Amplify and Forward Relaying; Channel Model and Outage Behaviour

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***Abstract*—Characteristic of receiver output SNR associated with Source-to-Relay-to-Destination link for Amplify and forward relaying systems is studied and theoretical expressions for probability density function and cumulative distribution function are derived. Thereby, closed-form solution for outage probability of the system is derived. The analysis is then extended to a scenario where the destination combines the signals corresponding to the source and the relay transmissions using maximal ratio combining technique. The statistics of the equivalent SNR is studied and some new closed form expressions at high SNR regime are derived. The comparison of theoretical and numerical results shows that the high SNR expressions for outage probability are adequately tight in low SNR too. The performance of the AF relaying system is further investigated at low SNR regime; the results show that at low SNR regime, the relay transmission does not play role on the the performance of the overall system, therefore, spending resources (e.g. power or bandwidth) for the relay transmission is only wasting the resources whereas at high SNR regime the relay transmission provides full diversity.**

I. Introduction

Fading is one of the most severe channel impairments affecting the performance of wireless communication systems. A variety of methods aim to combat fading by time, frequency or spatial diversity; multiple antennas are most effective to exploit spatial diversity [1]. However, despite the large theoretical performance gains of multiple antenna systems, practical limits in implementing transceivers (user terminals in particular) call for other methods of exploiting spatial diversity. cooperative communication is another method of exploiting spatial diversity which is obtained by cooperation among distributed nodes.

Several relaying protocols have been proposed in the literature, e.g. Amplify-and-Forward (AF), Decode-and-Forward (DF), Soft-DF, and Compress-and-Forward (CF) (e.g. [3], [4]), where, depending on the parameters of the network, each of them can be the method of choice. There is currently a lot of interest in AF relaying because of its simplicity in terms of analysis and its low complexity compared to other relaying protocols; hence, AF is also the focus of this work.

For an analytical performance evaluation of a relaying scheme, the statistical model of the Source-to-Relay-to-Destination (S-R-D) link is most important. Several papers consider the problem: in [5], an equivalent S-R-D channel model has been proposed in terms of modified Bessel functions of the second kind, but it is assumed that the direct Source-to-Destination (S-D) channel is in a deep fade, so the effect of the S-D link can be ignored. Moreover, only the high SNR regime is considered in [5] using the moment-generating function. A similar approach is discussed in [6] too.

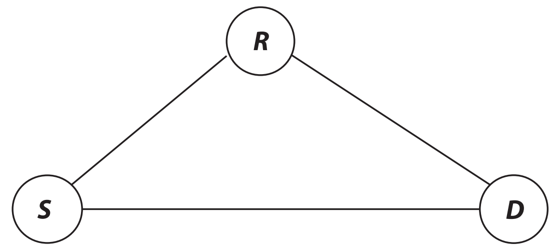

Fig. 1. System Model

In [3], the outage behaviour of different relaying protocols, including AF, has been studied at high SNR and low transmission rate, and in [7] outage capacity of different protocols, including AF, is studied in the low-SNR regime.

To the best of our knowledge the problem has not been investigated at moderate SNR regime. In fact, although, several papers evaluate the performance of the AF relaying system under high or low SNR assumption but, yet, there is not a general channel model which covers all the characteristics of the relaying system at low, moderate and high SNR.

In this paper we study the outage behaviour of an AF relaying system assuming a general S-R-D channel model when a direct link is available between the source and the destination.

The reminder of the paper is organized as follows. Section II introduces system model and derives a general equivalent channel model for S-R-D link. Section III studies the outage behaviour of the cooperative system. Section IV provides asymptotic outage behaviour of the system and compares it with the already existing results and finally the results are summarized at section V.

