CEJOR
DOI 10.1007/s10100-012-0260-9

ORIGINAL PAPER

Multiple equilibria and indifference-threshold points in a rational addiction model

Jonathan P. Caulkins · Gustav Feichtinger · Richard F. Hartl · Peter M. Kort · Andreas J. Novak · Andrea Seidl

Abstract Becker and Murphy (J Polit Econ 96(4):675–700, 1988) have established the existence of unstable steady states leading to threshold behavior for optimal consumption rates in intertemporal rational addiction models. In the present paper a simple linear-quadratic optimal control model is used to illustrate how their approach fits into the framework of multiple equilibria and indifference-threshold points. By changing the degree of addiction and the level of harmfulness we obtain a variety of behavioral patterns. In particular we show that when the good is harmful as well as very addictive, an indifference-threshold point, also known in the literature as a Skiba point, separates patterns converging to either zero or maximal consumption, where the latter occurs in the case of a high level of past consumption. This implicitly shows that an individual

J. P. Caulkins
Carnegie Mellon University, H. John Heinz III College,
5000 Forbes Avenue, Pittsburgh, PA 15213-3890, USA

G. Feichtinger (✉) · A. Seidl
Department for Operations Research and Control Systems,
Institute for Mathematical Methods in Economics,
Vienna University of Technology, Argentinierstr. 8, 1040 Vienna, Austria
e-mail: gustav@eos.tuwien.ac.at

R. F. Hartl · A. J. Novak
Department of Business Administration, University of Vienna,
Bruennerstr. 72, 1210 Vienna, Austria

P. M. Kort
Department of Econometrics and Operations Research and Center,
Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

P. M. Kort
Department of Economics, University of Antwerp, Prinsstraat 13, 2000 Antwerp, Belgium

Published online: 27 July 2012

needs to be aware in time of these characteristics of the good. Otherwise, he/she may start consuming so much that in the end he/she is totally addicted.

Keywords Optimal control · Indifference points · History-dependence · Rational addiction

1 Introduction

In the seventies and eighties a theory of rational addiction was developed in a series of papers. 'Rational' in this context means that the consumers maximize utility continuously over time. While some might see addiction as the antithesis of rational behavior and the strength of empirical support has been challenged (Auld and Grootendorst 2004), the majority of economists working in this field believe the models demonstrate the power of economic reasoning to produce real-world implications concerning addictive behavior; see Melberg and Rogeberg (2010).

An early forward-looking maximization of an intertemporal utility stream with stable preferences was presented by Ryder and Heal (1973). Their utility function does not depend only on the consumption rate, but also on the past (accumulated) consumption. They identified a special property of such a state-dependent utility called 'adjacent complementarity'.

Another important attempt to model rational addiction is by Stigler and Becker (1977); their analysis considers the concepts of beneficial and harmful addiction. If greater present consumption lowers future utility, habits or addictions are called harmful. Extending this approach, Iannaccone (1986) delivers a further clarification of this concept; see also Léonard (1989), Orphanides and Zervos (1994), Orphanides and Zervos (1995), Orphanides and Zervos (1998), and Becker (1992). More recent work is by Braun and Vanini (2003) and Gavrila et al. (2005).

Becker and Murphy (1988) pursued the idea that consumers anticipate the future consequence of their choices. Denoting a good as potentially addictive if increasing past consumption raises the utility of current consumption, they showed that steady-state consumption is *unstable* when the degree of addiction is strong. Becker and Murphy were the first authors stressing the importance of unstable equilibria to explain addictive behavior. They state this as follows: '... *powerful complementaries* [i.e. a substantial effect of past consumption on current consumption] *cause some steady states to be unstable... even small deviations from consumption of an unstable steady state can lead to large cumulative rises over time in addictive consumption or to rapid falls in consumption to abstention*'.

This analysis of Becker and Murphy is related to indifference-threshold points[1] without referring to them explicitly. Readers might have guessed from Becker and Murphy (1988) that an indifference-threshold point might occur, but still be worried that perhaps there is always convergence to one of the steady states. Here we determine the scenarios under which indifference-threshold points occur, and also determine in which cases we always have convergence to one of the steady states. The purpose of

[1] For a discussion of this topic see Grass et al. (2008, Ch. 5).

the paper is to draw attention to this important but neglected aspect. For this aim the solution structure of a simple linear-quadratic model is analyzed for varying model parameters. The paper by Gavrila et al. (2005) might be seen in this sense as a forerunner of the present paper; they consider the implications of a budget constraint on a consumer of an addictive substance.

The contribution of our paper is that within our framework we can exactly identify the scenarios under which in the long run individuals end up consuming nothing, consuming the maximal amount, or consuming at an intermediate level. Interestingly, the above described indifference-threshold behavior arises when the good is very addictive and harmful. An addictive good has the characteristic that the marginal utility of consumption goes up with past consumption. This implies that consumers have to be aware in time whether a good is strongly addictive and harmful. This is because the danger arises that initially they consume so much that they get addicted and end up consuming large amounts, which gives a lot of harm. On the other hand, when they know in time how addictive/harmful consuming the good is, in the end they will be safe by consuming nothing. Here the level of past consumption is crucial, and we show that an indifference-threshold point separates the regions of past consumption levels leading to these two different behaviors.

In our framework history-dependence can occur in two different ways. The first arises when the unstable steady state is an unstable node. Then this unstable node can be a threshold point itself. In this scenario the long-run solution always depends only on the initial state value. In the second case the unstable steady state is either a focus or a node and there is an overlap of solution paths in the state-space. Then it is more difficult to determine the location of the indifference-threshold point.

