Performance Analysis of Amplify-and-Forward Relaying using Fractional Calculus

Mehdi Mortazawi Molu, Norbert Goertz
Institute of Telecommunications, Vienna University of Technology

***Abstract*—The paper provides a simple approach for analysing the performance of Amplify-and-Forward relaying systems that are subject to block Rayleigh fading. The PDF of equivalent Source-to-Relay-to-Destination (S-R-D) link SNR involves modified Bessel functions of the second kind. Using fractional calculus mathematics, a simple, yet novel approach is introduced to rewrite the modified Bessel functions in series form using simple elementary functions. Then, we derive a novel S-R-D link SNR model which is mathematically convenient to work with. By obtaining a simple SNR model for S-R-D link, we derive the PDF of the equivalent total SNR which is observed by the destination (including both the S-R-D and S-D channels). By having a simple expression for the equivalent total SNR, performance analysis of the relaying system turns to be a simple task. Finally, we derive novel theoretical expressions for bit error probability of the system using simple elementary functions. The theoretical results are confirmed with Monte-Carlo simulations.**

I. Introduction

Cooperative communication to enhance the transmission rate of a communication system was first introduced in [1], and "distributed spatial diversity" turns out to be a promising method that exploits the antennas of several distributed user terminals to achieve transmit diversity in space. In order to establish a cooperative network, several users share resources, e.g. power or bandwidth, to communicate with a common receiver or even different receivers; those schemes are commonly subsumed under the term "relaying".

Several relaying protocols have been proposed in the literature, e.g. Amplify-and-Forward (AF), Decode-and-Forward (DF), Soft-DF, and Compress-and-Forward (CF) (e.g. [2], [3]), where, depending on the parameters of the network, each of them can be the method of choice. There is currently a lot of interest in AF relaying because of its simplicity in terms of analysis and its low complexity compared to other relaying protocols; hence, AF is also the focus of this work.

For an analytical performance evaluation of a relaying scheme, the statistical model of the Source-to-Relay-to-Destination (S-R-D) link is most important. Several papers consider the problem: in [4]–[6], an equivalent S-R-D channel model has been proposed in terms of modified Bessel functions of the second kind, but it is assumed that the direct Source-to-Destination (S-D) channel is in a deep fade, so the effect of the S-D link can be ignored. Moreover, only the high SNR regime is considered in [5] using the moment-generating function.

In [7], the PDF of the S-R-D link-SNR in the high-SNR regime is derived, and a relay selection scenario is investigated, based on the knowledge of the Source-to-Relay (S-R) channel. In [2], the outage behaviour of different relaying protocols, including AF, has been studied at high SNR and moderate transmission rate, and in [8] outage capacity of different protocols, including AF, is studied in the low-SNR regime.

The complexity of performance analysis of AF relaying systems is evident when considering a system model in which the destination employs an MRC receiver in order to combine the signals which correspond to the source and the relay transmissions. Indeed, although a closed form expression is available for the PDF of S-R-D channel SNR in the literature but, to the best of our knowledge, a closed form expression for the PDF of the total SNR at the output of MRC receiver is *not* yet available. This may be due to the modified Bessel functions in the PDF of the SNR of the S-R-D link that make further mathematical calculations a challenging task. In order to cope with the problem, we use a novel equivalent series representation of the modified Bessel functions. However, the choice of an *appropriate* equivalent representation is crucial: even though a series representation of the modified Bessel function of the second kind, $K_\nu(\cdot)$, is also available from [9, 8.446], this representation is much more complicated than the Bessel function itself. In [10] an equivalent representation for $K_\nu(\cdot)$ is also introduced, but this formulation again is not helpful for the analysis of the outage behaviour of AF relaying.

The remainder of the paper is organized as follows: in Section II the system model is introduced and a general equivalent channel model is derived for the S-R-D link. In Section III the fractional-calculus method is exploited to derive an equivalent series representation of $K_\nu(x)$ (the modified Bessel function of the second kind). Based on the equivalent model for $K_\nu(x)$, novel statistical model is provided for the distribution of the equivalent SNR at the output of MRC receiver in Section IV and based on that, average bit error probability is calculated. Some conclusion remarks are provided in Section V.

II. System and Channel Model

We consider a two-hop Amplify-and-Forward (AF) communication system as illustrated by Fig. 1. The source (S) sends data to the destination (D) by the help of an intermediate relay node (R). The destination might "hear" both the source and the relay transmissions and apply Maximal Ratio Combining (MRC) of the available information in the destination, or it can only "hear" the relay transmission (e.g. due to deep fading on the S-D channel) [5]: both the scenarios are evaluated.