II. System and Channel Model

We consider a two-hop Amplify-and-Forward (AF) communication system as illustrated by Fig. 1. The source (S) sends data to the destination (D) by the help of an intermediate relay node (R). The destination might "hear" both the source and the relay transmissions and apply Maximal Ratio Combining (MRC) of the available information in the destination, or it can only "hear" the relay transmission (e.g. due to deep fading on the S-D channel [5]): both the scenarios are evaluated. It is assumed that the relay operates in half-duplex mode, i.e. the relay can not receive and transmit simultaneously. Moreover, the overall system is orthogonal in time, i.e. the transmission

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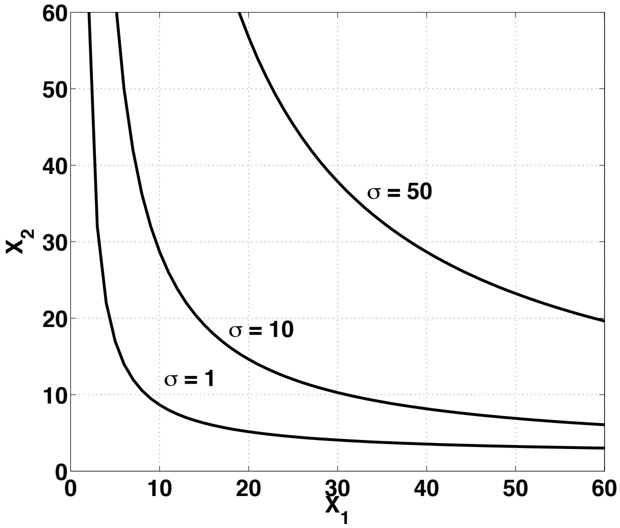

Fig. 2. The area under the curves corresponds to $\frac{X_1 X_2}{X1+X2+\sigma} < 2$ for different σs.

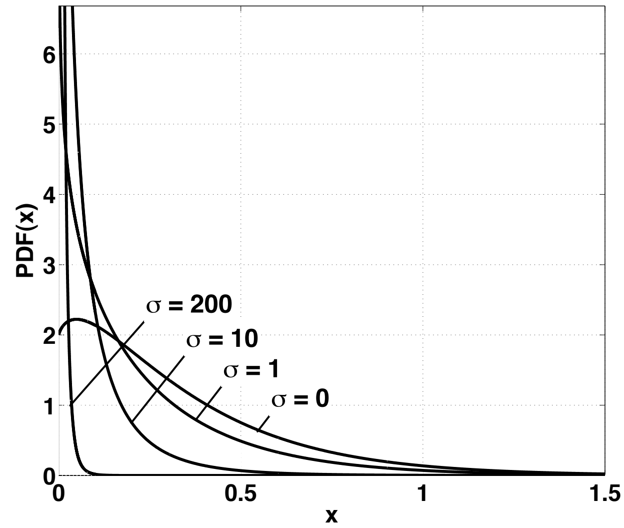

Fig. 3. The PDF of RV X, (6), for various values of σ and $\lambda_1 = \lambda_2 = 1$.

time is divided into two periodically repeated slots: the wireless channel is allocated for the source transmission during the first time slot and for the relay transmission in the second time slot. Of course, the orthogonality constraint induces the crucial need for full synchronization among the nodes (which is assumed). The channels are subject to Rayleigh fading and AWGN receiver noise. The signals[1] corresponding to the source transmission received at the destination ($\mathbf{y}_{\text{sd}}$) and the relay ($\mathbf{y}_{\text{sr}}$) are

$$\begin{aligned} \mathbf{y}_{\text{sd}} &= \sqrt{P_{\text{s}}} h_{\text{sd}}\, \mathbf{s} + \mathbf{n}_{\text{d}} \\ \mathbf{y}_{\text{sr}} &= \sqrt{P_{\text{s}}} h_{\text{sr}}\, \mathbf{s} + \mathbf{n}_{\text{r}} \end{aligned} \tag{1}$$

where $\mathbf{s}$ is transmit signal vector. The parameter P_{s} is the source power constraint, and h_{sd} and h_{sr} represent the channel coefficients corresponding to the S-D and the S-R links, respectively. The channel coefficients, which capture the effects of path-loss and fading, are zero-mean, white complex Gaussian processes with variances σ_{sd}^2 and σ_{sr}^2. The coefficients are constant during every time slot (or transmit block) and they vary independently from one block to another (block-fading model). Additive receiver noise is modelled by $\mathbf{n}_{\text{d}}$ and $\mathbf{n}_{\text{r}}$, which are sample-vectors from zero-mean, white complex Gaussian processes, for simplicity both with variance $N_0 = 1$. For Amplify and Forward (AF), the relay amplifies (without any further processing) the signal received from the source such that it fulfils the relay's power constraint, P_{r}, and retransmits the signal towards the destination; the channel coefficient h_{sr} is assumed to be available to the relay. The signal received at the destination corresponding to the relay transmission is given by

