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History-dependence in production-pollution-trade-off models: a multi-stage approach

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Abstract Multi-stage modeling provides powerful tools to study optimal switches between different technologies. In most of the related literature, however, it is assumed that the number of switches is a-priori fixed. In the present paper we allow for multiple optimally determined switches. Consequently, we are able to locate solution paths that not only lead to different long-run outcomes but also differ in the number of switches along these paths.

We present a simple production-pollution model in which a representative firm wants to maximize the profit gained out of production which, however, causes harmful pollution as by-product. The firm has the choice between two different technologies, one which is efficient in production but pollutive, and another one which is less efficient but environmentally friendly.

With this two stage-model we focus on the numerical investigation of the conditions determining when and how often it is optimal for the firm to switch between these different technologies. We show that for certain parameters even several switches can be optimal and that the height of the switching costs crucially influences the long-run outcome. In the course of these investigations, we discuss two different economic mechanisms related to the harm due to pollution which lead to the occurrence of multiple equilibria, history-dependence and so-called Skiba points.

Keywords Multi-stage modeling · Production-pollution trade-off · Optimal control · Pontryagin's maximum principle · History-dependence

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1 Introduction

In conventional optimal control models it is assumed that the underlying canonical system evolves continuously over time. To adequately deal with sudden changes in the system dynamics and/or in the objective one can apply multi-stage methods. They provide not only conditions for the optimal timing of switches between consecutive regimes, but also for how a switch has to be optimally anticipated. There is a growing number of literature where multi-stage modeling techniques are used. Applications include changes in (drug) markets (Bultmann et al. 2008), the adaption of counter-terror measures (Grass et al. 2008, Chap. 8.1), and changes in the organizational regimes within a firm (Vallée and Moreno-Galbis 2011; Caulkins et al. 2013).

The present paper deals with a firm's choice to adopt a different technology, which is either more efficient or more environmentally friendly. The trade-off between environmental quality and economic performance within a two-stage model was considered in Boucekkine et al. (2011). They studied the question whether to switch to a more environmentally friendly, but less productive technology. Questions related to the issue of when to adopt a new technology have been dealt in Boucekkine et al. (2004, 2012), Saglam (2011) and Grass et al. (2012) within a multi-stage framework. In most of the papers presenting multi-stage models, however, the number of possible switches is fixed and the only question is when these switches are supposed to happen and how should they be anticipated. An example for optimally determined numbers of switches can be found in Goeschl and Perino (2007), who studied the problem of a policy maker introducing a new, additional technology at the switching point. Multiple optimally determined switches were also considered in Boucekkine et al. (2013), who extended Boucekkine et al. (2011) by including a pollution threshold above which the natural cleaning rate with respect to pollution decay goes down.

In the present paper we study a simple model related to the problem of a firm which can decide between two different technologies. Besides considering the optimal number and timing of switches, we put a focus on the impact of multiple equilibria in a multi-stage framework. We show in this paper that there can exist solution paths that do not only lead to a different long-run outcome but also differ in the optimal number of switches.

This model considers an economy in which production comes along with pollution as by-product and hence entails some damage. The representative firm therefore faces a trade-off between production and the social responsibility for the caused pollution stock which has a negative impact on the firm's reputation and consequently can be interpreted as disutility. In contrast to many other set-ups, in which the damage of pollution is interpreted as losses in productivity and hence negative mixed partial derivatives of the utility often are assumed, we postulate separability in production and pollution so that the firm still faces a damage of the pollution generated in the past in the sense of a negative reputation even if the current production output is set to zero.

While this conventional production technology (in the following called 'brown' technology) is pollutive, there also exists a cleaner alternative technology (in the following called 'green' technology) that can be used as substitute in production, but is less efficient than the pollutive one and more expensive. In literature one can find various approaches for such 'green' alternatives, cf. Stimming (1999), Cassou and Hamilton (2004), Lange and Moslener (2004), Cunha-e Sá et al. (2010).