Our paper is organized as follows. In Sect. 2 a simple linear-quadratic model is presented. In Sect. 3 the first-order optimality conditions are stated and used for a phase portrait analysis. Further, different solution structures for varying parameters are calculated. Particular interest is laid on the discussion of thresholds. Section 4 concludes. A brief introduction into multiple equilibria, indifference-threshold points and path dependency can be found in the "Appendix".

2 The model

The model we analyze is a special case of the one studied by Becker and Murphy (1988). We do not explicitly consider budget constraints or the price for the addictive good, but constrain consumption by assuming that there is a maximum amount that an individual can consume. We also do not allow negative consumption. Utility u of an individual depends both on the current consumption rate c as well as on a stock S denoted as 'consumption capital'. The state variable S measures past (accumulated) consumption, sometimes also called habit. Thus,

$$u = u(c, S),$$

where $c = c(t)$, $S = S(t)$ are time-dependent variables connected by

$$\dot{S} = c - \delta S, \tag{1}$$

and the initial value

$$S(0) = S_0 \tag{2}$$

is given.

The instantaneous depreciation rate δ measures the exogenous rate of depreciation of the consumption stock and is assumed to be constant.

We assume that the maximum possible consumption is 1, thus we have the control constraint $0 \leq c \leq 1$.

Remark 1 The constraint $0 \leq c \leq 1$ implies that the state S is restricted to the interval $[0, 1/\delta]$, provided that $S_0 \in [0, 1/\delta]$.

Similar to Becker and Murphy (1988, p. 679, eq. (7)), we restrict attention to a quadratic utility function

$$u(c, S) = \alpha_c c + \frac{\alpha_{cc}}{2} c^2 + \alpha_S S + \frac{\alpha_{SS}}{2} S^2 + \alpha_{cS} c S. \tag{3}$$

The coefficients α have the following signs

$$\alpha_c > 0, \quad \alpha_{cc} < 0, \tag{4}$$

$$\alpha_S < 0, \quad \alpha_{SS} < 0, \tag{5}$$

$$\alpha_{cS} > 0. \tag{6}$$

The signs (4) are economically clear and reflect the concavity of u w.r.t. c and S, respectively. The price of the addictive good is contained in α_c.

The negativity of α_S and α_{SS}, i.e. assumption (5), is known as harmful addiction in the literature (see, e.g., Iannaccone 1986; Dockner and Feichtinger 1993). Consequently, present consumption leads to a lower future utility.

The positive interaction (6) is essential for addictive behavior. It says that

$$\alpha_{cS} = \frac{\partial}{\partial S} \frac{\partial u(c, S)}{\partial c} > 0,$$

i.e. the marginal utility of current consumption *increases* with past consumption.

We do not model budget constraints, the concavity of u with respect to c implicitly bounds consumption rates.

Then the optimization problem reads as follows:

$$\max_c \int_0^\infty e^{-rt} u(c, S) dt, \tag{7}$$

subject to (1) and (2) and $0 \leq c \leq 1$, where $r > 0$ denotes the time preference rate and $u(c, S)$ is given by (3).[2]

[2] Note that here and in the following the time argument t is mostly omitted.

3 Classifying solutions according to degree of addiction and harmfulness

We apply Pontryagin's maximum principle and show that it is possible to produce explicit analytical expressions for the steady states and also to gain insights concerning their stability properties. The current value Hamiltonian is

$$H = \alpha_c c + \alpha_{cS} c S + \frac{\alpha_{cc} c^2}{2} + \alpha_S S + \frac{\alpha_{SS} S^2}{2} + \lambda\,(c - \delta S)\,,$$

which gives the necessary optimality conditions

$$\begin{aligned} H_c &= \alpha_c + \alpha_{cS} S + \alpha_{cc} c + \lambda = 0, \\ \dot{\lambda} &= (r + \delta)\,\lambda - \alpha_S - \alpha_{SS} S - \alpha_{cS} c. \end{aligned}$$

We derive that the optimal consumption c^* is given by

$$c^* = \begin{cases} 0 & \text{if } \lambda < -\alpha_{cS} S - \alpha_c, \\ \frac{\alpha_c + \alpha_{cS} S + \lambda}{-\alpha_{cc}} & \text{if } -\alpha_{cS} S - \alpha_c < \lambda < -\alpha_{cc} - \alpha_{cS} S - \alpha_c, \\ 1 & \text{if } -\alpha_{cc} - \alpha_{cS} S - \alpha_c < \lambda. \end{cases} \qquad (8)$$

Since

$$\begin{pmatrix} H_{SS} & H_{cS} \\ H_{cS} & H_{cc} \end{pmatrix} = \begin{pmatrix} \alpha_{SS} & \alpha_{cS} \\ \alpha_{cS} & \alpha_{cc} \end{pmatrix},$$

the Hamiltonian is jointly concave in state and control iff $\alpha_{cc} < 0, \alpha_{SS} < 0$, and $\alpha_{cS}^2 < \alpha_{SS}\alpha_{cc}$.

First we analyze the canonical system along the boundary arcs with $c = 0$ and $c = 1$, respectively. The results are summarized in the following proposition.