$$\begin{aligned} \mathbf{y}_{\text{rd}} &= \sqrt{\frac{P_{\text{r}}}{\mathbb{E}(|y_{\text{sr}}|^2)}} h_{\text{rd}} \mathbf{y}_{\text{sr}} + \mathbf{n}_{\text{d}} \\ &= \sqrt{\frac{P_{\text{r}} P_{\text{s}}}{P_{\text{s}} |h_{\text{sr}}|^2 + 1}} h_{\text{sr}} h_{\text{rd}} \mathbf{s} + \sqrt{\frac{P_{\text{r}}}{P_{\text{s}} |h_{\text{sr}}|^2 + 1}} h_{\text{rd}} \mathbf{n}_{\text{r}} + \mathbf{n}_{\text{d}} \end{aligned} \tag{2}$$

[1]The signals $\mathbf{y}_{\text{sd}}$ and $\mathbf{y}_{\text{sr}}$ are vectors of matched-filter output samples; their dimensions depend on the number of channel uses within a time slot.

The R-D channel (Rayleigh fading with variance σ_{rd}^2) and the noise characteristics ($N_0 = 1$) are similar to those of the S-D and the S-R links.

From inspection of (2) it is clear that the equivalent S-R-D link can not be modelled as a Rayleigh fading channel. However, due to the block-fading assumption, the equivalent noise at the destination corresponding to the relay transmission (middle term in the second line of (2)) is Gaussian per block and another Gaussian receiver noise $\mathbf{n}_{\text{d}}$ is added. Hence, a substitute additive Gaussian noise model can be used. The corresponding equivalent receiver-output Signal-to-Noise Ratio (SNR) at the destination will be one of the major parameters governing the performance of the overall system, as this output SNR can directly be related to the capacity, diversity, throughput, error rate and other performance measures of the overall system. Therefore, the statistics of the equivalent S-R-D SNR (*SNR*$_{\text{srd}}$) will be derived. Assuming $P_{\text{s}} = P_{\text{r}} = P$ for simplicity, the instantaneous *SNR*$_{\text{srd}}$ using (2) is

$$\mathit{SNR}_{\text{srd}} = \frac{P |h_{\text{sr}}|^2 |h_{\text{rd}}|^2}{|h_{\text{sr}}|^2 + |h_{\text{rd}}|^2 + \sigma} \tag{3}$$

where $N_0 = 1$ is assumed and $\sigma \doteq N_0/P = 1/P$. With the channel coefficients known at the receivers (both at the relay and the destination) coherent detection can be used, and the squared magnitudes $|h_{ij}|^2$ of the Rayleigh-distributed channel coefficients that appear in (3) are exponentially distributed with parameter $\lambda_{ij} \doteq 1/\sigma_{ij}^2$, $i \in \{\text{s,r}\}$, $j \in \{\text{r,d}\}$, $i \neq j$.

In the rest of this section the cumulated density function (CDF) and the probability density function (PDF) of the random variable (RV) $\frac{|h_{\text{sr}}|^2 |h_{\text{rd}}|^2}{|h_{\text{sr}}|^2 + |h_{\text{rd}}|^2 + \sigma}$ are derived, negelecting the factor P in (3).

Theorem 1. *CDF of the RV* $X = \frac{X_1 X_2}{X_1 + X_2 + \sigma}$

Let X_1 and X_2 be two independent exponential RVs with the PDFs $f_{X_i}(x_i) = \lambda_i e^{-\lambda_i x_i}, x_i \geq 0$, $i \in \{1, 2\}$, and the parameters $\lambda_1, \lambda_2 > 0$, and let $\sigma > 0$ be a real constant. Then, the CDF of the RV $X = \frac{X_1 X_2}{X_1 + X_2 + \sigma}$ is given by

$$\begin{aligned} F_X(x) &= 1 - 2e^{-(\lambda_1 + \lambda_2)x} \sqrt{\lambda_1 \lambda_2 x (x + \sigma)} \times \\ &\qquad K_1 \left(2\sqrt{\lambda_1 \lambda_2 x (x + \sigma)} \right) \end{aligned} \tag{4}$$

with $K_\nu(\cdot)$ the modified Bessel function of the second kind and ν-th order.

