Intermag

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A general integrator for the Landau-Lifshitz-Gilbert equation

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joint work with

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General LLG

Problem Formulation

LLG equation

$$\mathbf{m}_t = -\frac{1}{1+\alpha^2}\mathbf{m} \times \mathbf{h}_{\mathsf{eff}} - \frac{\alpha}{1+\alpha^2}\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\mathsf{eff}})$$

- constraints:

$$\mathbf{m}(0) = \mathbf{m}_0 \in H^1(\Omega, \mathbb{S}^2),$$
$$\partial_n \mathbf{m} = 0 \quad \text{on } (0,T) \times \partial\Omega.$$

- effective field
- exchange
- general field contribution
  - pointwise (uniaxial-, cubic anisotropy)
  - linear and continuous (strayfield)
  - uniformly monotone (nonlinear, multiscale)
  - m-independent (applied field)

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$$\mathbf{h}_{\text{eff}} = C_{\text{exch}} \Delta \mathbf{m} + \boldsymbol{\pi}(\mathbf{m})$$

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Weak Formulation

- $\mathbf{m} \in H^1(\Omega_\tau)$ with $|\mathbf{m}| \equiv 1$ a.e. on Ω_τ;
- for all $\phi \in C^\infty(\Omega_\tau)$;

$$\int_{\Omega_\tau} \mathbf{m}_t \cdot \phi - \alpha \int_{\Omega_\tau} (\mathbf{m} \times \mathbf{m}_t) \cdot \phi = \int_{\Omega_\tau} (\mathbf{h}_{\text{eff}} \times \mathbf{m}) \cdot \phi$$

- $\mathbf{m}(0, \cdot) = \mathbf{m}_0$ in the sense of traces;
- for a.e. $t' \in [0, T]$

$$\|\nabla \mathbf{m}(t')\|^2_{L^2(\Omega)} + \|\mathbf{m}_t\|^2_{L^2(\Omega_{t'})} \leq C;$$

Equivalent formulation

$\mathbf{m}_t - \alpha \mathbf{m} \times \mathbf{m}_t = \mathbf{h}_{\text{eff}} \times \mathbf{m}$

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Abstract algorithm

(i) Find $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$ s.t. for all $\boldsymbol{\psi}_h \in \mathcal{K}_{\mathbf{m}_h^j}$,

$$\alpha\langle \mathbf{v}_h^j, \boldsymbol{\psi}_h\rangle + \langle (\mathbf{m}_h^j \times \mathbf{v}_h^j), \boldsymbol{\psi}_h\rangle = -C_{\text{exch}}\langle \nabla(\mathbf{m}_h^j + \theta k \mathbf{v}_h^j), \nabla\boldsymbol{\psi}_h\rangle + \langle \boldsymbol{\pi}(\mathbf{m}_h^j), \boldsymbol{\psi}_h\rangle$$

(ii) Define $\mathbf{m}_h^{j+1}$ nodewise by $\mathbf{m}_h^{j+1}(z) = \frac{\mathbf{m}_h^j(z)+k\mathbf{v}_h^j(z)}{|\mathbf{m}_h^j(z)+k\mathbf{v}_h^j(z)|}$

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$$\alpha\mathbf{m}_t + \mathbf{m} \times \mathbf{m}_t = \mathbf{h}_{\text{eff}} - (\mathbf{m} \cdot \mathbf{h}_{\text{eff}})\mathbf{m}$$

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Convergence Result

For $t_j \le t \le t_{j+1}$ define:

$$\mathbf{m}_{hk}(t,\mathbf{x}) := \frac{t-jk}{k}\mathbf{m}_h^{j+1}(\mathbf{x}) + \frac{(j+1)k-t}{k}\mathbf{m}_h^{j}(\mathbf{x})$$

$$\mathbf{m}_{hk}^{-}(t,\mathbf{x}) := \mathbf{m}_h^{j}(\mathbf{x})$$

Convergence theorem

- Let $\theta \in (1/2, 1]$
- $\mathbf{m}_0 \in H^1(\Omega, \mathbb{S}^2)$ with $\mathbf{m}_h^0 \rightharpoonup \mathbf{m}_0$ as $h \to 0$.
- Let $\mathcal{T}_h$ satisfy certain angle condition
- $\|\boldsymbol{\pi}(\boldsymbol{n})\|_{L^2(\Omega)} \le C_{\boldsymbol{\pi}} \quad \forall \boldsymbol{n}$ with $|\boldsymbol{n}| \le 1$

  $\Longrightarrow \mathbf{m}_{hk}^{-} \to \mathbf{m}$ in $L^2(\Omega_\tau)$

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Maxwell-LLG

MLLG

- Setting: $\omega \Subset \Omega \subseteq \mathbb{R}^3$
  $\Omega \backslash \overline{\omega} \ldots$ vacuum

