac{1}{2} - \mathfrak{K})u_0\rangle_\Gamma \end{aligned}$$

holds for all $(v, \psi) \in \mathcal{H}$. The unknown ϕ is related to the interface problem via $\phi = \partial_\nu u^{\mathrm{ext}}$. We use the lowest-order Galerkin discretization and approximate $u \approx u_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ by continuous piecewise affine functions and $\phi \approx \phi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell^\Gamma)$ by piecewise constant functions. $\mathcal{T}_\ell$ is a regular triangulation of Ω and $\mathcal{E}_\ell^\Gamma$ is a regular triangulation of Γ. Here, $\mathfrak{V}$ denotes the simple-layer potential, whereas $\mathfrak{K}$ denotes the double-layer potential.

Numerical experiment



Adaptive algorithm & Convergence

We follow the concept of estimator reduction from [3]: Suppose that Dörfler marking is used to mark edges in an adaptive algorithm of the type

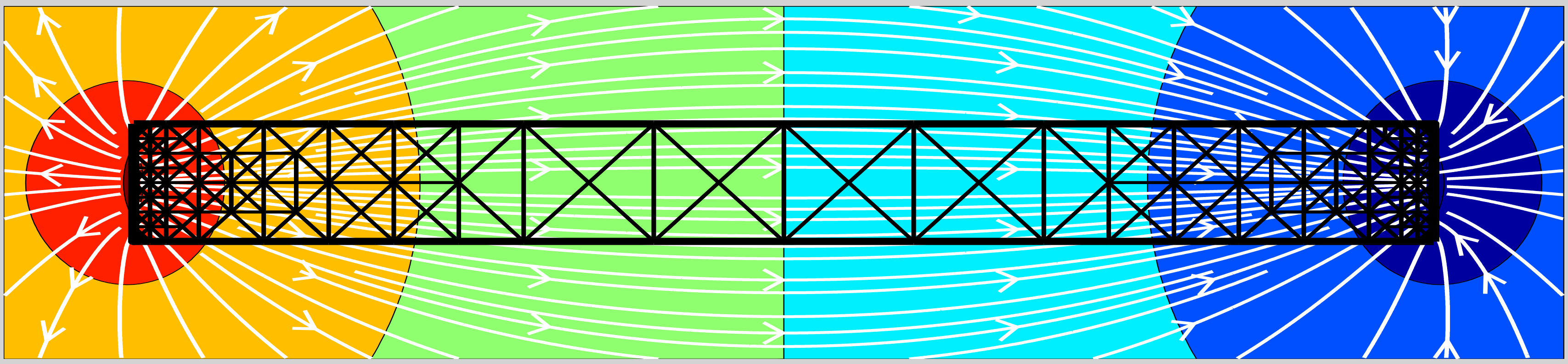
`Solve` $\longmapsto$ `Estimate` $\longmapsto$ `Mark` $\longmapsto$ `Refine`

and suppose that at least all marked edges are refined to obtain the new meshes $\mathcal{T}_{\ell+1}$ and $\mathcal{E}_{\ell+1}^\Gamma$. Then, the residual-based error estimator ρ_ℓ from [2] satisfies the *reduction estimate*

$$\rho_{\ell+1}^2 \le \kappa \rho_\ell^2 + C(\|u_{\ell+1} - u_\ell\|_{H^1(\Omega)}^2 + \|\phi_{\ell+1} - \phi_\ell\|_{H^{-1/2}(\Gamma)}^2)$$

with certain ℓ-independent constants $0 < \kappa < 1$ and $C > 0$. With the Céa lemma and nestedness of the discrete spaces at hand, one obtains convergence of u_ℓ and ϕ_ℓ towards certain — yet unknown — limits. Now, elementary calculus and the reduction estimate prove $\rho_\ell \to 0$ as $\ell \to 0$. Together with reliability of ρ_ℓ, this shows convergence of the adaptive scheme.

Strayfield computation



Strayfield in- and outside of a ferromagnetic bar with constant magnetization. Adaptive algorithm leads to high refinement near the corners.

Generalizations

- Other FEM-BEM coupling methods, such as the Bielak-MacCamy coupling and Costabel's symmetric coupling in 2D and 3D.
- Certain types of nonlinearities in the interior domain Ω.
- Higher-order Galerkin discretizations.

References:

[1] M. Aurada, M. Feischl, T. Führer, M. Karkulik, J.M. Melenk, D. Praetorius *Inverse estimates for elliptic integral operators and application to the adaptive coupling of FEM and BEM*, (in progress)

[2] M. Aurada, M. Feischl, M. Karkulik, D. Praetorius *A posteriori error estimates for the Johnson-Nédélec coupling*, Eng. Anal. Bound. Elem., 36 (2012), 255–266

[3] M. Aurada, S. Ferraz-Leite, D. Praetorius *Estimator reduction and convergence of adaptive BEM*, Appl. Numer. Math, published online (2011)