It is assumed that the relay operates in half-duplex mode, i.e. the relay can not receive and transmit simultaneously. Moreover, the overall system is orthogonal in time, i.e. the transmission time is divided into two periodically repeated slots: the wireless channel is allocated for the source transmission during the first time slot and for the relay transmission in the second time slot. Of course, the orthogonality constraint induces the crucial need for full synchronization among the nodes (which is assumed). The channels are subject to Rayleigh fading and AWGN receiver noise. The signals corresponding to the source transmission received at the destination ($\mathbf{y}_{\text{sd}}$) and the relay ($\mathbf{y}_{\text{sr}}$) are

$$\begin{aligned} \mathbf{y}_{\text{sd}} &= \sqrt{P_{\text{s}}} h_{\text{sd}}\, \mathbf{s} + \mathbf{n}_{\text{d}} \\ \mathbf{y}_{\text{sr}} &= \sqrt{P_{\text{s}}} h_{\text{sr}}\, \mathbf{s} + \mathbf{n}_{\text{r}} \end{aligned} \tag{1}$$

where $\mathbf{s}$ is transmit signal vector. The parameter P_{s} is the source power constraint, and h_{sd} and h_{sr} represent the channel coefficients corresponding to the S-D and the S-R links, respectively. The channel coefficients, which capture the effects of path-loss and fading, are zero-mean, white complex Gaussian processes with variances σ^2_{sd} and σ^2_{sr}. The coefficients are constant during every time slot (or transmit block) and they vary independently from one block to another (block-fading model). Additive receiver noise is modelled by $\mathbf{n}_{\text{d}}$ and $\mathbf{n}_{\text{r}}$, which are sample-vectors from zero-mean, white complex Gaussian processes, for simplicity both with variance $N_0 = 1$. For Amplify and Forward (AF), the relay amplifies (without any further processing) the signal received from the source such that it fulfils the relay's power

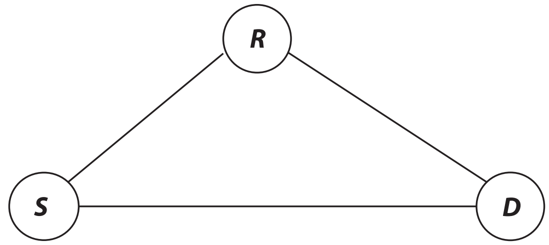

Fig. 1. System Model

constraint, P_r, and retransmits the signal towards the destination; the channel coefficient h_sr is assumed to be available to the relay. The signal received at the destination corresponding to the relay transmission is given by

$$\begin{aligned}\mathbf{y}_\text{rd} &= \sqrt{\frac{P_\text{r}}{\mathbb{E}(|y_\text{sr}|^2)}} h_\text{rd}\mathbf{y}_\text{sr} + \mathbf{n}_\text{d} \quad (2)\\ &= \sqrt{\frac{P_\text{r}P_\text{s}}{P_\text{s}|h_\text{sr}|^2+1}} h_\text{sr}h_\text{rd}\mathbf{s} + \sqrt{\frac{P_\text{r}}{P_\text{s}|h_\text{sr}|^2+1}} h_\text{rd}\mathbf{n}_\text{r} + \mathbf{n}_\text{d}\end{aligned}$$

The R-D channel (Rayleigh fading with variance σ^2_rd) and the noise characteristics ($N_0 = 1$) are similar to those of the S-D and the S-R links. From inspection of (2) it is clear that the equivalent S-R-D link can not be modelled as a Rayleigh fading channel. However, due to the block-fading assumption, the equivalent noise at the destination corresponding to the relay transmission (middle term in the second line of (2)) is Gaussian per block and another Gaussian receiver noise $\mathbf{n}_\text{d}$ is added. Hence, a substitute additive Gaussian noise model can be used, and the corresponding equivalent receiver-output Signal-to-Noise Ratio (SNR) at the destination will be one of the major parameters governing the performance of the overall system, as this output SNR can directly be related to the capacity, diversity, throughput, error rate and other performance measures of the overall system. Therefore, the statistics of the equivalent S-R-D SNR (γ_srd) will be derived. Assuming $P_\text{s} = P_\text{r} = P$ for simplicity, the instantaneous γ_srd using (2) is

$$\gamma_\text{srd} = \frac{P|h_\text{sr}|^2|h_\text{rd}|^2}{|h_\text{sr}|^2 + |h_\text{rd}|^2 + \sigma} \quad (3)$$

where $N_0 = 1$ is assumed and $\sigma \doteq N_0/P = 1/P$. With the channel coefficients known at the receivers (both at the relay and the destination) coherent detection can be used, and the squared magnitudes $|h_{ij}|^2$ of the Rayleigh-distributed channel coefficients that appear in (3) are exponentially distributed with parameter $\lambda_{ij} \doteq 1/\sigma^2_{ij}$, $i \in \{\text{s,r}\}$, $j \in \{\text{r,d}\}$, $i \neq j$.

In the rest of this section the cumulated density function (CDF) and the probability density function (PDF) of the random variable (RV) $\frac{|h_\text{sr}|^2|h_\text{rd}|^2}{|h_\text{sr}|^2+|h_\text{rd}|^2+\sigma}$ are derived, neglecting the factor P in (3).