Proof: Fig. 2 illustrates the RV $X < 2$ for various values of σ, where, accordingly, one of the RVs X_1 or X_2 can take any value larger than 0; e.g. let say $X_1 \in [0, \infty)$, then $X_2 \in [0, \frac{(x_1+\sigma)x}{(x_1-x)})$. Therefore we have

$$\begin{aligned} F_X(x) &= \mathbb{P}\,(\frac{X_1X_2}{X1+X2+\sigma} < x) \\ &= \int_{x_1=0}^{\infty}\int_{x_2=0}^{\frac{(x_1+\sigma)x}{(x_1-x)}} \lambda_2 e^{-\lambda_2 x_2}\cdot \lambda_1 e^{-\lambda_1 x_1} dx_2 dx_1 \\ &= \int_{x_1=0}^{\infty}\left(1-e^{-\lambda_2\frac{(x_1+\sigma)x}{(x_1-x)}}\right)\cdot \lambda_1 e^{-\lambda_1 x_1} dx_1 \\ &= 1-\lambda_1\int_{x_1=x}^{\infty} e^{-\lambda_2\frac{(x_1+\sigma)x}{x_1-x}}\cdot e^{-\lambda_1 x_1} dx_1 \\ &= 1-\lambda_1\int_{u=0}^{\infty} e^{-\lambda_2\frac{(u+x+\sigma)x}{u}}\cdot e^{-\lambda_1 (u+x)} du \\ &= 1-\lambda_1 e^{-(\lambda_1+\lambda_2)x}\int_{u=0}^{\infty} e^{-\lambda_2\frac{(x+\sigma)x}{u}}\cdot e^{-\lambda_1 u} du \\ &= 1-2e^{-(\lambda_1+\lambda_2)x}\sqrt{\lambda_1\lambda_2 x(x+\sigma)} \qquad (5) \\ &\qquad \times K_1\left(2\sqrt{\lambda_1\lambda_2 x(x+\sigma)}\right) \end{aligned}$$

where the last equality follows from [8, 3.471.9]. ■

One can obtain PDF of RV X, specified in Theorem 1 by taking the derivative of $F_X(x)$, (4), with respect to x as

$$\begin{aligned} f_X(x) &= 2e^{-\lambda_S x}\Big[\lambda_P(2x+\sigma)K_0(2\sqrt{\lambda_P x(x+\sigma)}) \qquad (6) \\ &\qquad +\lambda_S\sqrt{\lambda_P x(x+\sigma)}K_1(2\sqrt{\lambda_P x(x+\sigma)})\Big] \end{aligned}$$

where $\lambda_P = \lambda_1\lambda_2$ and $\lambda_S = \lambda_1+\lambda_2$. Fig. 3 illustrates $f_X(x)$, derived in (6) for various values of σ and assuming that $\lambda_{sr} = \lambda_{rd} = 1$. It is clear that for larger values of σ (i.e. lower SNR per hop) the PDF curve moves left, towards the PDF(x) axis, consequently, when $\sigma \to \infty$, the entire power of the PDF will shrink at the point $x = 0^+$, which implies that communicating via relay is impossible.

III. Outage Analysis

Outage probability is a common standard criterion, characterizing the performance of communication systems operating in fading environments. By definition, (e.g. [9], [10]), outage probability, p_{out}, is the probability that the instantaneous receiver output SNR falls below a certain threshold which is specified according the expected performance (e.g. BER, capacity, etc.) of the system. In order to evaluate the outage behaviour of the system illustrated in Fig. 1, we consider two scenarios: we first investigate outage behaviour of AF relaying system when there is not a direct link available between the source and the destination, then the problem is further investigated when a direct link between the source and the destination is available. In the following, let define $N_0 = 1$ and $SNR = \frac{P}{N_0} = P$. Suppose that input signal vector is i.i.d. circularly symmetric complex Gaussian with identity covariance matrix, i.e. $\mathbb{E}(\mathbf{s}\mathbf{s}^H) = I$. Moreover we assume that the perfect Channel State Information (CSI) of all the links, i.e. h_{sr}, h_{sd} and h_{rd}, are available at the destination but only h_{sr} is known for the relay.

