Degrees of Freedom of Certain Interference Alignment Schemes with Distributed CSI

Paul de Kerret*, Maxime Guillaud§, and David Gesbert*
* Eurecom, Campus SophiaTech, 450 Route des Chappes, 06410 Biot, France
{dekerret,gesbert}@eurecom.fr
§ Institute of Telecommunications - Vienna University of Technology
Gußhausstraße 25 / E389, A-1040 Vienna, Austria
guillaud@tuwien.ac.at

***Abstract*—In this work, we consider the use of interference alignment (IA) in a MIMO interference channel (IC) under the assumption that each transmitter (TX) has access to channel state information (CSI) that generally differs from that available to other TXs. This setting is referred to as *distributed CSIT*. In a setting where CSI accuracy is controlled by a set of power exponents, we show that in the static 3-user MIMO square IC, the number of degrees-of-freedom (DoF) that can be achieved with distributed CSIT is at least equal to the DoF achieved with the worst accuracy taken across the TXs *and* across the interfering links. We conjecture further that this represents exactly the DoF achieved. This result is in strong contrast with the *centralized CSIT* configuration usually studied (where all the TXs share the same, possibly imperfect, channel estimate) for which it was shown that the DoF achieved at receiver (RX) i is solely limited by the quality of its *own* feedback. This shows the critical impact of CSI discrepancies between the TXs, and highlights the price paid by distributed precoding.**

I. Introduction

It has recently been shown that an improvement in the DoF achieved over certain multi-user channels could be obtained by designing the transmission scheme such that interference aligns at the RXs [1], [2], [3]. Considering static MIMO channels, a large number of iterative IA algorithms have then been developed (see [4], [5], [6], among others).

One of the main obstacles to the practical use of IA comes from the need to gather the CSI relative to the global multi-user channel. Indeed, the resources available for feedback are very limited and make the obtaining of the multi-user CSI at the TX (CSIT) in a timely manner especially challenging [7].

Consequently, the study of how CSIT requirements for IA methods can somehow be alleviated has become an active research topic in its own right [4], [8], [9], [5], [10]. Another line of work consists in studying the minimal number of CSI quantization bits that should be conveyed to the TXs to achieve some given DoF using IA [8], [9], [11]. It should be noted that in all these works, every one of the TXs is assumed to be provided with the *same* quantized CSIT, meaning that the imperfect estimates are perfectly shared between the TXs, which we call the *centralized* CSIT configuration, since this setting is equivalently obtained when all the precoders are computed centrally and then shared to the TXs.

Since the interfering TXs in an IC are usually not colocated, this assumption is likely to be breached, for instance because each TX receives its channel estimate via a different feedback channel. For example, if the CSIT is obtained via an analog feedback broadcast from the RXs, as in [10], each TX receives a different estimation of the multi-user channel with a priori different accuracies. An alternative possibility, currently envisioned for future LTE systems, consists in letting each RX feeds back its CSI to its serving TX which then forwards it to the other TXs [7], [12]. The sharing step leads in most cases to CSIT aging, or requires further quantization. In both scenarios, each TX receives its *own* estimate of the multi-user channel based on which it computes its precoder without additional communications with the other TXs. This case has been first denoted in [13], [14] as the *distributed CSIT* configuration.

Thus, we investigate here how the works [8], [9] dealing with IA in the centralized CSIT configuration extend to the distributed CSIT case. Specifically, our main contributions are as follows:

- In a general MIMO IC, we provide a sufficient criterion on the accuracy of the precoder design to achieve the full (perfect CSI) DoF.
- Studying the particular 3-User MIMO square setting, we provide a closed-form expression for the achieveable DoF. It is shown to depend on the worst accuracy across the TXs and the channel elements.

Notations: We write $x \doteq y$ to represent the exponential equality in the SNR P, i.e., $\lim_{P\rightarrow\infty} \log_2(x)/\log_2(P) = \lim_{P\rightarrow\infty} \log_2(y)/\log_2(P)$. The inequalities $\dot{\leq}$ and $\dot{\geq}$ are defined similarly. $\lambda_i(\mathbf{A})$ denotes the ith largest eigenvalue of the diagonalizable matrix $\mathbf{A}$ while $\lambda_{\min}(\mathbf{A})$ denotes its minimal eigenvalue. $\mathrm{E}_{\mathcal{A}}[\cdot]$ denotes the expectation over the subspace $\mathcal{A}$.