The pollution stock itself is modeled as a state variable which accumulates with the production rate and declines due to natural processes. Although a linear specification of these environmental dynamics, as sometimes used in literature (cf. Nahorski and Ravn 2000), would be mathematically more tractable, natural processes are rather supposed to behave in a non-linear way. As the cleaning rate represents the environment's capacity to recover, it is assumed in many papers dealing with pollution, cf. Heijdra and Heijnen (2009), Mäler et al. (2003), Xepapadeas (2005), that a pollution stock beyond a critical level affects this capacity. In this cases a non-linear, and often also non-convex feedback (cf. Brock and Starrett 2003) leads to an environmental disaster, even if the source of pollution declines again. This so-called hysteresis effect sometimes is considered to be even irreversible. Because a linear specification would omit this feedback effects, the natural cleaning rate rather should depend on the pollution stock, as also discussed in Tahvonen and Withagen (1996) and El Ouardighi et al. (2011). While a monotonously declining convex natural cleaning rate therefore is quite common in literature, cf. Forster (1975), Tahvonen and Withagen (1996), Hediger (2009), we follow Tahvonen and Salo (1996) and even use a monotonously declining convex-concave natural cleaning rate. The convex-concave specification of the cleaning rate in our model means that the environment's capacity to recover is high as long as pollution is low, however, when pollution is high, this capacity converges towards zero. The convex-concave setting admits under certain conditions history-dependence and a multiplicity of the optimal long-run solution within a single stage. This allows us to gain insights on mechanisms responsible for whether an environment is clean or polluted in the long run. Another approach to include feedback effects was followed in Boucekkine et al. (2013), who considered within a multi-stage framework a pollution-threshold above which the natural cleaning rate goes down.

As far as the damage of pollution is concerned, it is conventional to model this as a monotonously increasing convex function so that the damage is unbounded, cf. Tahvonen and Withagen (1996), Boucekkine et al. (2012). In our approach, however, the damage due to pollution is understood as the loss in the firm's reputation. We assume that the pollution of the firm has a negative externality on the public and hence the public reacts with negative pressure for example via their buying pattern or legislative processes (cf. Henriques and Sadorsky 1996), of which the occurring costs are represented by this damage function. The higher the pollution, the higher is this negative pressure and the more the firm is forced to include this environmental aspect in its production planing. The same argument can be found in Cohen and Konar (2000), where the increasing impact of pollution is explained by the growing publicity the firm has. Hence, the damage function is supposed to be convex for a high pollution levels. However, in order to model the complete effect, we even go a step further and do not only include the negative impact of pollution on reputation, but also the positive impact of the firm's efforts into the reduction of pollution. We assume that a high effort in keeping pollution low leads to an improvement in reputation at an increasing rate, meaning that the damage function declines at an increasing rate. Therefore, we consider the damage function as a concave-convex function in the pollution stock which is also consistent with Mas-Colell et al. (1995), where it is stated that externalities often lead to non-convexities.

Given this trade-off between production and environmental pollution, the goal of this paper is to study under which circumstances and at what time the firm switches to a less pollutive technology in order to keep the polluting impact low. In order to gain some basic understanding about the general implications of our assumptions on the optimal solution, we always study first a one-stage version of each model scenario before we proceed to the multi-stage framework. One state optimal control models of accumulative pollution have been extensively studied in the literature. Early references are Forster (1975, 1977), Plourde (1972), Keeler et al. (1972); see also Hediger (2009) for a more complete list of references.

In the present model we are able to find multiple equilibria and can show that the optimal long-run solution is history-dependent, i.e. it depends on the initial pollution level. At so-called Skiba points (in literature often also referred to as DNSS points or indifference-threshold points) the firm is indifferent between two different solution paths. Forster (1975) seems to be the first who pointed out that multiple equilibria can exist for optimal pollution control models with biological assimilation functions; cf. also Pearce (1976). Another important contribution with respect to Skiba points in environmental models is Wagener (2003), who studied the occurrence of such points in a model dealing with the pollution of shallow lakes (cf. Mäler et al. 2003). For a recent exposition on indifference points see Chap. 5 of Grass et al. (2008).

Due to the chosen convex-concave setting in our model, the occurrence of Skiba points does not come as a surprise, but this analysis should give an idea where the multiple equilibria in the multi-stage problem originate. In the multi-stage approach itself it turns out that even more than one switch can be optimal for the firm. The incentive for switching back to brown technology is given if the gain of a cheaper production technology is higher than the damage through a reduced reputation. Different scenarios, in which we adapt step by step the mentioned non-linearities and non-convexities of the natural cleaning rate and the damage function, respectively, show that the optimal solution heavily relies on these assumptions. One main result is that for certain initial state values a firm might have the option to choose between solution paths leading not only to different long-run outcomes, but also with different numbers of switches between the technologies. We even find a scenario where it is optimal to switch three times between green and brown technologies. In addition also switching costs turn out to have a big impact on whether and when it is optimal to change the technology.