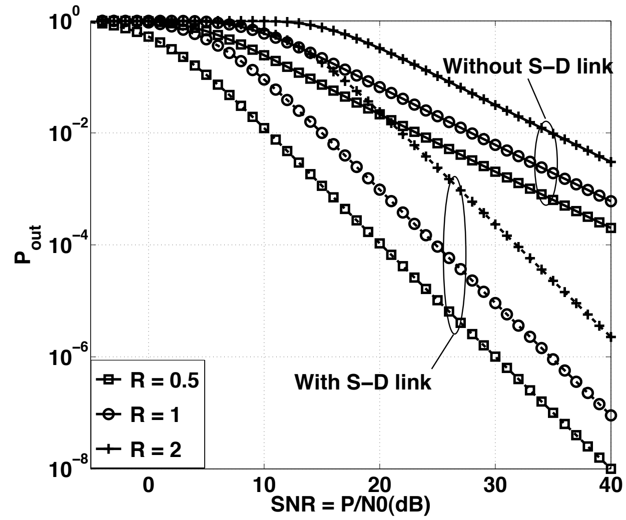

Fig. 4. Outage probability for various R. $\lambda_{sr} = \lambda_{rd} = \lambda_{sd} = 1$

A. Without Source to Destination Link

Let assume that the destination terminal does *not* hear the source transmission, which can be e.g due to occurrence of deep fading in S-D channel. Therefore the source communicates with the destination only via the relay. Therefore, the mutual information, assuming (3), is

$$I = \frac{1}{2}\log_2\left(1+\frac{SNR|h_{sr}|^2|h_{rd}|^2}{|h_{sr}|^2+|h_{rd}|^2+\sigma}\right) \text{bps/Hz} \qquad (7)$$

where the factor $\frac{1}{2}$ reflects the fact that information is conveyed to the destination in two time slots. The outage probability is defined as

$$\begin{aligned} p_{out}(R, SNR) &= \mathbb{P}\left(\frac{1}{2}\log_2(1+\frac{SNR|h_{sr}|^2|h_{rd}|^2}{|h_{sr}|^2+|h_{rd}|^2+\sigma}) \leq R\right) \\ &= \mathbb{P}\left(\frac{|h_{sr}|^2|h_{rd}|^2}{|h_{sr}|^2+|h_{rd}|^2+\sigma} \leq \frac{2^{2R}-1}{SNR}\right) \qquad (8) \end{aligned}$$

The probability specified in (8) was already derived in (4). By substituting $x = r = \frac{2^{2R}-1}{SNR}$ in (4), $p_{out}(R, SNR)$ is

$$\begin{aligned} p_{out}(R, SNR) &= 1-2e^{-(\lambda_{sr}+\lambda_{rd})r}\sqrt{\lambda_{sr}\lambda_{rd} r(r+\sigma)} \\ &\qquad \times K_1\left(2\sqrt{\lambda_{sr}\lambda_{rd} r(r+\sigma)}\right) \qquad (9) \end{aligned}$$

Fig. 4 (solid lines) illustrates the theoretical outage probability for the non S-D link scenario, for various values of transmission rate when $\lambda_{sr} = \lambda_{rd} = 1$.

B. With Source to Destination Link

Suppose that the destination hears both the source and the relay transmissions and employs MRC technique for combining them. Therefore the receiver output SNR at the destination can be written as

$$SNR_{tot} = \left(\frac{SNR|h_{sr}|^2|h_{rd}|^2}{|h_{sr}|^2+|h_{rd}|^2+\sigma} + SNR|h_{sd}|^2\right) \qquad (10)$$

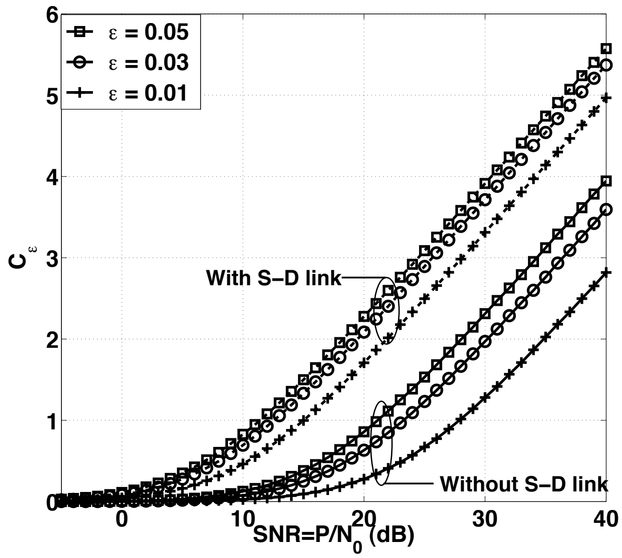

Fig. 5. ϵ-outage capacity for various ϵ values. $\lambda_{\text{sr}} = \lambda_{\text{rd}} = \lambda_{\text{sd}} = 1$