Proposition 1

1. *There exists a boundary steady state along the arc with* $c \equiv 0$ *iff* $\alpha_S < -\alpha_c(r+\delta)$ *and it is given by*

$$\left(S_0^\infty, \lambda_0^\infty, c_0^\infty\right) = \left(0, \frac{\alpha_S}{r + \delta}, 0\right) \qquad (9)$$

2. *The second boundary steady state is given by*

$$\left(S_0^\infty, \lambda_0^\infty, c_0^\infty\right) = \left(\frac{1}{\delta}, \frac{\delta(\alpha_{cS} + \alpha_S) + \alpha_{SS}}{\delta(r + \delta)}, 1\right) \qquad (10)$$

and it is feasible iff

$$\frac{-\delta(r + \delta)(\alpha_{cc} + \alpha_c) - \delta\alpha_S - \alpha_{SS}}{r + 2\delta} < \alpha_{cS} \qquad (11)$$

3. *Both boundary steady states (provided they exist) are saddle path stable with real eigenvalues, one positive and one negative.*

Proof

1. Along the boundary arc with $c \equiv 0$ the canonical system reduces to:

$$\begin{aligned}\dot{S} &= -\delta S\\ \dot{\lambda} &= (r+\delta)\lambda - \alpha_S - \alpha_{SS}S\end{aligned}$$

The boundary steady state is therefore given by (9). It is feasible, i.e. matches the optimality condition (8) iff $\alpha_S < -\alpha_c(r+\delta)$.

2. Setting $c \equiv 1$ the canonical system along the second boundary arc is given by

$$\begin{aligned}\dot{S} &= 1 - \delta S\\ \dot{\lambda} &= (r+\delta)\lambda - \alpha_S - \alpha_{SS}S - \alpha_{cS}\end{aligned}$$

leading to the steady state (10).
It matches the optimality condition (8) iff

$$-(r+\delta)[\delta(\alpha_{cc}+\alpha_c)+\alpha_{cS}] < \delta(\alpha_{cS}+\alpha_S)+\alpha_{SS}$$

which is equivalent to (11).

3. Both boundary steady states are stable saddle-points as the eigenvalues of the Jacobian

$$J = \begin{pmatrix} -\delta & 0 \\ -\alpha_{SS} & r+\delta \end{pmatrix} \quad \text{are } -\delta, r+\delta.$$

□

Next we analyze the canonical system in the interior, i.e. for c given by

$$c^* = \frac{\alpha_c + \alpha_{cS}S + \lambda}{-\alpha_{cc}},$$

this yields

$$\begin{aligned}\dot{S} &= -\left(\frac{\alpha_{cS}}{\alpha_{cc}} + \delta\right)S - \frac{\lambda}{\alpha_{cc}} - \frac{\alpha_c}{\alpha_{cc}}\\ \dot{\lambda} &= \left(\frac{\alpha_{cS}^2}{\alpha_{cc}} - \alpha_{SS}\right)S + \left(r+\delta+\frac{\alpha_{cS}}{\alpha_{cc}}\right)\lambda - \alpha_S + \frac{\alpha_{cS}\alpha_c}{\alpha_{cc}}.\end{aligned} \tag{12}$$

The dynamic system gives rise to the following proposition.

Proposition 2 *The canonical system (12) possesses an interior steady state at*

$$S^\infty = \frac{-\alpha_S - \alpha_c(r+\delta)}{(r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS}} \tag{13}$$

iff either

$$0 < \delta[-\alpha_S - \alpha_c(r+\delta)] < (r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS} \tag{14}$$

or

$$(r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS} < \delta[-\alpha_S - \alpha_c(r+\delta)] < 0. \tag{15}$$

Proof

1. From the canonical system (12) we compute the $\dot{S} = 0-$isocline

$$\lambda = -(\delta\alpha_{cc} + \alpha_{cS})S - \alpha_c,$$

as well as the $\dot{\lambda} = 0-$ isocline

$$\lambda = \frac{\alpha_S\alpha_{cc} - \alpha_c\alpha_{cS} - (\alpha_{cS}^2 - \alpha_{cc}\alpha_{SS})S}{\alpha_{cc}(r+\delta) + \alpha_{cS}},$$

To calculate the interior steady state we obtain

$$-(\delta\alpha_{cc} + \alpha_{cS})S - \alpha_c = \frac{\alpha_S\alpha_{cc} - \alpha_c\alpha_{cS} - (\alpha_{cS}^2 - \alpha_{cc}\alpha_{SS})S}{\alpha_{cc}(r+\delta) + \alpha_{cS}}$$

which implies

$$S\{\underbrace{\alpha_{cS}^2 - \alpha_{cc}\alpha_{SS} - (\delta\alpha_{cc} + \alpha_{cS})[\alpha_{cc}(r+\delta) + \alpha_{cS}]}_{=-\alpha_{cc}[\alpha_{SS}+2\delta\alpha_{cS}+\delta^2\alpha_{cc}+r\delta\alpha_{cc}+r\alpha_{cS}]}\} = \alpha_{cc}[\alpha_c(r+\delta) + \alpha_S],$$

leading to (13).

2. An interior feasible steady state requires $0 < S^\infty < 1/\delta$, i.e.

$$0 < \frac{\delta[-\alpha_S - \alpha_c(r+\delta)]}{(r+2\delta)\alpha_{cS} + \delta\alpha_{cc}(r+\delta) + \alpha_{SS}} < 1.$$

If the denominator is positive, this implies (14); if the denominator is negative it leads to (15). □

Proposition 3 *The feasibility of the interior steady state as well as its stability is related to the existence of the boundary steady states. The following four cases can be distinguished based on the level and harmfulness of the addiction.*

	Less harmful $-\alpha_c(r+\delta)<\alpha_S$	More harmful $\alpha_S<-\alpha_c(r+\delta)$
Low addiction $\alpha_{cS}<\frac{1}{r+2\delta}[\delta(-\alpha_{cc}-\alpha_c)(r+\delta)$ $-\delta\alpha_S-\alpha_{SS}]$	Region 1 only (13) is feasible It is stable	Region 3 (9) is the only feasible steady state
High addiction $\alpha_{cS}>\frac{1}{r+2\delta}[\delta(-\alpha_{cc}-\alpha_c)(r+\delta)$ $-\delta\alpha_S-\alpha_{SS}]$	Region 2 (10) is the only feasible steady state	Region 4 both (9) & (10) are feasible, (13) is unstable