Theorem 1. *CDF of the RV* $X = \frac{X_1X_2}{X_1+X_2+\sigma}$

Let X_1 and X_2 be two independent exponential RVs with the PDFs $f_{X_i}(x_i) = \lambda_i e^{-\lambda_i x_i}, x_i \geq 0$, $i \in \{1, 2\}$, and the parameters $\lambda_1, \lambda_2 > 0$, and let $\sigma > 0$ be a real constant. Then, the CDF of the RV $X = \frac{X_1X_2}{X_1+X_2+\sigma}$ is given by

$$F_X(x) = 1 - 2e^{-(\lambda_1+\lambda_2)x}\sqrt{\lambda_1\lambda_2 x(x+\sigma)} \times K_1\left(2\sqrt{\lambda_1\lambda_2 x(x+\sigma)}\right) \quad (4)$$

with $K_\nu(\cdot)$ the modified Bessel function of the second kind and ν-th order.

Proof: Assuming $F_X(x)$ is the CDF of the RV X, we have by the definition of the CDF

$$\begin{aligned}F_X(x) &= \mathbb{P}\left(\frac{X_1X_2}{X_1+X_2+\sigma} < x\right)\\ &= \int_{x_1=0}^{\infty}\int_{x_2=0}^{\frac{(x_1+\sigma)x}{(x_1-x)}} \lambda_2 e^{-\lambda_2 x_2}\cdot\lambda_1 e^{-\lambda_1 x_1}dx_2dx_1\\ &= \int_{x_1=0}^{\infty}\left(1 - e^{-\lambda_2\frac{(x_1+\sigma)x}{(x_1-x)}}\right)\cdot\lambda_1 e^{-\lambda_1 x_1}dx_1\\ &= 1 - \lambda_1\int_{x_1=x}^{\infty} e^{-\lambda_2\frac{(x_1+\sigma)x}{x_1-x}}\cdot e^{-\lambda_1 x_1}dx_1\\ &= 1 - \lambda_1\int_{u=0}^{\infty} e^{-\lambda_2\frac{(u+x+\sigma)x}{u}}\cdot e^{-\lambda_1(u+x)}du\\ &= 1 - \lambda_1 e^{-(\lambda_1+\lambda_2)x}\int_{u=0}^{\infty} e^{-\lambda_2\frac{(x+\sigma)x}{u}}\cdot e^{-\lambda_1 u}du\\ &= 1 - 2e^{-(\lambda_1+\lambda_2)x}\sqrt{\lambda_1\lambda_2 x(x+\sigma)} \times\\ &\qquad K_1\left(2\sqrt{\lambda_1\lambda_2 x(x+\sigma)}\right) \quad (5)\end{aligned}$$

where the last equality is obtained from [9, 3.471.9] with

$$\int_0^\infty x^{\nu-1}e^{-\frac{\beta}{x}}\cdot e^{-\alpha x}dx = 2\left(\frac{\beta}{\alpha}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\alpha}\right), \quad (6)$$

where β, α are positive real values and $K_\nu(\cdot)$ is the modified Bessel function of the second kind and ν^th order. ■

Corollary 1. *PDF of the RV* $X = \frac{X_1X_2}{X_1+X_2+\sigma}$

Let X_1 and X_2 be two independent exponential RVs with parameters λ_1 and λ_2, respectively, and $\sigma > 0$ be a real constant. Then, the PDF of the RV $X = \frac{X_1X_2}{X_1+X_2+\sigma}$ is given by

$$\begin{aligned}f_X(x) &= 2e^{-\lambda_\text{S}x}\Big[\lambda_\text{P}(2x+\sigma)K_0(2\sqrt{\lambda_\text{P}x(x+\sigma)}) \quad (7)\\ &\quad + \lambda_\text{S}\sqrt{\lambda_\text{P}x(x+\sigma)}K_1(2\sqrt{\lambda_\text{P}x(x+\sigma)})\Big]\end{aligned}$$

where $\lambda_\text{P} = \lambda_1\lambda_2$ and $\lambda_\text{S} = \lambda_1 + \lambda_2$.

Proof: The PDF $f_X(x)$, (7), is obtained by taking the derivative of $F_X(x)$, (4), with respect to x, using the calculation rules for derivatives of the Bessel functions (e.g. [11, p. 439,10.1.23]). ■

Fig. 2 illustrates $f_X(x)$ derived in (7) for various values of σ and assuming that $\lambda_1 = \lambda_2 = 1$.

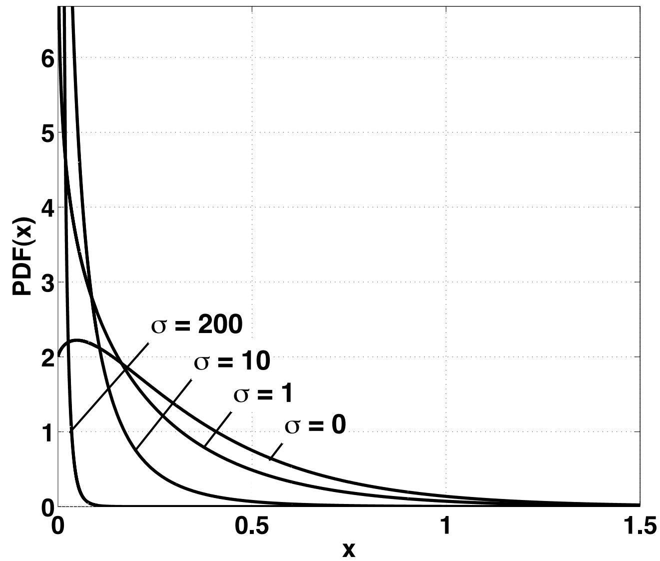

Fig. 2. The PDF of RV X, (7), for various values of σ when $\lambda_1 = \lambda_2 = 1$.