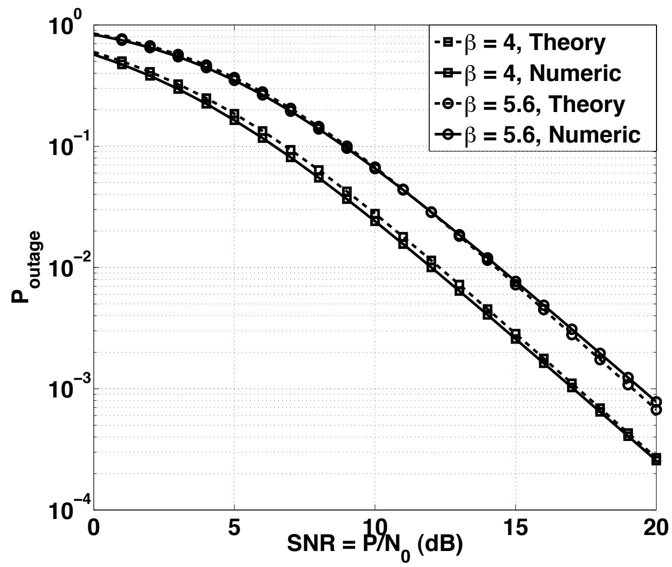

Fig. 6. Outage probability, theoretical and numerical results, R = 0.5.

Therefore, the outage probability is

$$p_{\text{out}}(R, \mathit{SNR}) = \mathbb{P}\,(\underbrace{\frac{|h_{\text{sr}}|^2|h_{\text{rd}}|^2}{|h_{\text{sr}}|^2+|h_{\text{rd}}|^2+\sigma}}_{X} + \underbrace{|h_{\text{sd}}|^2}_{Y} \leq \frac{2^{2R}-1}{\mathit{SNR}}) \quad (11)$$

where the RV X is distributed according (6) and RV Y is exponentially distributed with parameter λ_{sd}. Then, p_{out} in (11) can be written as

$$\begin{aligned} p_{\text{out}}(R, \mathit{SNR}) &= \mathbb{P}\,(X+Y \leq r) \\ &= \int_0^r \int_0^{r-x} f_Y(y)\cdot f_X(x)dydx \\ &= F_X(r) - e^{-\lambda_{\text{sd}}r}\int_0^r e^{\lambda_{\text{sd}}x}\cdot f_X(x)dx \\ &= \lambda_{\text{sd}}e^{-\lambda_{\text{sd}}r}\int_0^r e^{\lambda_{\text{sd}}x}\cdot F_X(x)dx \qquad (12) \\ &= 1 - e^{-\lambda_{\text{sd}}r} - 2\lambda_{\text{sd}}e^{-\lambda_{\text{sd}}r} \\ &\quad \times \int_0^r e^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}}-\lambda_{\text{sd}})x}\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}x(x+\sigma)} \\ &\qquad \times K_1\left(2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}x(x+\sigma)}\right)dx \end{aligned}$$

where the fourth equality is obtained using integration by parts algorithm. The outage probability in (12), or equivalently, the statistics of the receiver output SNR at the destination, becomes more challenging to obtain. In fact the integral appeared in (12) is non-trivial and seems not to have general closed form solution in the form of known mathematical functions. However, we develop some theoretical closed form solution in section IV-B. We also resort to the numerical integration methods for solving the problem.We employ Gauss-Kronrod quadrature adaptive integration algorithm [11] to obtain p_{out} from (12). Fig. 4 (dashed lines) present the outage probability for various values of *SNR* and σ using Gauss-Kronrod algorithm. The theoretical, closed form results will follow in section IV-B.
Fig. 5 illustrates the outage capacity for the scenarios explained in III-A (solid lines) and III-B (dashed lines).