Proof To determine the stability of this interior steady state we compute the Jacobian, which is given by

$$J=\begin{pmatrix} -\frac{\alpha_{cS}}{\alpha_{cc}}-\delta & -\frac{1}{\alpha_{cc}} \\ -\alpha_{SS}+\frac{\alpha_{cS}^2}{\alpha_{cc}} & r+\delta+\frac{\alpha_{cS}}{\alpha_{cc}} \end{pmatrix}$$

with $\mathrm{tr}J=r$ and $\det J=-\frac{\alpha_{cS}}{\alpha_{cc}}[r+2\delta]-\delta(r+\delta)-\frac{\alpha_{SS}}{\alpha_{cc}}$.

As the eigenvalues are given by $\lambda_{1,2}=\frac{\mathrm{tr}J\pm\sqrt{(\mathrm{tr}J)^2-4\det J}}{2}$ the interior steady state (13) is therefore a/an

$$\begin{aligned} \text{saddle point (node)} &\Leftrightarrow \det J<0 \\ \text{unstable node} &\Leftrightarrow 0<\det J<\left(\frac{\mathrm{tr}J}{2}\right)^2 \\ \text{unstable focus} &\Leftrightarrow \left(\frac{\mathrm{tr}J}{2}\right)^2<\det J \end{aligned}$$

□

The case of an unstable interior steady state (south-east case of the table above) can be further divided into two sub-cases. The interior steady state is an unstable node, iff

$$0<\delta[-\alpha_S-\alpha_c(r+\delta)]<\alpha_{cS}(r+2\delta)+\alpha_{cc}\delta(r+\delta)+\alpha_{SS}<\frac{-\alpha_{cc}r^2}{4}.$$

It is an unstable focus, iff

$$0<\delta[-\alpha_S-\alpha_c(r+\delta)]<\alpha_{cS}(r+2\delta)+\alpha_{cc}\delta(r+\delta)+\alpha_{SS}, \quad \text{and}$$
$$\frac{-\alpha_{cc}r^2}{4}<\alpha_{cS}(r+2\delta)+\alpha_{cc}\delta(r+\delta)+\alpha_{SS}$$

In the model section we noted that α_{cS} stands for the effect of past consumption on the marginal utility of current consumption. It is clear that the larger α_{cS} is, the more addictive the good is. Therefore, in Proposition 3 we denote $\alpha_{cS}<\frac{1}{r+2\delta}[\delta((\alpha_{cc}-\alpha_c)(r+\delta)-\alpha_S)-\alpha_{SS}]$ as a "low addiction" scenario, whereas $\alpha_{cS}>\frac{1}{r+2\delta}[\delta((\alpha_{cc}-\alpha_c)(r+\delta)-\alpha_S)-\alpha_{SS}]$ is a "high addiction scenario". We know from the literature that the (negative) value of α_S is a measure for how harmful the addiction is. This explains that in Proposition 3 we have to deal with a less harmful addiction when $\alpha_S>-\alpha_c(r+\delta)$ and a more harmful one when $\alpha_S<-\alpha_c(r+\delta)$.

4 Numerical example

Our aim is to investigate the effects of a good's addictiveness and harmfulness on the individual's consumption behavior. For this reason we let the values of α_{cS} (addictiveness) and α_S (harmfulness) vary, while keeping the other parameter values fixed. Concerning the latter we choose the following values: $\delta = 0.1$, $r = 0.05$, $\alpha_c = 5.0$, $\alpha_{cc} = -10$, $\alpha_{SS} = -0.15625$. By changing the parameters α_S and α_{cS} we are able to capture all possible configurations with respect to the optimal solution. Changing the values of the other parameters would not lead to any more configurations.

	Less harmful $-0.75 < \alpha_S$	More harmful $\alpha_S < -0.75$
Weak addiction $\alpha_{cS} < 0.925 - 0.4\alpha_S$	Only interior steady state is feasible Stable	$(S = 0, c = 0)$ is the only Feasible steady state
Strong addiction $0.925 - 0.4\alpha_S < \alpha_{cS}$	$(S = 10, c = 1)$ is the only Feasible steady state	Both boundary steady states feasible, Interior steady state unstable

Figure 1 shows the 4 Regions in the $\alpha_S - \alpha_{cS}$ parameter space. In the upper left corner (tiny triangle) of Region 4 the interior steady state is an unstable node; in the remaining part of Region 4 it is an unstable focus. Since the utility function is jointly concave to the left of the dotted line, the optimal solution is unique in Region 4a and the saddle point corresponds to the threshold point separating two long-run solutions. In Region 4 three steady states are admissible; this does not mean, however, that all play a role with respect to the optimal solution within this region. Within the dashed line of Region 4b, indifference-threshold points occur at which the decision maker has the choice to either approach the high or the no addiction steady state. At the dashed line the indifference-threshold point coincides with one of the boundary steady states. Thus in the part of Region 4 close to Region 3 it is optimal to approach the high consumption steady state. Similarly, in Region 4, close to Region 2, it is optimal to always approach the no consumption steady state even though there are other candidates for the optimal solution.

Figures 2, 3, 4 and 5 show optimal solutions in the state-control space. Dashed lines depict the isoclines, dotted lines show where control constraints become active.