Maxwell-LLG system

$$\mathbf{m}_t - \alpha \mathbf{m} \times \mathbf{m}_t = \mathbf{h}_{\text{eff}} \times \mathbf{m} \quad \text{in } \omega_T := (0,T) \times \omega$$

$$\varepsilon_0 \mathbf{E}_t - \nabla \times \mathbf{H} + \sigma \chi_\omega \mathbf{E} = -\mathbf{J} \quad \text{in } \Omega_T := (0,T) \times \Omega$$

$$\mu_0 \mathbf{H}_t + \nabla \times \mathbf{E} = -\mu_0 \mathbf{m}_t \quad \text{in } \Omega_T$$

- $\mathbf{h}_{\text{eff}} = C_{\text{exch}} \Delta \mathbf{m} + \mathbf{H} + \boldsymbol{\pi}(\mathbf{m})$
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$$\mathbf{m}(0) = \mathbf{m}_0 \text{ in } \omega, \quad \mathbf{E}(0) = \mathbf{E}_0, \mathbf{H}(0) = \mathbf{H}_0 \text{ in } \Omega$$

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MLLG algorithm

- Find $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$ s.t. for all $\phi_h \in \mathcal{K}_{\mathbf{m}_h^j}$

$$\alpha\langle\mathbf{v}_h^j, \phi_h\rangle + \langle(\mathbf{m}_h^j \times \mathbf{v}_h^j), \phi_h\rangle = -C_e\langle\nabla(\mathbf{m}_h^j + \theta k \mathbf{v}_h^j), \nabla\phi_h\rangle + \langle\mathbf{H}_h^j, \phi_h\rangle + \langle\boldsymbol{\pi}(\mathbf{m}_h^j), \phi_h\rangle$$

- Find $(\mathbf{E}_h^{j+1}, \mathbf{H}_h^{j+1}) \in (\mathcal{X}_h, \mathcal{Y}_h)$ s.t. for all $(\psi_h, \zeta_h) \in \mathcal{X}_h \times \mathcal{Y}_h$

$$\varepsilon_0\langle d_t\mathbf{E}_h^{j+1}, \psi_h\rangle - \langle\mathbf{H}_h^{j+1}, \nabla \times \psi_h\rangle + \sigma\langle\chi_\omega\mathbf{E}_h^{j+1}, \psi_h\rangle = -\langle\mathbf{J}^j, \psi_h\rangle$$

$$\mu_0\langle d_t\mathbf{H}_h^{j+1}, \zeta_h\rangle + \langle\nabla \times \mathbf{E}_h^{j+1}, \zeta_h\rangle = -\mu_0\langle\mathbf{v}_h^j, \zeta_h\rangle$$

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- Define $\mathbf{m}_h^{j+1} \in \mathcal{M}_h$ nodewise by $\mathbf{m}_h^{j+1}(z) = \frac{\mathbf{m}_h^j(z) + k\mathbf{v}_h^j(z)}{|\mathbf{m}_h^j(z) + k\mathbf{v}_h^j(z)|}$

MLLG algorithm

- Find $\mathbf{v}_h^j \in \mathcal{K}_{\mathbf{m}_h^j}$ s.t. for all $\phi_h \in \mathcal{K}_{\mathbf{m}_h^j}$

$$\alpha\langle \mathbf{v}_h^j, \phi_h\rangle + \langle(\mathbf{m}_h^j \times \mathbf{v}_h^j), \phi_h\rangle = -C_e\langle\nabla(\mathbf{m}_h^j + \theta k \mathbf{v}_h^j), \nabla\phi_h\rangle \\ + \langle \mathbf{H}_h^j, \phi_h\rangle + \langle \boldsymbol{\pi}(\mathbf{m}_h^j), \phi_h\rangle$$

- Find $(\mathbf{E}_h^{j+1}, \mathbf{H}_h^{j+1}) \in (\mathcal{X}_h, \mathcal{Y}_h)$ s.t. for all $(\psi_h, \zeta_h) \in \mathcal{X}_h \times \mathcal{Y}_h$

$$\varepsilon_0\langle d_t\mathbf{E}_h^{j+1}, \psi_h\rangle - \langle \mathbf{H}_h^{j+1}, \nabla\times\psi_h\rangle + \sigma\langle\chi_\omega \mathbf{E}_h^{j+1}, \psi_h\rangle = -\langle \mathbf{J}^j, \psi_h\rangle$$
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- Unconditionally convergent integrator for general LLG
- Inclusion of full Maxwell system
- Decoupled system ⇒ only 2 linear systems per timestep

Extensions & Future work

- Improved energy bounds
- Inclusion of Magnetostrictive effects
- Treatment of Eddy-Current equation
- General coupling operator
- Higher order extension

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Thanks for Listening!

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