II. System Model

A. MIMO interference channel

We consider a conventional static MIMO IC with K users [6] and assume that each TX has its *own* CSIT in the form

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of an imperfect estimate of the whole multi-user channel state. TX j is equipped with M_j antennas and RX i with N_i antennas. The antenna configuration is supposed to be tightly-feasible in the sense that the number of antennas available is the minimal one which allows to achieve the DoF desired at every user [15]. The channel from TX j to RX i is represented by the channel matrix $\mathbf{H}_{i,j} \in \mathbb{C}^{N_i \times M_j}$ with its elements distributed according to a continuous distribution which ensures that all the sub-matrices are almost surely full rank. We denote by $\mathcal{H}$ the space of all possible channel realizations. Since interference alignment is invariant by scaling (by a non-zero complex scalar) of the channel matrices, we further define $\tilde{\mathbf{H}}_{i,j} \triangleq \mathrm{e}^{\jmath\phi_{i,k}} \frac{\mathbf{H}_{i,j}}{\|\mathbf{H}_{i,j}\|_{\mathrm{F}}}$, where $\phi_{i,k} \in \mathbb{R}$ is chosen so as to let the first element of $\mathrm{vect}(\tilde{\mathbf{H}}_{i,k})$ be real valued.

The global multi-user channel matrix is denoted by $\mathbf{H} \in \mathbb{C}^{N_{\mathrm{tot}} \times M_{\mathrm{tot}}}$ with $N_{\mathrm{tot}} \triangleq \sum_{i=1}^{K} N_i$ and $M_{\mathrm{tot}} \triangleq \sum_{i=1}^{K} N_i$, and defined as

$$\mathbf{H} \triangleq \begin{bmatrix} \mathbf{H}_{1,1} & \mathbf{H}_{1,2} & \dots & \mathbf{H}_{1,K} \\ \mathbf{H}_{2,1} & \mathbf{H}_{2,2} & \dots & \mathbf{H}_{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{K,1} & \mathbf{H}_{K,2} & \dots & \mathbf{H}_{K,K} \end{bmatrix}. \tag{1}$$

The matrix $\tilde{\mathbf{H}}$ is defined similarly from the matrices $\tilde{\mathbf{H}}_{i,k}$.

Assume that TX j uses the precoder $\mathbf{T}_j \triangleq \sqrt{P}\mathbf{U}_j \in \mathbb{C}^{M_j \times d_j}$ with $\|\mathbf{U}_j\|_{\mathrm{F}}^2 = 1$ to transmit the data symbol $\boldsymbol{s}_j \in \mathbb{C}^{d_j}$ (i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$) to RX j. Hence, the precoder fulfills the per-TX power constraint $\|\mathbf{T}_j\|_{\mathrm{F}}^2 = P$.

The received signal $\boldsymbol{y}_i \in \mathbb{C}^{N_i}$ at RX i is

$$\boldsymbol{y}_i = \sqrt{P}\mathbf{H}_{i,i}\mathbf{U}_i\boldsymbol{s}_i + \sqrt{P} \sum_{j=1, j\neq i}^{K} \mathbf{H}_{i,j}\mathbf{U}_j\boldsymbol{s}_j + \boldsymbol{\eta}_i \tag{2}$$

where $\boldsymbol{\eta}_i \in \mathbb{C}^{N_i}$ is the noise at RX i and has its elements i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$. The received signal $\boldsymbol{y}_i$ is then processed by a RX filter $\mathbf{G}_i^{\mathrm{H}} \in \mathbb{C}^{d_i \times N_i}$ with $\|\mathbf{G}_i\|_{\mathrm{F}}^2 = 1$.

The average rate achieved at user i is written as

$$R_i = \mathrm{E}_{\mathcal{H},\mathcal{W}}\left[\log_2\left|\mathbf{I}_{d_i} + P\bar{\mathbf{R}}_i^{-1}\mathbf{G}_i^{\mathrm{H}}\mathbf{H}_{i,i}\mathbf{U}_i\mathbf{U}_i^{\mathrm{H}}\mathbf{H}_{i,i}^{\mathrm{H}}\mathbf{G}_i\right|\right] \tag{3}$$

where

$$\bar{\mathbf{R}}_i = \mathbf{I}_{d_i} + P \sum_{\ell=1, \ell\neq i}^{K} \mathbf{G}_i^{\mathrm{H}}\mathbf{H}_{i,\ell}\mathbf{U}_\ell\mathbf{U}_\ell^{\mathrm{H}}\mathbf{H}_{i,\ell}^{\mathrm{H}}\mathbf{G}_i \tag{4}$$