IV. Asymptotic Outage Behaviour

We will evaluate both the scenarios explained in III-A and III-B at high and low *SNR* regimes.

A. Low SNR

For the relay scenario explained in III-A: the outage probability is given in (9). When $\sigma \to \infty$ (i.e. $\mathit{SNR} \to 0$), $K_1(\cdot) \to 0$, which means that the $p_{\text{out}} \to 1$. That is intuitively true, because at low *SNR* ($\sigma \to \infty$), the relay, almost, only forwards the receiver noise towards the destination.

For the relay scenario explained in (III-B): the outage probability is given in (11). At low *SNR* regime, $\frac{|h_{\text{sr}}|^2|h_{\text{rd}}|^2}{|h_{\text{sr}}|^2+|h_{\text{rd}}|^2+\sigma} \to 0$ as $\sigma \to \infty$. Then, outage probability can be written as

$$\begin{aligned} p_{\text{out}}(R, \mathit{SNR}) &= \mathbb{P}(|h_{\text{sd}}|^2 \leq \frac{2^{2R}-1}{\mathit{SNR}}) \\ &= 1 - e^{-\lambda_{\text{sd}}r} \qquad (13) \end{aligned}$$

where $r = \frac{2^{2R}-1}{\mathit{SNR}}$.Therefore, the outage probability of a AF relaying systems subject to Rayleigh fading is equivalent to the outage probability of the direct source to destination transmission. Then, the ϵ-outage capacity can be calculated from

$$\mathbb{P}(|h_{\text{sd}}|^2 < \frac{2^{2C_\epsilon}-1}{\mathit{SNR}}) = \epsilon. \qquad (14)$$

Assuming $\lim_{x\to 0}\log(1+x) = x$, and some manipulations, (14) will be simplified to

$$\lim_{\substack{\epsilon\to 0 \\ \mathit{SNR}\to 0}} \frac{C_\epsilon}{\epsilon \mathit{SNR}} = \frac{1}{2\lambda_{\text{sd}}} \quad \text{bps/Hz.} \qquad (15)$$

From MIMO theory it is well known that the optimum transmission strategy at low SNR is to allocate all the available power to the best antenna (the antenna corresponding to the strongest eigenmode of MIMO channel). Likewise, since the relay terminal does not improve the capacity at low SNR, the best policy is to allocate all available power to the source transmission, instead of splitting it between the source and the relay. The channel capacity will be doubled if the source terminal transmit with full power. (i.e. $P_{\text{r}} = 0$ and $P_{\text{s}} = 2P = 2\,SNR$)

$$\lim_{\substack{\epsilon\to 0\\ SNR\to 0}} \frac{C_\epsilon}{\epsilon SNR} = \frac{1}{\lambda_{\text{sd}}} \quad \text{bps/Hz}. \tag{16}$$

From inspection of (15) and (16) it is clear that splitting total available power between the source and the relay is not optimum at very low *SNR*. In fact at very low *SNR* regime, the best policy is to *avoid* cooperation. However, the minimum instantaneous *SNR* which one can benefit cooperation is out of scope of the paper.

B. High SNR

For the relay scenario explained in III-A: the outage probability at high *SNR* from (9) is

$$\begin{aligned} p_{out}(R, SNR) &= 1 - 2r\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}e^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}})r} \\ &\quad \times K_1\left(2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}r\right) \end{aligned} \tag{17}$$

where $r = \frac{2^{2R}-1}{SNR}$. The equation in (17) is derived by substituting $\sigma = 0$ in (9) and assuming that r is arbitrary value depending on R and *SNR*. Furthermore, suppose that $r \to 0$, i.e. low transmission rate [3]. Then, using [12, 10.30.2], $K_1(2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}r) \approx \frac{1}{2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}r}$. Therefore, (17) will be further simplified to

$$\begin{aligned} p_{out}(R, SNR) &= \lim_{\substack{r\to 0\\ SNR\to\infty}} 1 - e^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}})r} \quad (18)\\ &= (\lambda_{\text{sr}} + \lambda_{\text{rd}})r \\ &= (\lambda_{\text{sr}} + \lambda_{\text{rd}})\left(\frac{2^{2R}-1}{SNR}\right)^1. \quad (19) \end{aligned}$$

Note that (18) is equivalent to the CDF of an exponential RV with parameter $\lambda_{\text{sr}} + \lambda_{\text{rd}}$. Consequently, at high *SNR* and low transmission rate regime, the S-R-D link can be modelled by an exponential random variable with parameter $\lambda_{\text{sr}} + \lambda_{\text{rd}}$. Interestingly, this result is reported in [3] too, in which obtains the same results taking a totally different approach. The exponent "1" in (19) proves that the relay path at high SNR provides full diversity order.