Although the results in (4) and (7) represent closed form solutions for the CDF and PDF of the equivalent S-R-D channel-SNR, the appearance of the modified Bessel functions in (4) and (7) makes them hard to handle e.g. for outage or bit error analysis. For instance, integrations including (4) and (7) will not have a closed form solution. Therefore, in the following an equivalent representation of $K_\nu(\cdot)$ is derived that is based on a series-representation involving simple mathematical functions of the form $x^n e^{-x}$. This novel equivalent representation of $K_\nu(\cdot)$ paves the way for further theoretical analysis of AF relaying systems.

III. Equivalent Representation of Modified Bessel Functions of Second Kind

The mathematical concept of integration and differentiation of arbitrary (non-integer) order is called "fractional calculus"; foundations of the theory are discussed e.g. in [12], [13]. It will be used below to derive an equivalent representation of $K_\nu(\beta x)$.

Theorem 2. *Equivalent representation of the modified Bessel function $K_\nu(\beta x)$ of the second kind and ν^{th} order*

A modified Bessel function $K_\nu(\beta x)$ of the second kind, ν^{th} order, can be represented by an infinite series as:

$$K_\nu(\beta x) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \Lambda \cdot x^{i-\nu} e^{-\beta x}\,, \tag{8}$$

where

$$\Lambda = \sum_{j=0}^{i} \frac{\sqrt{\pi}(-1)^{n+i+j}(2\beta)^{i-\nu}\Gamma(\frac{1}{2}+n-\nu)\Gamma(2\nu)\,(1+j-i-n)_n}{n!\,j!\,(i-j)!\,\Gamma(\frac{1}{2}-\nu)\,\Gamma(\frac{1}{2}+n+\nu)} \tag{9}$$

Proof: Let s be a real non-negative number, i.e. $s > 0$ and $s \in \mathbb{C}$. Let $f(x)$ be continuous on $x \in [0,\infty)$ and integrable on any finite subinterval of $x \geqslant 0$. Then the Riemann-Liouville operator (e.g. [13]) of fractional integration is defined as

$$I^s\{f(x)\} \doteq \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1} f(t)dt\,. \tag{10}$$

On the other hand, from [9, 3.471.4] we have

$$\int_0^x (x-t)^{s-1} t^{-2s} e^{-\beta/t} dt = \frac{\Gamma(s)\beta^{\frac{1}{2}-s}}{\sqrt{\pi x}} e^{\frac{-\beta}{2x}} K_{s-\frac{1}{2}}(\frac{\beta}{2x})\,. \tag{11}$$

Assuming $f(t) = t^{-2s}e^{-\beta/t}$, the two integrals in (10) and (11) are identical: this motivates the novel approach to derive an equivalent expression for $K_\nu(\beta x)$ by use of fractional integration.

It follows from (10) and (11) that

$$I^s\left\{x^{-2s}e^{-\beta/x}\right\} = \frac{\beta^{\frac{1}{2}-s}}{\sqrt{\pi x}} e^{\frac{-\beta}{2x}} K_{s-\frac{1}{2}}(\frac{\beta}{2x})\,. \tag{12}$$

The Leibniz rule for the Riemann-Liouville operator (see appendix for a proof) is given by

$$I^s\{h(x)g(x)\} = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(n+s)}{n!\,\Gamma(s)} I^{(s+n)}\{h(x)\} D^n\{g(x)\} \tag{13}$$

where n is a non-negative integer, $s+n$ is a non-negative fractional number and $D^n \doteq \frac{d^n}{dx^n}$. By solving $I^{(s+n)}\{h(x)\}$ for $h(x) = x^{-2s}$ and $D^n\{g(x)\}$ for $g(x) = e^{-\beta/x}$, the equivalent Bessel model (8) will be derived. Let $h(x) = x^p$, then

$$\begin{aligned} I^\alpha x^p &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t^p dt, \qquad (\alpha > 0) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (1-\frac{t}{x})^{\alpha-1} x^{\alpha-1} t^p dt \\ &= \frac{x^{\alpha+p}}{\Gamma(\alpha)}\int_0^1 u^p(1-u)^{\alpha-1}du, \quad (u = \frac{t}{x}) \\ &= \frac{\Gamma(1+p)}{\Gamma(1+p+\alpha)} x^{p+\alpha}\,. \end{aligned} \tag{14}$$