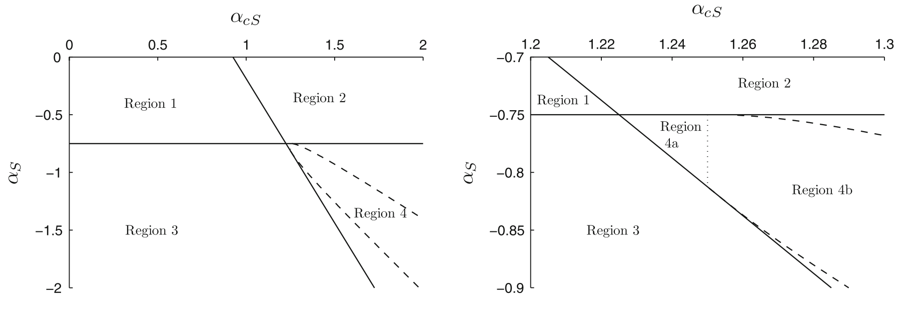

Fig. 1 Four different parameter regions can be distinguished, the *right panel* shows a zooming of the *left panel*

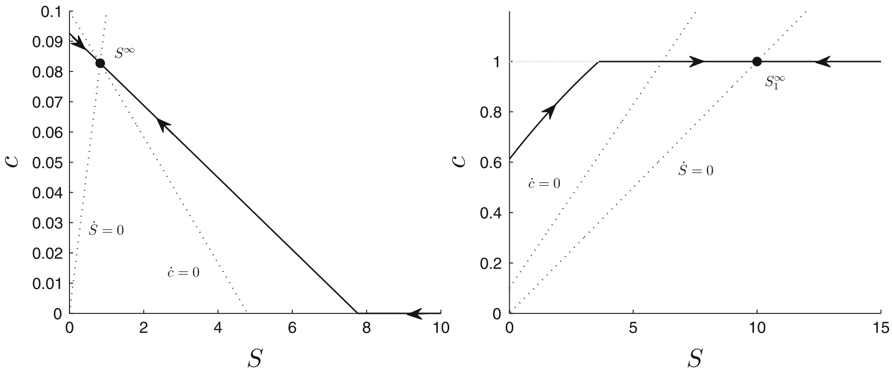

Fig. 2 Less harmful habits that are not (*left panel*; region 1) or are (*right panel*; region 2) very addictive

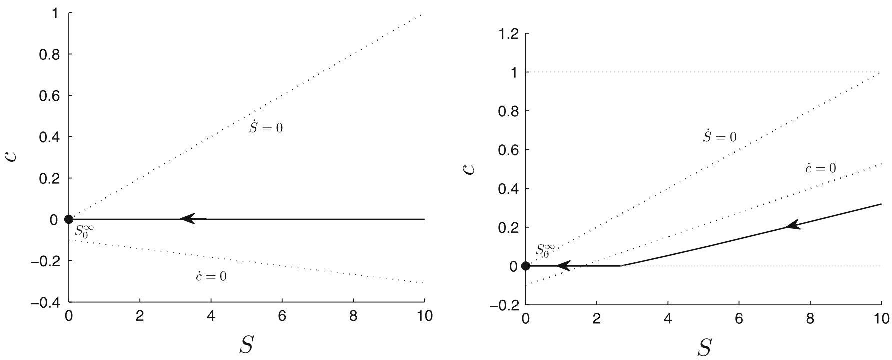

Fig. 3 More harmful habits that are not very addictive; region 3 (parameters for *right panel* $\alpha_S = -0.9$ and $\alpha_{cS} = 1$)

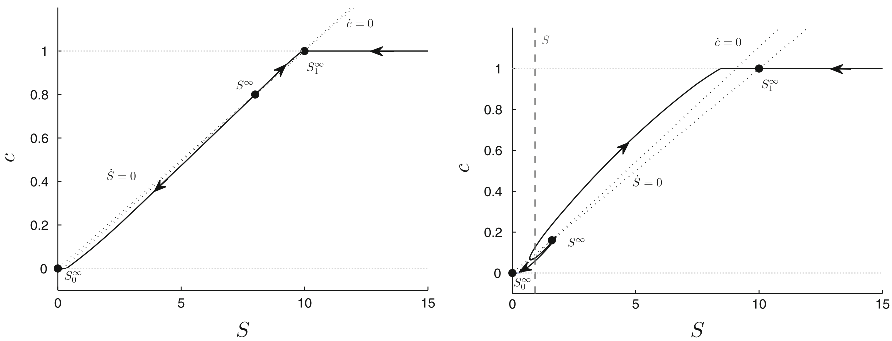

Fig. 4 Harmful, addictive habits: region 4

– **Region 1:** (less harmful, low addiction) Only interior steady state, which is a saddle, is feasible.
The parameters chosen for the numerical calculations are $\alpha_S = -0.6$, $\alpha_{cS} = 0.5$. We can find a steady state at $(0.827586, -4.58621, 0.0827586)$.

Here we have a kind of normal situation, where, after a possible adjustment phase, the individual consumes a fixed amount, corresponding to a unique steady state. This situation is depicted in the left panel of Fig. 2.

- **Region 2:** (less harmful, strong addiction) Only the steady state with $S = 1/\delta$, $c = 1$ is feasible.
  We choose $\alpha_S = -0.6, \alpha_{cS} = 1.5$. The steady state is located at (10., −4.41667, 1.).
  The good is strongly addictive, but this is relatively less harmful. Therefore, it makes sense that in the end we are in a situation with maximal consumption; see right panel of Fig. 2.
- **Region 3:** (more harmful, low addiction) The only feasible steady state can be found at $S = 0$, $\lambda = \alpha_S/(r + \delta)$, $c = 0$.
  Here we use $\alpha_S = -0.9, \alpha_{cS} = 0.5$ and can calculate the steady state at (0., −6., 0.).
  Here we have the opposite situation to Region 2. Now the good is less addictive but more harmful. It makes sense that a rational individual does not consume at all in this case, at least in the long term, which is confirmed in Fig. 3.
- **Region 4:** (harmful, strong addiction) All steady states are feasible.
  - **Region 4a:** The interior steady state is an unstable node.
    The used parameters are $\alpha_S = -0.78, \alpha_{cS} = 1.24$ and the steady states (0., −5.2, 0.), (10., −7.3, 1.), and (8., −6.92, 0.8).
  - **Region 4b:** The interior steady state is an unstable focus.
    We use $\alpha_S = -0.78, \alpha_{cS} = 1.3$ and find steady states at (0., −5.2, 0.), (10., −6.95, 1.), and (1.6, −5.48, 0.16).