and $\mathrm{E}_{\mathcal{H},\mathcal{W}}[\cdot]$ denotes the expectation over the channel matrices and the channel estimation errors according to the feedback model described in Subsection II-B. The DoF at user i, or prelog factor, is then defined as

$$\mathrm{DoF}_i = \lim_{P\to\infty} \frac{R_i}{\log_2(P)}. \tag{5}$$

B. Distributed CSIT and distributed precoding

Let us assume that TX j receives its own estimate of the channel from TX k to RX i. We denote this estimate by $\tilde{\mathbf{H}}_{i,k}^{(j)}$, assumed to have the same properties (unit norm and real-valued first coefficient) as $\tilde{\mathbf{H}}_{i,k}$. Furthermore, similar to (1), we let $\tilde{\mathbf{H}}^{(j)}$ denote the channel state information available at TX j. In the sequel, we assume that each TX independently computes its own solution of the IA problem based on its own CSI. Specifically, TX j computes the solution (in terms of the precoders and receive filters $\mathbf{U}_k^{(j)}, k = 1 \dots K$ and $\mathbf{G}_i^{(j)}, i = 1 \dots K$) of its own IA problem based on $\tilde{\mathbf{H}}^{(j)}$,

$$(\mathbf{G}_i^{(j)})^{\mathrm{H}}\tilde{\mathbf{H}}_{i,k}^{(j)}\mathbf{U}_k^{(j)} = \mathbf{0}_{d_i \times d_j} \quad \forall k \neq i \tag{6}$$

where $\mathbf{U}_k^{(j)}$ is the precoder designed to be used by TX k and $\mathbf{G}_i^{(j)}$ is the receive filter assumed at RX i. However, since the TXs are not colocated and do not exchange further informations, only $\mathbf{U}_j^{(j)}$ is used for the actual transmission at TX j, while the $\mathbf{U}_i^{(j)}, i \neq j$ are discarded. Considering all the TXs, this gives

$$\mathbf{U}_j = \mathbf{U}_j^{(j)}, \qquad \forall j. \tag{7}$$

C. Imperfect CSIT model

We assume that $\tilde{\mathbf{H}}_{i,k}^{(j)}$ results from the quantization of $\tilde{\mathbf{H}}_{i,k}$, using a quantization scheme using $B_{i,k}^{(j)}$ bits according to

$$\tilde{\mathbf{H}}_{i,k}^{(j)} = \underset{\mathrm{vect}(\mathbf{W})\in\mathcal{W}_{i,k}^{(j)}}{\mathrm{argmin}} \left\|\tilde{\mathbf{H}}_{i,k} - \mathbf{W}\right\|_{\mathrm{F}}, \qquad \forall k,i,j, \tag{8}$$

where $\mathcal{W}_{i,k}^{(j)}$ contains $2^{B_{i,k}^{(j)}}$ vectors of size $\mathbb{C}^{N_i M_k}$ isotropically distributed over the unit-sphere and rotated to have their first element real-valued. We further define

$$(\sigma_{i,k}^{(j)})^2 \triangleq \mathrm{E}_{\mathcal{H},\mathcal{W}}\left[\left\|\tilde{\mathbf{H}}_{i,k}^{(j)} - \tilde{\mathbf{H}}_{i,k}\right\|_{\mathrm{F}}^2\right] \quad \text{and} \tag{9}$$

$$\mathbf{N}_{i,k}^{(j)} \triangleq \frac{\tilde{\mathbf{H}}_{i,k}^{(j)} - \tilde{\mathbf{H}}_{i,k}}{\sigma_{i,k}^{(j)}}, \tag{10}$$

where $\mathrm{E}_{\mathcal{W}}[\cdot]$ denotes the expectation over the random codebooks. It then gives

$$\tilde{\mathbf{H}}_{i,k}^{(j)} = \tilde{\mathbf{H}}_{i,k} + \sigma_{i,k}^{(j)}\mathbf{N}_{i,k}^{(j)}. \tag{11}$$

Since there is no confusion possible we use the short notation $\mathrm{E}_{\mathcal{H}}[\cdot]$ instead of $\mathrm{E}_{\mathcal{H},\mathcal{W}}[\cdot]$.

Due to the adopted normalization, the quantization scheme (8) corresponds to the Grassmannian quantization over the Grassmannian space, similar to that used in [16], [9]. Using this property and the results from [17], [14], the variance of the estimation error can be related to the number of quantization bits as follows.