Suppose that $p = -2s$ and $\alpha = s+n$, then

$$I^{(s+n)}\left\{x^{-2s}\right\} = \frac{\Gamma(1-2s)}{\Gamma(1-s+n)} x^{n-s}\,, \tag{15}$$

and assuming $g(x) = e^{-\beta/x}$ in (13), $D^n\left\{e^{-\beta/x}\right\}$ can be computed as

$$D^n\left\{e^{-\beta/x}\right\} = \frac{d^n}{dx^n} e^{-\beta/x} = x^{-n}e^{-\beta/x}\sum_{i=0}^{n}\sum_{j=0}^{i} \frac{(-1)^{i+j}\,(1+j-i-n)_n\,(\beta/x)^i}{j!\,(i-j)!} \tag{16}$$

where $(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)}$ is the Pochhammer symbol.

By substituting (15) and (16) into (12) and (13) it is straightforward to obtain (17), at the top of next page. Changing the variable $x \to \frac{1}{2x}$, assuming $1-2s = 2\nu$, and exploiting $K_{-\nu} = K_\nu$, the result is the infinite series defined in (8) ■

It should be made clear that the above representation of $K_\nu(\beta x)$ is *not* valid for $\nu = \left\{0, \frac{1}{2}, \frac{3}{2}, \cdots\right\}$. That is because $\Gamma(2\nu)$ and $\Gamma(\frac{1}{2}+n-\nu)$ in (9) diverge to $\pm\infty$. However, one can compute $K_0(\beta x)$ using the equivalent representation of $K_1(\beta x)$ and $K_2(\beta x)$ by $K_\nu(x) = K_{\nu-2}(x) + \frac{2(\nu-1)}{x}K_{\nu-1}(x)$ that is obtained from [14, 10.38.4].

Finite Series Representation of $K_\nu(\beta x)$: The equivalent representation of $K_\nu(\beta x)$ may significantly simplify computations involving $K_\nu(\beta x)$, as the series in (8) contains the variable x only in the simple function-template $x^{i-\nu}e^{-\beta x}$ that can, e.g., be easily integrated. The series representation contains, however, an infinite number of terms that can't be computed in practical applications.

Fortunately, the series representation of $K_\nu(\beta x)$ is rather accurate

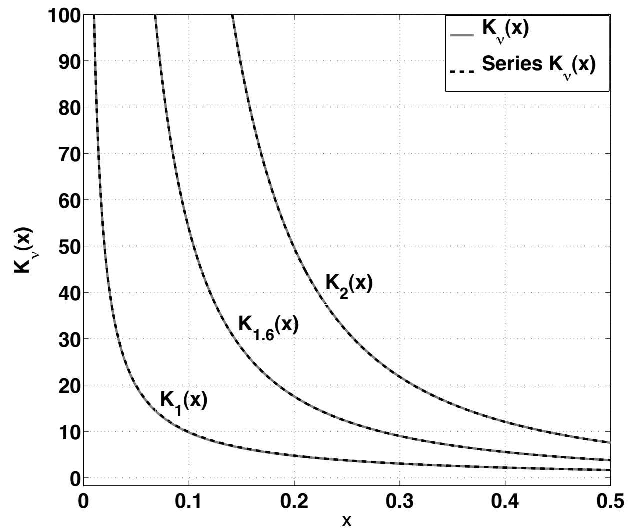

Fig. 3. $K_\nu(x)$ vs. finite series representation of $K_\nu(x)$ with $k = 2$ in (18).

$$\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+s)}{n!\,\Gamma(s)}\frac{\Gamma(1-2s)}{\Gamma(1+n-s)}x^{n-s}\cdot x^n e^{-\beta/x}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^j(-\beta/x)^i\,(1+j-n-i)_n}{j\,!(i-j)\,!} = \frac{\beta^{\frac{1}{2}-s}}{\sqrt{\pi x}}e^{-\frac{\beta}{2x}}K_{s-\frac{1}{2}}(\frac{\beta}{2x}) \quad (17)$$

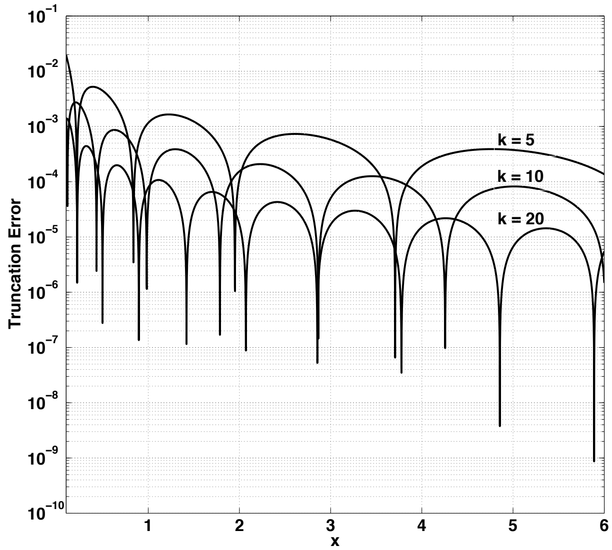

Fig. 4. Truncation error of $K_1(x)$ for various values of k.