  The parameters in most of Region 4 pose a difficult situation for the consumer. Consumption is harmful, but at the same time very addictive. The latter implies that marginal utility of consumption is strongly increasing in past consumption (S). This in turn implies that, although the good is harmful, the individual still increases consumption to the maximal level, or keeps it at the maximal level, when S is large. In such a case the individual is addicted to the good. However, once the individual did not consume a lot of it in the past, it is optimal to reduce consumption and refrain from consuming it at all in the end. Here we observe history dependent behavior: having consumed a lot of the good in the past leads to large consumption levels in the future, whereas otherwise we converge to a situation of no consumption. An indifference-threshold point, $\bar{S}$, separates the levels of past consumption that gives this different behavior, where of course the addicted person starts with an $S_0 > \bar{S}$.

  There are two different types of threshold points. In the left panel of Fig. 4 the threshold point is located at the unstable steady state, i.e. $\bar{S} = S^{\infty}$, which is an unstable node. Hence, by determining the unstable steady state we have also determined the location of the threshold point. In this case it holds that consumption, c, is a continuous function of past consumption, S. A sufficient (but not necessary) condition for such history dependent behavior to occur is that the Hamiltonian is concave in the unstable steady state. This was detected for the first time in Wirl and Feichtinger (2005) and extensively analyzed in Hartl et al. (2004) by employing a capital accumulation model.

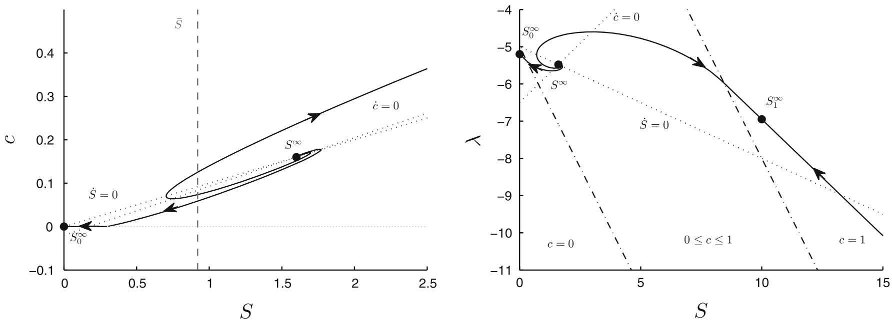

Fig. 5 Region 4b: zooming of the indifference-threshold point and unstable focus; phase portrait in the state-costate space

In the right panel of Fig. 4 and in Fig. 5, respectively, the unstable steady state is an unstable focus. Here the location of the threshold point is not so clear: it lies somewhere "close" to the unstable focus. While in principle the model is simple enough that it can be studied analytically, the resulting expressions are rather long and hard to interpret. Thus, numerical investigations are undertaken in order to find the exact location of the Skiba point. Another difference with the case of Fig. 4 is that here consumption is not a continuous function of S: right at the indifference-threshold point there is a discontinuity, i.e. the decision maker has the choice between two different strategies.

In the left panel of Fig. 4 the solution path remains closer to both isoclines than in the right panel. Consequently, the dynamics is faster in the right panel.

Note that the concave utility function already limits consumption. If in addition there was a budget constraint, set sufficiently low, that would effectively reduce the upper bound of consumption. This would lead to solutions with the same properties, qualitatively speaking. Only, the boundary steady state would be located differently, and the indifference-threshold point would move to the right.

5 Conclusions

According to Becker (1992) habits exist if current consumption is positively related to former consumption. Addiction is essentially the manifestation of a strong habit. Depending on their welfare effects, habits may be harmful or beneficial. Examples for harmful addiction are regular consumption of legal or illicit drugs, overeating, gambling, etc.

In a nutshell, Becker and Murphy's main result was that unstable steady states are crucial to understanding rational addictive behavior. Our analysis confirms that an increase of the interaction term α_{cS} measuring the degree of (potential) addiction, combined with a significant negative value of α_S, increases the likelihood that the interior steady state is unstable.

Becker and Murphy (1988) were well aware of the multiplicity of steady states resulting from their approach to addiction. They stress that with two long-run equilibria, '*consumption diverges from the unstable state toward zero or toward the* *sizable steady-state level*' (compare also Fig. 1 in Becker and Murphy 1988, p. 681). As illustrated in the "Appendix", much progress has been made since the early days of Skiba points to clarify the occurrence of multiple steady states.

Our aim in this paper was to illustrate how saddle point equilibria are separated by thresholds. For explanatory reasons we have selected a simple linear-quadratic scenario. For more complex models, the analysis proceeds essentially in the same manner. If we consider, e.g., models with two state variables, not only limit cycles can be established but also indifference-threshold curves. By starting at a point on such an indifference curve, we are indifferent between converging in the long run to strong addictive behavior or to abstinence.

Acknowledgments The authors like to thank Dieter Grass for his helpful comments. This research was supported by the Austrian Science Fund (FWF) under Grant P21410-G16.