Proposition 1 ([17, Theorem 2])**.** *When the size $L_{i,k}^{(j)} = 2^{B_{i,k}^{(j)}}$ of the random codebook is sufficiently large, it then holds that*

$$(\sigma_{i,k}^{(j)})^2 = C_{i,k}^{(j)} 2^{-B_{i,k}^{(j)}/(N_i M_k - 1)} \tag{12}$$

for some constant $C_{i,k}^{(j)} > 0$.

With centralized CSIT, it is well known [18], [8] that the number of quantization bits should scale with the SNR in order to achieve a positive DoF. Hence, we define the *CSIT scaling coefficients* $A_{i,k}^{(j)}$ as

$$\forall k,i, \quad A_{i,k}^{(j)} \triangleq \lim_{P\to\infty} \frac{B_{i,k}^{(j)}}{B_{i,k}^{\star}} \tag{13}$$

where we have defined

$$B^{\star}_{i,k} \triangleq (N_i M_k - 1)\log_2(P). \tag{14}$$

The pre-log coefficient $N_i M_k - 1$ corresponds to the number of channel coefficients to feedback after normalization of the channel matrix. $B^{\star}_{i,k}$ is a number of bits giving a quantization error decreasing as P^{-1}, which is essentially perfect in terms of DoF [16], [14]. Hence, $A^{(j)}_{i,k}$ can be seen as the fraction of the feedback requirements to achieve the maximal DoF.

Remark 1. We consider here a codebook-based quantization of the channel vectors but the results can be easily translated to a setting where analog feedback is used [18], [10] by making the quantization error a function of the SNR. In fact, the digital quantization used in this work is simply a model for the errors in the channel estimates resulting from the limited feedback. Furthermore, only CSIT requirements are investigated, and different scenarios can be envisaged (e.g., direct broadcasting from the RXs to all the TXs, sharing through a backhaul, ...) [7], [12]. □

III. DoF Analysis with Static Coefficients and Distributed CSI

Let us now focus on the situation where every TX designs its precoder based on a different multi-user channel estimate. Hence, the precoding matrices used for the transmission do not form exactly an IA solution for any imperfect estimate of the multi-user channel. This is in contrast to the centralized case studied in [8], [9]. Hence, the analysis done in these works does not hold in the setting considered here and a new approach is required.

The analysis of this situation is complicated by the fact that the function that gives the precoders as a function of the channel coefficients can not be assumed to be continuous. This can be seen by observing that there are in general multiple solutions to the IA equations [19], while iterative algorithms, such as the iterative leakage minimization from [4], converge to one of the IA solutions. So far this convergence is not fully understood, and it can not be ruled out that a small change in the CSI (as in the case in the distributed CSI considered here) leads to a convergence to completely different solutions across the users.

A. Sufficient condition for an arbitrary IA scheme

Let us denote by $\mathbf{U}^{\star}_i$ and $\mathbf{G}^{\star}_i$ the precoder and the RX filter at TX i and RX i, respectively, when perfect CSIT is available at the TXs for *an arbitrary IA scheme*, i.e., verifying $(\mathbf{G}^{\star}_i)^{\mathrm{H}}\mathbf{H}_{ij}\mathbf{U}^{\star}_j = \mathbf{0}_{d_i\times d_j}, \forall i \neq j$. We further define

$$\mathbf{\Delta U}^{(j)}_i \triangleq \mathbf{U}^{(j)}_i - \mathbf{U}^{\star}_i, \qquad \forall i,j. \tag{15}$$

We now characterize the DoF achieved as a function of the precoder accuracy.

Proposition 2. *In the IC with distributed CSIT as described in Section II, if the CSIT is such that*

$$\mathrm{E}_{\mathcal{H}}[\|\mathbf{\Delta U}^{(j)}_j\|^2_F] \doteq P^{-\beta_j}, \qquad \forall j, \tag{16}$$

with $\beta_j \in [0,1]$*, then*

$$\mathrm{DoF}_i \geq d_i \min_{j\neq i} \beta_j, \qquad \forall i. \tag{17}$$

Proof: Since we want to derive a lower bound for the DoF, we can choose $\mathbf{G}_k = \mathbf{G}^{\star}_k, \forall k$. Following a classical derivation [16], [18], we can write

$$R_i \geq R^{\star}_i - \mathrm{E}_{\mathcal{H}}[\log_2|\mathbf{I}_{d_i} + P\sum_{j=1,j\neq i}^{K}(\mathbf{G}^{\star}_i)^{\mathrm{H}}\mathbf{H}_{i,j}\mathbf{U}^{(j)}_j(\mathbf{U}^{(j)}_j)^{\mathrm{H}}\mathbf{H}^{\mathrm{H}}_{i,j}\mathbf{G}^{\star}_i|] \tag{18}$$