For the relay scenario explained in III-B: , the outage probability will be calculated from (12) when $\sigma \to 0$

$$\begin{aligned} p_{\text{out}}(R, SNR) &= 1 - e^{-\lambda_{\text{sd}}r} - 2\lambda_{\text{sd}}\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}\, e^{-\lambda_{\text{sd}}r} \\ &\quad \times \int_0^r xe^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}}-\lambda_{\text{sd}})x}K_1(2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}x)dx \end{aligned} \tag{20}$$

As explained before, the integral in (20) is non-trivial but for the case when $(\lambda_{\text{sr}} + \lambda_{\text{rd}} - \lambda_{\text{sd}}) = 2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}$, the closed form solution is available as follows

$$\begin{aligned} p_{\text{out}}(R, SNR) &= 1 - \left(1 + \frac{2}{3\beta}\right)e^{-\lambda_{\text{sd}}r} \\ &\quad - \frac{\beta\lambda_{\text{sd}}}{3}r^2e^{-(\lambda_{\text{sr}}+\lambda_{\text{rd}})r}(K_1(\beta r) - K_2(\beta r)) \end{aligned} \tag{21}$$

where $\beta = (\lambda_{\text{sr}} + \lambda_{\text{rd}} - \lambda_{\text{sd}}) = 2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}$. In order to obtain (21) from (20), we employ [12, 10.43.9] in which

$$\begin{aligned} &\int_0^r e^{-x}x^\nu K_\nu(x)dx \\ &= \frac{e^{-r}r^{\nu+1}}{2\nu+1}(K_\nu(r) - K_{\nu+1}(r)) + \frac{2^\nu\Gamma(\nu+1)}{2\nu+1} \end{aligned} \tag{22}$$

Fig. 6 shows the theoretical outage probability obtained using (21) and also using numerical Gauss-Kronrod adaptive algorithm in (12) for various values of β and wide *SNR* regime. The perfect merge of numerical and theoretical curves confirms the correctness of the calculations.

Moreover, when $r \to 0$ (i.e. low and moderate transmission rate and high *SNR*), (20) can be further simplified to

$$\begin{aligned} p_{\text{out}}(R, SNR) &= \lim_{\substack{r\to 0\\ SNR\to\infty}} 1 - \frac{(\lambda_{\text{sr}} + \lambda_{\text{rd}})e^{-\lambda_{\text{sd}}r} - \lambda_{\text{sd}}e^{-(\lambda_{\text{sr}}+\lambda_{\text{sr}})r}}{\lambda_{\text{sr}} + \lambda_{\text{rd}} - \lambda_{\text{sd}}} \\ &= \frac{1}{2}\lambda_{\text{sd}}(\lambda_{\text{sr}} + \lambda_{\text{rd}})r^2 \\ &= \frac{1}{2}\lambda_{\text{sd}}(\lambda_{\text{sr}} + \lambda_{\text{rd}})\left(\frac{2^{2R}-1}{SNR}\right)^2 \end{aligned} \tag{23}$$

where we again apply the assumption that $K_1(2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}r) \approx \frac{1}{2\sqrt{\lambda_{\text{sr}}\lambda_{\text{rd}}}r}$ in (20) and exploit series expansion of $e^{\alpha r}$ at $r \to 0$. The exponent "2" in (23) shows the diversity order where one of the orders is provided by S-D link and the other one by the S-R-D link.

V. Conclusion

An equivalent S-R-D channel model is derived for AF relaying protocol. Using the channel model, the outage probability (or equivalently outage capacity) is derived for the S-D-R link. the problem is then extended to a more complicated scenario where a direct link between the source and the destination is available. Theoretical results on outage probability are derived. The asymptotic (high and low SNR) behaviour of the communication systems of consideration are investigated. The comparison of theoretical and numerical results confirms the correctness of calculations.

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