6 Appendix: Multiple equilibria, points of indifference, and thresholds

In the sixties of the last century optimal control theory started its applications to economic analysis. A common feature of those early intertemporal optimization models is the existence of a *unique* long-run equilibrium. Or, to put it more precisely: the necessary optimality conditions resulting from Pontryagin's maximum principle, i.e. the canonical system, exhibits a *unique* steady state.

A well-known illustration is the golden rule of Ramsey-type optimal growth model. The neoclassical growth model of Cass (1965) and Koopmans (1965) predicts that countries will converge to a common standard of living.

In the seventies, however, this scenario has been enriched by the *multiplicity* of equilibria. This means that for given initial states (e.g., capital endowment) there exist *multiple* optimal solutions, i.e. the decision maker is *indifferent* about which to chose. The possibility of multiple equilibria provide a basis for the empirically observed heterogeneity of growth patterns.

By using a convex-concave production function, Skiba (1978) extended the Ramsey model and obtained an unstable steady state separating two saddle point equilibria. In the book of Brock and Malliaris (1989, Chap. 6) the threshold separating the basins of attraction of the two saddles was denoted as '*Skiba point*'.

However, there are forerunners. The first reference describing such a situation seems to be Clark (1971) dealing with a renewable resource model; see also Clark (1976). Forster (1975) found multiple equilibria in a pollution control problem. Sethi (1977, 1979) was another pioneer in this field. A first existence proof of a Skiba point was given by Dechert and Nishimura (1983).

A first wave of applications appeared in the eighties, e.g., Lewis and Schmalensee (1982) on renewable resources, Brock (1983) on lobbying, Dechert (1983) on regulated firms, Brock and Dechert (1985) on dynamic Ramsey pricing.

In the endogenous growth literature multiple equilibria have been used to explain the occurrence of development miracles and poverty traps (see, e.g., Lucas 1988).

More recent applications may be found in environmental economics. Here the literature on the so-called shallow lake model is particularly interesting; see Mäler (2000), Brock and Starrett (2003), Wagener (2003).

For a survey of further applications compare Grass et al. (2008, pp. 272–276). Deissenberg et al. (2004) examine different economic mechanisms that can generate multiple equilibria.[3]

In policy making, it may be important to recognize whether a given problem exhibits multiple equilibria. In *one-state* models two saddle point steady states are separated by an unstable equilibrium. The latter gives rise to a threshold separating basins of attraction surrounding the saddles. At such a threshold, a rational agent is indifferent between moving toward one or the other steady state. Small movements away from this threshold will lead to different optimal courses of actions depending on the direction of the slight change. It is this history-dependence (sometimes also denoted as path-dependence) which has the discussion of various economic problems considerably enriched. Note, however, that the mere existence and admissibility of multiple equilibria does not guarantee that the optimal solution is history-dependent, it might still be optimal to always approach a certain steady state.

To summarize: the optimal long-run stationary solution toward which an optimally controlled system converges can depend on the initial conditions.

Let us now put this scenarios to a formal ground.

Definition 1 (*Solution sets*) Let $x^*(\cdot)$ be the optimal state trajectory starting at x_0 then

$$\mathcal{S}(x_0) = \{x^*(\cdot) : x^*(0) = x_0\},$$

is called the *solution set*, and

$$\mathcal{S}^\infty(x_0) = \{x(x_0, \infty) : x(\cdot) \in \mathcal{S}(x_0)\},$$

where $x(x_0, \infty)$ is the limit set, is called the *asymptotic solution set*.

Definition 2 (*Indifference point*) Let $\mathcal{S}(x_0)$ be the solution set of an optimal control problem. If there exist $x_1^*(\cdot), x_2^*(\cdot) \in \mathcal{S}(x_0)$ and $t \in [0, T]$ satisfying

$$x_1^*(t) \neq x_2^*(t),$$

then x_0 is called an *indifference point*. If the solution set $\mathcal{S}(x_0)$ has cardinality $k < \infty$, then x_0 is called of *order* k; otherwise it is of *infinite order*.

Wagener's conjecture: In an optimal control model with n states, the maximal order of an indifference point is $n + 1$. An example for a threefold indifference point for a two-state production-inventory model is given in Feichtinger and Steindl (2006).

Definition 3 (*Threshold point*) Let us consider an infinite time horizon problem with $x(0) = x_0$. Then x_0 is called a *threshold point* if for every neighborhood U of x_0 there

[3] There, the nice medieval story of Buridan's donkey is mentioned. A donkey stands at equal distance from two identical and equidistant bales of hay, unable to decide toward which bale to go. As rational economic agent the donkey is indifferent between moving toward the one or other bale, i.e. two optimal long-run stationary solutions.

exist $x_{10}, x_{20} \in U$ satisfying

$$\mathcal{S}^{\infty}(x_{10}) \cap \mathcal{S}^{\infty}(x_{20}) = \emptyset, \quad \mathcal{S}^{\infty}(x_{i0}) \neq \emptyset, \quad i = 1, 2.$$

Definition 4 (*Indifference-threshold point*) Let us consider an infinite time horizon problem with $x(0) = x_0$. Then x_0 is called an *indifference-threshold point* if it is both an indifference point and a threshold point.

Remark 1 At the moment there exists no canonical nomenclature of these points. However what is denoted here as indifference-threshold point is also known as Skiba or DNSS point; see Grass et al. (2008).

Remark 2 Note that in a one-dimensional infinite time horizon model the occurrence of an unstable focus plays a crucial role regarding the existence of an indifference-threshold point described in Definition 4. An unstable node might be a threshold point. In this case the control is continuous, unlike the first case where it jumps at the indifference point.