where we have defined

$$R^{\star}_i \triangleq \mathrm{E}_{\mathcal{H}}\left[\log_2\left|\mathbf{I}_{d_i} + P(\mathbf{G}^{\star}_i)^{\mathrm{H}}\mathbf{H}_{i,i}\mathbf{U}^{(i)}_i(\mathbf{U}^{(i)}_i)^{\mathrm{H}}\mathbf{H}^{\mathrm{H}}_{i,i}(\mathbf{G}^{\star}_i)\right|\right]. \tag{19}$$

It is easily seen that $R^{\star}_i \doteq d_i\log_2(P)$, such that it remains to study the second term of (18), which we denote by $\mathcal{I}_i$. Since $(\mathbf{G}^{\star}_i)^{\mathrm{H}}\mathbf{H}_{i,j}\mathbf{U}^{\star}_j = \mathbf{0}_{d_i\times d_j}$ for $i\neq j$, it holds that

$$\mathcal{I}_i = \mathrm{E}_{\mathcal{H}}[\log_2|\mathbf{I}_{d_i} + P\sum_{j=1,j\neq i}^{K}(\mathbf{G}^{\star}_i)^{\mathrm{H}}\mathbf{H}_{i,j}\mathbf{\Delta U}^{(j)}_j(\mathbf{\Delta U}^{(j)}_j)^{\mathrm{H}}\mathbf{H}^{\mathrm{H}}_{i,j}\mathbf{G}^{\star}_i|]. \tag{20}$$

Using that $\|\mathbf{G}^{\star}_i\|^2_{\mathrm{F}} = 1$, we can upper bound the interference to write

$$\begin{aligned}
\mathcal{I}_i &\leq \mathrm{E}_{\mathcal{H}}[\log_2|\mathbf{I}_{d_i} + (P\sum_{j=1,j\neq i}^{K}\|\mathbf{H}_{i,j}\|^2_{\mathrm{F}}\|\mathbf{\Delta U}^{(j)}_j\|^2_{\mathrm{F}})\mathbf{I}_{d_i}|] \\
&\overset{(a)}{\leq} d_i\Bigg(\mathrm{E}_{\mathcal{H}}[\log_2(1+P\sum_{j=1,j\neq i}^{K}\|\mathbf{H}_{i,j}\|^2_{\mathrm{F}})] \\
&\qquad + \mathrm{E}_{\mathcal{H}}[\log_2(1+P\sum_{j=1,i\neq j}^{K}\|\mathbf{\Delta U}^{(j)}_j\|^2_{\mathrm{F}})]\Bigg) \qquad (21) \\
&\overset{(b)}{\leq} d_i\Bigg(\mathrm{E}_{\mathcal{H}}[\log_2(1+P\sum_{j=1,j\neq i}^{K}\|\mathbf{H}_{i,j}\|^2_{\mathrm{F}})] \\
&\qquad + \log_2(1+P\sum_{j=1,j\neq i}^{K}\mathrm{E}_{\mathcal{H}}[\|\mathbf{\Delta U}^{(j)}_j\|^2_{\mathrm{F}}])\Bigg)
\end{aligned}$$

where inequality (a) can be seen to hold since only positive terms have been added and we have used Jensen's inequality to obtain inequality (b). Using (16), we can write that

$$\sum_{j=1,j\neq i}^{K}\mathrm{E}_{\mathcal{H}}\left[\|\mathbf{\Delta U}^{(j)}_j\|^2_{\mathrm{F}}\right] \doteq P^{-\min_{j\neq i}\beta_j}. \tag{22}$$

Inserting (22) inside (21) and (18) gives

$$\begin{aligned}
R_i &\dot{\geq} d_i\left(\log_2(P) - \log_2(1+PP^{-\min_{j\neq i}\beta_j})\right) && (23)\\
&\dot{\geq} d_i(\min_{j\neq i}\beta_j)\log_2(P), && (24)
\end{aligned}$$

which concludes the proof. ■

Proposition 2 provides some insights into the performance by relating the accuracy with which the precoder is computed to the achieved DoF. However, the accuracy of the precoder design is difficult to relate to the accuracy of the CSIT. Indeed, the relation is dependent on the precoding method used and some precoding schemes might be more or less robust to imperfections in the CSIT. Hence, the structure of the IA algorithm has to be studied to observe what is the impact of the CSIT imperfection over the precoding at each TX.