for a finite number of terms as defined as follows:

$$K_\nu(\beta x) \quad = \quad \sum_{n=0}^{k}\sum_{i=0}^{n}\Lambda\cdot x^{i-\nu}e^{-\beta x}+\epsilon \qquad (18)$$

with

$$\epsilon \quad = \quad \sum_{n=k+1}^{\infty}\sum_{i=0}^{n}\Lambda\cdot x^{i-\nu}e^{-\beta x}\,. \qquad (19)$$

The first term on the right-hand side of (18) represents the actual function to approximate $K_\nu(\beta x)$, and ϵ represents the truncation error. Fig. 3 illustrates numerical values of the finite series representation with $k=2$ (in (18)) of $K_\nu(x)$ for various values of ν (dashed lines) and also the theoretical fully accurate values of $K_\nu(x)$ (solid lines). It is clear from the figure that the finite series for $K_\nu(x)$ with only $k=2$ produces only a rather small error.

Truncation Error: In the Appendix it is proved that the Leibniz rule, (13), for the Riemann-Liouville operator is a direct result of a Taylor-series expansion of some function, say $h(t)$, at $t=x$. Consequently, the equivalent infinite series representation of $K_\nu(\beta x)$ in (8) is also a result of some Taylor expansion at point x. Therefore, it is expected that the equivalent infinite series representation of $K_\nu(\beta x)$ can be truncated with high accuracy with only few terms. Fig. 4 shows the absolute value of the truncation error, i.e. $|\epsilon|$, for $k=5,10$ and 20. It is obvious from Fig. 4 that the error is as low as about 10^{-4} for $k=10$ and as low as about 10^{-5} for $k=20$. The truncation error is about 10^{-3} when $x\to 0$, but considering that $K_\nu(x)\to\infty$ as $x\to 0$, the truncation error of 10^{-3} is negligible. In the remainder of the paper we assume $k=10$, although even much lower values of, e.g. $k=2$, turn out to produce accurate results.

IV. Distribution of Equivalent Received SNR at the Destination

Assuming an MRC receiver at the destination, the total receive SNR is the sum of SNRs corresponding to the S-D and the S-R-D links, i.e.

$$\gamma_{\text{tot}} = \gamma_{\text{sd}} + \gamma_{\text{srd}} \qquad (20)$$

Therefore, for the CDF of γ_{tot} we obtain

$$\begin{aligned} F_{\gamma_{\text{tot}}}(x) &= \mathbb{P}\left(\gamma_{\text{tot}}\le x\right) = \mathbb{P}\left(\gamma_{\text{srd}}+\gamma_{\text{srd}}\le x\right) \\ &= \int_0^x\int_0^{x-u} f_{\gamma_{\text{sd}}}(t)\cdot f_{\gamma_{\text{srd}}}(u)dtdu \\ &= \int_0^x (1-\mathrm{e}^{-\lambda_{\text{sd}}(x-u)})\cdot f_{\gamma_{\text{srd}}}(u)du \\ &= F_{\gamma_{\text{srd}}}(x)-\mathrm{e}^{-\lambda_{\text{sd}}x}\int_0^x \mathrm{e}^{\lambda_{\text{sd}}u}\cdot f_{\gamma_{\text{srd}}}(u)du \\ &= \lambda_{\text{sd}}\mathrm{e}^{-\lambda_{\text{sd}}x}\int_0^x \mathrm{e}^{\lambda_{\text{sd}}u}\cdot F_{\gamma_{\text{srd}}}(u)du\,, \end{aligned} \qquad (21)$$

where the last equality follows from integration by parts with $\int_0^x p(u)q'(u)du = p(u)q(u)|_0^x - \int_0^x p'(u)q(u)du$ and $p(u)\doteq \mathrm{e}^{\lambda_{\text{sd}}u}$ and $q'(u)\doteq f_{\gamma_{\text{srd}}}(u)$.

The CDF $F_{\gamma_{\text{srd}}}(\cdot)$ and the PDF $f_{\gamma_{\text{srd}}}(\cdot)$ were derived in (4) and (7), respectively. Note that the results in this paper are restricted to the high "transmit-SNR" regime ($\sigma\to 0$) but it will be demonstrated by Fig. 5 that this is justified because it leads to very accurate numerical results, even in the low-SNR region. By substituting (4) in (21), one can write

$$\begin{aligned} F_{\gamma_{\text{tot}}}(x) \quad = \quad & 1-\mathrm{e}^{-\lambda_{\text{sd}}x}-2\lambda_{\text{sd}}\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}\mathrm{e}^{-\lambda_{\text{sd}}x} \\ & \times\int_0^x u\mathrm{e}^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}}-\lambda_{\text{sd}})u}K_1(2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}u)du\,. \end{aligned} \qquad (22)$$