The above discussion on multiple equilibria and Skiba points refers to *deterministic* optimal control problems with *infinite* time horizon. The multiplicity of optimal solutions may occur in models with *finite* time horizon. For a recent work in that context see Caulkins et al. (2010).

Finally, there are interesting extensions of multiple equilibria and threshold behavior to a stochastic framework; see, e.g., Bultmann et al. (2010).

References

Auld M, Grootendorst P (2004) An empirical analysis of milk addiction. J Health Econ 23(6):1117–1133
Becker GS (1992) Habits, addictions, and traditions. Kyklos 45(3):327–346
Becker GS, Murphy KM (1988) A theory of rational addiction. J Polit Econ 96(4):675–700
Braun N, Vanini P (2003) On habits and addictions. J Inst Theor Econ 159:603–626
Brock WA (1983) Pricing, predation and entry barriers in regulated industries. In: Evans DS (ed) Breaking up bell. North-Holland, New York, pp 191–229
Brock WA, Dechert WD (1985) Dynamic Ramsey pricing. Int Econ Rev 26(3):569–591
Brock WA, Malliaris AG (1989) Differential equations, stability and chaos in dynamic economics. North-Holland, Amsterdam
Brock WA, Starrett D (2003) Nonconvexities in ecological managment problems. Environ Resour Econ 26(4):575–624
Bultmann R, Feichtinger G, Tragler G (2010) Stochastic skiba sets: an example from models of illicit drug consumption. In: Lirkov I, Margenov S, Wasniewski J (eds) Large-scale scientific computing. Springer, Heidelberg, pp 239–246
Cass D. (1965) Optimum growth in an aggregative model of capital accumulation. Rev Econ Stud 32(3):233–240
Caulkins JP, Feichtinger G, Grass D, Hartl RF, Kort PM, Seidl A (2010) Skiba points in free end time problems: the option to sell the firm (in submission)
Clark C (1971) Economically optimal policies for the utilization of biologically renewable resources. Math Biosci 12(3–4):245–260
Clark CW (1976) Mathematical bioeconomics, the optimal management of renewable resources. Wiley-Interscience, New York
Dechert WD (1983) Increasing returns to scale and the reverse flexible accelerator. Econ Lett 13(1):69–75
Dechert WD, Nishimura K (1983) A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. J Econ Theory 31(2):332–354

Deissenberg C, Feichtinger G, Semmler W, Wirl F (2004) Multiple equilibria, history dependence, and global dynamics in intertemporal optimization models. In: Barnett WA, Deissenberg C, Feichtinger G (eds) Economic complexity: non-linear dynamics, multi-agents economies and learning. Elsevier, Amsterdam, pp 91–122

Dockner EJ, Feichtinger G (1993) Cyclical consumption patterns and rational addiction. Am Econ Rev 83(1):256–263

Feichtinger G, Steindl A (2006) DNS curves in a production/inventory model. J Optim Theory Appl 128(2):295–308

Forster BA (1975) Optimal pollution control with a nonconstant exponential rate of decay. J Environ Econ Manag 2:1–6

Gavrila C, Feichtinger G, Tragler G, Hartl RF, Kort PM (2005) History-dependence in a rational addiction model. Math Soc Sci 49(3):273–293

Grass D, Caulkins JP, Feichtinger G, Tragler G, Behrens DA (2008) Optimal control of nonlinear processes: with applications in drugs, corruption and terror. Springer, Heidelberg

Hartl RF, Kort PM, Feichtinger G, Wirl F (2004) Multiple equilibria and thresholds due to relative investment costs. J Optim Theory Appl 123(1):49–82

Iannaccone LR (1986) Addiction and satiation. Econ Lett 21(1):95–99

Koopmans TC (1965) On the concept of optimal economic growth. Pontificiae Academiae Scientiarum Scripta Varia 28(1):225–300

Léonard D (1989) Market behaviour of rational addicts. J Econ Psychol 10(1):117–144

Lewis TR, Schmalensee R (1982) Optimal use of renewable resources with nonconvexities in production. In: Mirman LJ, Spulber DF (eds) Essays in the economics of renewable resources. North-Holland, Amsterdam, pp 95–111

Lucas RE (1988) On the mechanics of economic development. J Monet Econ 22(1):3–42

Mäler KG (2000) Development, ecological resources and their management: a study of complex dynamic systems. Eur Econ Rev 44(4–6):645–665

Melberg HO, Rogeberg OJ (2010) Rational addiction theory: a survey of opinions. J Drug Policy Anal 3(1):5

Orphanides A, Zervos D (1994) Optimal consumption dynamics with non-concave habit-forming utility. Econ Lett 44(1–2):67–72

Orphanides A, Zervos D (1995) Rational addiction with learning and regret. J Polit Econ 103(4):739–758

Orphanides A, Zervos D (1998) Myopia and addictive behaviour. Econ J 108(446):75–91

Ryder HE, Heal GM (1973) Optimal growth with intertemporally dependent preferences. Rev Econ Stud 40:1–33

Sethi SP (1977) Nearest feasible paths in optimal control problems: theory, examples, and counterexamples. J Optim Theory Appl 23(4):563–579

Sethi SP (1979) Optimal advertising policy with the contagion model. J Optim Theory Appl 29(4):615–627

Skiba AK (1978) Optimal growth with a convex-concave production function. Econometrica 46(3): 527–539

Stigler GJ, Becker GS (1977) De gustibus non est disputandum. Am Econ Rev 67(2):76–90

Wagener FOO (2003) Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. J Econ Dyn Control 27(9):1533–1561

Wirl F, Feichtinger G (2005) History dependence in concave economies. J Econ Behav Organ 57(4): 390–407