Remark 2. This follows from the fact that the precoders $\mathbf{U}_j^{(j)}, \forall j$ do not form (a priori) together an alignment solution for any of the multi-user channel estimates available at the TXs. □

B. DoF analysis in the 3-user square MIMO IC

a) Perfect CSIT Solution: We consider now a 3-user IC with $M_i = N_j, \forall i,j$ and $d_i = d, \forall i$. We also assume for the description of the IA scheme that perfect CSIT is available such that we denote the precoder used at TX j by $\mathbf{U}_j^{\star}$. Since we consider the tightly-feasible case [15], we have $M = N = 2d$. In that case, the IA constraints can be written as [20]

$$\begin{aligned} \operatorname{span}\left(\tilde{\mathbf{H}}_{3,1}\mathbf{U}_1^{\star}\right) &= \operatorname{span}\left(\tilde{\mathbf{H}}_{3,2}\mathbf{U}_2^{\star}\right), \\ \operatorname{span}\left(\tilde{\mathbf{H}}_{1,2}\mathbf{U}_2^{\star}\right) &= \operatorname{span}\left(\tilde{\mathbf{H}}_{1,3}\mathbf{U}_3^{\star}\right), \\ \operatorname{span}\left(\tilde{\mathbf{H}}_{2,3}\mathbf{U}_3^{\star}\right) &= \operatorname{span}\left(\tilde{\mathbf{H}}_{2,1}\mathbf{U}_1^{\star}\right). \end{aligned} \tag{25}$$

In particular, this system of equations can be easily seen to be fulfilled if the precoders verify

$$\begin{aligned} \mathbf{U}_1^{\star}\mathbf{\Lambda}_1 &= \tilde{\mathbf{H}}_{3,1}^{-1}\tilde{\mathbf{H}}_{3,2}\tilde{\mathbf{H}}_{1,2}^{-1}\tilde{\mathbf{H}}_{1,3}\tilde{\mathbf{H}}_{2,3}^{-1}\tilde{\mathbf{H}}_{2,1}\mathbf{U}_1^{\star} \\ \mathbf{U}_3^{\star} &= (\tilde{\mathbf{H}}_{2,3})^{-1}\tilde{\mathbf{H}}_{2,1}\mathbf{U}_1^{\star} \\ \mathbf{U}_2^{\star} &= (\tilde{\mathbf{H}}_{1,2})^{-1}\tilde{\mathbf{H}}_{1,3}\mathbf{U}_3^{\star} \end{aligned} \tag{26}$$

for some diagonal matrix $\mathbf{\Lambda}_1$. We also define for clarity the matrix $\mathbf{Y}^{\star}$ equal to

$$\mathbf{Y}^{\star} \triangleq \tilde{\mathbf{H}}_{3,1}^{-1}\tilde{\mathbf{H}}_{3,2}\tilde{\mathbf{H}}_{1,2}^{-1}\tilde{\mathbf{H}}_{1,3}\tilde{\mathbf{H}}_{2,3}^{-1}\tilde{\mathbf{H}}_{2,1}. \tag{27}$$

The system of equations (26) is then fulfilled by setting

$$\begin{aligned} \mathbf{U}_1^{\star} &= \frac{1}{\sqrt{d}}\operatorname{EVD}(\mathbf{Y}^{\star})\left[\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d\right] \\ \mathbf{U}_3^{\star} &= \frac{1}{\|(\tilde{\mathbf{H}}_{2,3})^{-1}\tilde{\mathbf{H}}_{2,1}\mathbf{U}_1^{\star}\|_{\mathrm{F}}}(\tilde{\mathbf{H}}_{2,3})^{-1}\tilde{\mathbf{H}}_{2,1}\mathbf{U}_1^{\star} \\ \mathbf{U}_2^{\star} &= \frac{1}{\|(\tilde{\mathbf{H}}_{1,2})^{-1}\tilde{\mathbf{H}}_{1,3}\mathbf{U}_3^{\star}\|_{\mathrm{F}}}(\tilde{\mathbf{H}}_{1,2})^{-1}\tilde{\mathbf{H}}_{1,3}\mathbf{U}_3^{\star}. \end{aligned} \tag{28}$$

b) Distributed CSIT Solution: With distributed CSIT, TX j computes using its channel estimate $\tilde{\mathbf{H}}^{(j)}$ the matrix

$$\mathbf{Y}^{(j)} = (\tilde{\mathbf{H}}_{3,1}^{(j)})^{-1}(\tilde{\mathbf{H}}_{3,2}^{(j)})(\tilde{\mathbf{H}}_{1,2}^{(j)})^{-1}(\tilde{\mathbf{H}}_{1,3}^{(j)})(\tilde{\mathbf{H}}_{2,3}^{(j)})^{-1}(\tilde{\mathbf{H}}_{2,1}^{(j)}). \tag{29}$$