The integral in (22) is non-trivial and does not seem to have closed-form solution. However, using the results from Section III, the integral can be rewritten as follows

$$\begin{aligned} \eta \quad &\doteq \quad \int_0^x u\mathrm{e}^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}}-\lambda_{\text{sd}})u}K_1(\,2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}u)du \qquad (23) \\ &\cong \quad \sum_{n=0}^{k}\sum_{i=0}^{n}\Lambda\int_0^x u^i\mathrm{e}^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}}+2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}-\lambda_{\text{sd}})u}du \\ &\cong \quad \sum_{n=0}^{k}\sum_{i=0}^{n}\frac{\Lambda\,i!}{(\lambda_{\text{srd}}-\lambda_{\text{sd}})^{i+1}}\Big(1-\sum_{c=0}^{i}\frac{(\lambda_{\text{srd}}-\lambda_{\text{sd}})^c}{c!}x^c\mathrm{e}^{-(\lambda_{\text{srd}}-\lambda_{\text{sd}})x}\Big) \end{aligned}$$

where $\lambda_{\text{srd}} = (\sqrt{\lambda_{\text{sr}}}+\sqrt{\lambda_{\text{rd}}})^2$. The second equality is obtained by using the series representation of $K_1(2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}x)$ derived in (18) and the last equality follows from identity [9, 3.351.1] where

$$\int_0^x u^q\mathrm{e}^{-\lambda u}du = \frac{q!}{\lambda^{q+1}}(1-\sum_{c=0}^{q}\frac{\lambda^c}{c!}x^c\mathrm{e}^{-\lambda x}). \qquad (24)$$

Note that we use approximation ($\cong$) instead of strict equality ($=$) in (23) because of truncation in the series form, however, one can keep truncation error arbitrary small, as explained in III. By substituting (23) into (22) it is straightforward to obtain $F_{\gamma_{\text{tot}}}(x)$ as

$$F_{\gamma_{\text{tot}}}(x)\cong 1-\mathcal{A}\mathrm{e}^{-\lambda_{\text{sd}}x}+\sum_{n=0}^{k}\sum_{i=0}^{n}\sum_{c=0}^{i}\mathcal{B}x^c\mathrm{e}^{-\lambda_{\text{srd}}x} \qquad (25)$$

where coefficients $\mathcal{A}$ and $\mathcal{B}$ are independent of x and can, easily, be calculated from (22) and (23).

The PDF of γ_{tot} is the derivative of $F_{\gamma_{\text{tot}}}(x)$ in (25) w.r.t x, which is easy to calculate as the series representation involves simple

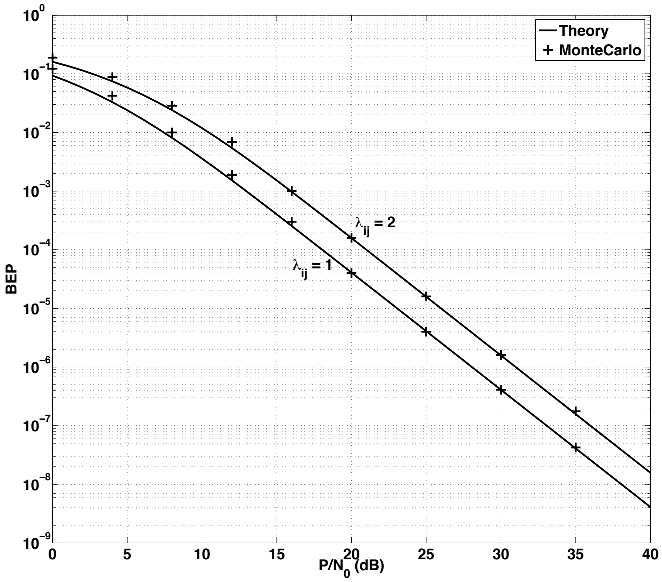

Fig. 5. Bit Error Probability for various λ_{ij}, $i \in \{$s,r$\}$ with $j \in \{$r,d$\}$ and $\alpha = 1$

elementary functions only; therefore

$$f_{\gamma_{\text{tot}}}(x) \cong \mathcal{A}\lambda_{\text{sd}}\mathrm{e}^{-\lambda_{\text{sd}}x} + \sum_{n=0}^{k}\sum_{i=0}^{n}\sum_{c=0}^{i}\mathcal{B}(cx^{c-1} - \lambda_{\text{srd}}x^{c})\mathrm{e}^{-\lambda_{\text{srd}}x} \,. \tag{26}$$

Bit Error Probability

The average BEP of BPSK-modulated transmission over an AWGN channel, given γ_{tot}, is

$$\begin{aligned} p_{\text{b}}(E) &= \frac{1}{2}\mathbb{E}_{\gamma_{\text{tot}}}\{\text{erfc}(\sqrt{\gamma_{\text{tot}}})\} \\ &= \frac{1}{2}\int_{0}^{\infty}\text{erfc}(\sqrt{x})f_{\gamma_{\text{tot}}}(x)dx \end{aligned} \tag{27}$$

From [11, 7.1.19], $\frac{d}{dx}\,\text{erfc}(\sqrt{x}) = -\mathrm{e}^{-x}/\sqrt{\pi x}$. Then, using integration by parts, $p_{\text{b}}(E)$ in (27) can be written as

$$p_{\text{b}}(E) = \frac{1}{2\sqrt{\pi}}\int_{0}^{\infty}x^{-1/2}\mathrm{e}^{-x}F_{\gamma_{\text{tot}}}(x)dx. \tag{28}$$

By substituting $F_{\gamma_{\text{tot}}}(x)$ from (25) into (28), then exploiting (24) and some basic manipulations, one can obtain the closed form solution for average BEP as

$$p_{\text{b}}(E) = \frac{1}{2}\Big(1 - \frac{\mathcal{A}}{\sqrt{1+\lambda_{\text{sd}}}} + \sum_{n=0}^{k}\sum_{i=0}^{n}\sum_{c=0}^{i}\frac{\mathcal{B}\,\Gamma(c+\frac{1}{2})}{\sqrt{\pi}(1+\lambda_{\text{srd}})^{c+\frac{1}{2}}}\Big). \tag{29}$$

Fig. 5 illustrates the BEP obtained from (29) with the outer summation truncated at $k = 4$. According to Fig. 5, regardless of the specific values of the fading parameters λ_{ij}, the theoretical BEP perfectly matches the Monte Carlo simulations at high SNR, whereas at low SNR the theoretical results show a very small (in fact insignificant) difference to the numerical results.

V. Conclusions

Based on fractional calculus, a novel series representation of the modified Bessel functions of the second kind and ν-th order has been presented. Based on the novel series representation for $K_\nu(.)$, the PDF and the CDF of the total SNR at the output of MRC receiver is calculated. The novel statistical model allows for performance analysis of Amplify-and-Forward cooperative systems. Novel closed-form solutions for the bit error probability of AF relaying system is provided in this paper.

Appendix
Proof of the Leibniz Rule for the Riemman-Liouville Integration Operator

Let $s > 0$. The Riemman-Liouville intagration operator is defined as $I^s\{h(x)g(x)\} = \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1}h(t)g(t)dt$. It is straightforward to derive the Leibniz rule by performing a Taylor series expansion of $h(t)$ at $t = x$, i.e. $h(t) = \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-t)^n D^n\{h(x)\}$. We obtain

$$\begin{aligned} &I^s\{h(x)g(x)\} \\ &= \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1}h(t)g(t)dt \\ &= \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-t)^n D^n\{h(x)\}\,g(t)dt \\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}D^n\{h(x)\}\int_0^x (x-t)^{n+s-1}g(t)dt \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+s)}{n!\Gamma(s)}I^{n+s}\{g(x)\}\,D^n\{h(x)\} \end{aligned} \tag{30}$$

where $D^n\{h(x)\} = \frac{d^n}{dx^n}h(x)$.

References

[1] E. C. van der Meulen, "Three-terminal communication channels," *Advances in Applied Probability*, vol. 3, p. 121, 1971.

[2] J. Laneman, D. Tse, and G. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Transactions on Information Theory*, vol. 50, pp. 3062–3080, Dec. 2004.

[3] M. M. Molu and N. Goertz, "A study on relaying soft information with error prone relays," *49th Annual Allerton Conference on Communication, Control, and Computing*, September 2010.

[4] M. Hasna and M.-S. Alouini, "Performance analysis of two-hop relayed transmissions over rayleigh fading channels," in *Vehicular Technology Conference, 2002. Proceedings. VTC 2002-Fall. 2002 IEEE 56th*, vol. 4, pp. 1992 – 1996 vol.4, 2002.

[5] M. O. Hasna and M. S. Alouini, "End-to-end performance of transmission systems with relays over Rayleigh fading channels," *IEEE Transactions on Wireless Communication*, vol. 2, November 2003.

[6] K. Seddik, A. Sadek, W. Su, and K. Liu, "Outage analysis of multi-node amplify-and-forward relay networks," in *Wireless Communications and Networking Conference, 2006. WCNC 2006. IEEE*, vol. 2, pp. 1184 –1188, april 2006.

[7] I. Krikidis, J. Thompson, S. McLaughlin, and N. Goertz, "Amplify-and-forward with partial relay selection," *IEEE Communication Letters*, vol. 12, April 2008.

[8] A. S. Avestimehr and D. N. C. Tse, "Outage capacity of the fading relay channel in the low SNR regime," *IEEE Transactions on Information Theory*, vol. 53, April 2007.

[9] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*. Academic, 7th ed., 2007.

[10] K. Maslanka, "Series representation of the modified bessel functions." http://arxiv.org/abs/math-ph/0104018v1, April 2001.

[11] Abramowitz and Stegun, *Handbook of Mathematical Functions*. Dover Publications, June 1965.

[12] R. Hilfer, *Applications Of Fractional Calculus In Physics*. World Scientific Publishing, March 2000.

[13] R. Gorenflo and F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, vol. 378 of CISM Courses and Lectures. Springer-Verlag, 1997.

[14] F. W. J. Oliver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*. Cambridge Press, July 2010.