The precoding matrices are then obtained from

$$\begin{aligned} \mathbf{U}_1^{(j)} &= \frac{1}{\sqrt{d}}\operatorname{EVD}(\mathbf{Y}^{(j)})\left[\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d\right] \\ \mathbf{U}_3^{(j)} &= \frac{1}{\|(\tilde{\mathbf{H}}_{2,3}^{(j)})^{-1}\tilde{\mathbf{H}}_{2,1}^{(j)}\mathbf{U}_1^{(j)}\|_{\mathrm{F}}}(\tilde{\mathbf{H}}_{2,3}^{(j)})^{-1}\tilde{\mathbf{H}}_{2,1}^{(j)}\mathbf{U}_1^{(j)} \\ \mathbf{U}_2^{(j)} &= \frac{1}{\|(\tilde{\mathbf{H}}_{1,2}^{(j)})^{-1}\tilde{\mathbf{H}}_{1,3}^{(j)}\mathbf{U}_3^{(j)}\|_{\mathrm{F}}}(\tilde{\mathbf{H}}_{1,2}^{(j)})^{-1}\tilde{\mathbf{H}}_{1,3}^{(j)}\mathbf{U}_3^{(j)}. \end{aligned} \tag{30}$$

In that case, we can give the following result on the DoF achieved.

Theorem 1. *Using the 3-User IA scheme described above with distributed CSIT, the DoF achieved at user i is denoted by $\mathrm{DoF}_i^{\mathrm{DCSI}}$ and verifies*

$$\mathrm{DoF}_i^{\mathrm{DCSI}} \geq d \min_{j\neq i} \min_{k,\ell,k\neq \ell} A_{k,\ell}^{(j)}. \tag{31}$$

Proof: Only a sketch of the proof is provided for lack of space. The detailed proof is publicly available in [21]. For a given $\varepsilon > 0$, we define the following channel subsets:

$$\mathcal{X}^{\varepsilon} \triangleq \{\tilde{\mathbf{H}}|\forall i,k,\ \lambda_{\min}(\tilde{\mathbf{H}}_{i,k}) \geq \varepsilon\} \tag{32}$$

$$\mathcal{Y}^{\varepsilon} \triangleq \{\tilde{\mathbf{H}}|\forall i\neq j,\ |\lambda_i(\mathbf{Y}^{\star}) - \lambda_j(\mathbf{Y}^{\star})| \geq \varepsilon\} \tag{33}$$

and $\mathcal{H}^{\varepsilon} \triangleq \mathcal{X}^{\varepsilon}\bigcap\mathcal{Y}^{\varepsilon}$. It can be easily seen from the continuous distribution of the channel matrices that $\forall \eta > 0, \exists \varepsilon > 0, \Pr(\mathcal{H}^{\varepsilon}) \geq 1-\eta$. Hence, considering only the channel realizations in $\mathcal{H}_{\varepsilon}$, we can then write after some steps

$$R_i \dot{\geq} (1-\eta)d_i\log_2(P) - d_i(\log_2(1+P\sum_{j=1,j\neq i}^{K}\mathrm{E}_{\mathcal{H}^{\varepsilon}}[\|\mathbf{\Delta}\mathbf{U}_j^{(j)}\|_{\mathrm{F}}^2])). \tag{34}$$

Exploiting the properties of the channel matrices in $\mathcal{H}^{\varepsilon}$, we can then show that $\mathrm{E}_{\mathcal{H}^{\varepsilon}}[\|\mathbf{\Delta}\mathbf{U}_j^{(j)}\|_{\mathrm{F}}^2] \dot{\leq} P^{-\min_{\ell\neq k} A_{\ell,k}^{(j)}}$. Inserting this result in (34) and taking η arbitrarily small concludes the proof. ■

We have shown that for the 3-user IA closed-form alignment scheme, the achieved DoF is larger than the worst accuracy of the channel estimates across the TXs. This bound is in fact conjectured to be tight.

Interestingly, the lower bound at RX j is limited by the accuracy of the estimates relative to the channels of all the *other* RXs. This result is in strong contrast with the centralized setting where the DoF of user i depends *solely* on the accuracy with which the channel matrices from the TXs to RX i are fed back. This show how IA becomes more sensitive to CSIT errors when the precoding is done based on distributed CSIT. Note that this result is reminiscent of [14] where it was shown in a K-user MISO BC with single-antenna RXs and with distributed CSIT, that the DoF was limited by the worst accuracy across the TXs and across the channel vectors.

IV. Simulations

In this section, we validate by Monte-Carlo simulations the results in the 3-user square IC channel studied in Subsection III-B. We consider $M = N = 4$ and $d = 2$ and we average the performance over 10000 realizations of a Rayleigh fading channel. We consider the distributed CSIT configuration described in Section II. The quantization error is modeled using (11) with $(\sigma_{i,k}^{(j)})^2 = 2^{-B_{i,k}^{(j)}/(N_iM_k-1)}$ and $\mathbf{N}_{i,k}^{(j)}$ having its elements i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1)$. We choose the CSIT scaling coefficients as

$$\forall (i,k,j) \in \{1,2,3\}^3 \setminus \{(3,2,2),(3,2,3)\}, \quad A_{i,k}^{(j)} = 1, \\ A_{3,2}^{(2)} = 0.5, \quad A_{3,2}^{(3)} = 0. \tag{35}$$

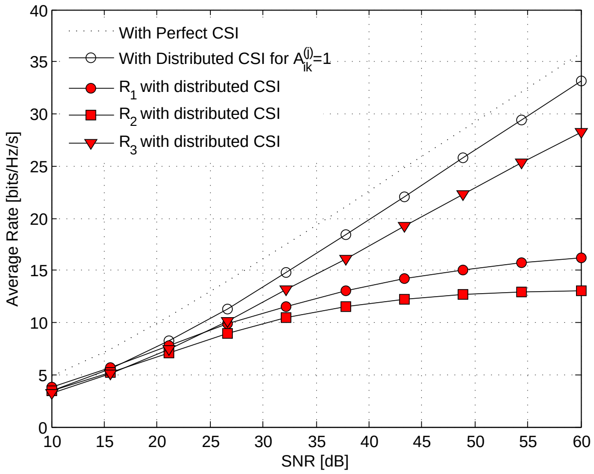

Fig. 1: Average rate per user in the square setting $M = N = 4$ with $d = 2$ for the CSIT scaling coefficients given in (35).

Following Theorem 1, we have for the CSIT configuration described in (35) that $\mathrm{DoF}_1 \geq 0$, $\mathrm{DoF}_2 \geq 0$, and $\mathrm{DoF}_3 \geq 0.5d = 1$. The average rate achieved is shown for each user in Fig. 1. For comparison, we have also simulated the average rate per-user achieved based on perfect CSIT and with distributed CSIT when the CSIT scaling coefficients are set equal to 1 for every TX ($\forall i, k, j, A_{i,k}^{(j)} = 1$). It can then be verified that having all CSIT scaling coefficients equal to one allows to achieve the maximal DoF.

With the CSIT configuration described in (35), the slope of the rate of user 3 decreases as the SNR increases, revealing a very slow convergence to the DoF. This makes it difficult to accurately observe the DoF achieved. Yet, it can be seen that having only $A_{3,2}^{(3)}$ equal to zero leads already to the saturation of the rates of users 1 and 2 (i.e., their DoF is equal to 0), which tends to confirm our conjecture.

V. Extension to Time-Alignment and Iterative Interference Alignment

We have studied the DoF in a particular antenna configuration for the case of static MIMO channels. This antenna configuration has been considered both because it is believed to be a simple, yet practically relevant configuration, and because the knowledge of a closed-form precoding formula is necessary for our analysis. In fact, our approach is expected to easily extend to numerous scenarios where a closed form expression exists for the IA precoding, under the condition that the precoding scheme is "robust" enough to the quantization errors, e.g., it consists of matrix inversions or matrix multiplications where the matrices have their elements distributed according to a continuous distribution. This in particular the case of the original time-alignment IA scheme from [2], [3]. Hence, our results can be trivially extended to this setting. Obtaining the DoF achieved with an iterative IA algorithm like the min-leakage algorithm or the max-SINR algorithm [4], [6] is a challenging open problem which will be investigated in subsequent works. As a prerequisite step, it requires deriving some basic properties of the IA algorithm, such as convergence properties, which have remained out of reach until now.

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