The paper is organized as follows. In Sect. 2 we present the model formulation and the derivation of the optimality conditions in a general notation, both for the one-stage as well as for the multi-stage approach. In Sect. 3 we apply these conditions on the first model version in which we assume a constant natural cleaning rate but a concave-convex damage function. In contrast, we replace in Sect. 4 the concave-convex damage function by a quadratic one, so that only the negative impact on reputation is considered, but instead include a pollution dependent natural cleaning rate. Finally, we combine these two mechanisms in the model in Sect. 5. For the numerical calculations we use the Matlab toolbox OCMat which is available at http://orcos.tuwien.ac.at/research/ocmat_software/ and relies on numerical methods described in Grass et al. (2008).

2 The model

2.1 Model formulation

Before starting with the multi-stage problem we consider the fundamental decision problem with only one type of technology to show how the optimal solutions behave under this internalized pollution aspect. In our model we consider an economy in which production output is generated at each time t with the production rate $u(t)$. During the production process, however, pollution $P(t)$ occurs as by-product, like for example waste water or waste air, which increases proportionally with $u(t)$ and has a negative impact on the surrounding environment. This environmental damage, however, comes along with some disutility in form of a bad reputation. But the stock of pollutants does not only increase with production, it also declines again due to natural processes. With the natural cleaning rate $\alpha(P(t))$, that might also depend on the stock of pollution, the environment is able to recover again.

In this paper we consider a representative firm maximizing its output by optimally choosing the production rate $u(t)$. Additionally, it faces the polluting impact of production for which it has to take social responsibility. Therefore, the optimal control problem of the firm is given as

$$\max_{u(t)} \int_0^\infty e^{-rt} U\big(u(t), P(t)\big)\, dt \tag{1}$$

$$\text{s.t.: } \dot{P}(t) = f\big(u(t)\big) - \alpha\big(P(t)\big)P(t), \tag{1a}$$

$$u \geq 0, \tag{1b}$$

where $U(.,.)$ defines the utility of the firm including the trade-off between production output and the social responsibility for the environmental damage. $U(.,.)$ is therefore increasing in the production rate and declining in the pollution stock, where $f(.)$ describes the pollution-accumulating impact of $u(t)$. Note that as of here, we will often omit the time argument t for the ease of exposition. Summing up, we have

$$\frac{\partial U(u, P)}{\partial u} > 0, \tag{2}$$

$$\frac{\partial U(u, P)}{\partial P} < 0, \tag{3}$$

$$\frac{\partial f(u)}{\partial u} > 0. \tag{4}$$

For the following analysis we assume that the function $U(.,.)$ is separable in its two components and is given as

$$U(u, P) = au^\gamma - D(P), \tag{5}$$

where the damage function $D(.)$ describes the social responsibility for pollution, $\gamma \leq 1$ is the production elasticity and a defines the marginal factor productivity. Following Forster (1977) we assume that the dynamics are convex and increasing in u. This comes along with the aspect of inefficient machines whose marginal polluting impact increases with the degree of utilization. Therefore, we have

$$f(u) = \frac{1}{2}\beta u^2, \tag{6}$$

where β describes the pollution intensity. $\alpha(.)$ will first be considered to be constant, while later on a concave-convex declining slope is assumed. Note that generally, we assume that $\alpha(.)$ is a continuously differentiable function for which the conditions

$$\alpha(P) \geq 0 \quad \forall P, \tag{7}$$

$$\alpha'(P) \leq 0 \quad \forall P, \tag{8}$$

hold. Summing up, the optimal control model of the firm is given as

$$\max_u \int_0^\infty e^{-rt}\big(au^\gamma - D(P)\big)\, dt \tag{9}$$

$$\text{s.t.: } \dot{P} = \frac{1}{2}\beta u^2 - \alpha(P)P \tag{9a}$$

$$u \geq 0. \tag{9b}$$

In order to investigate the non-linear feedback effects in environmental quality, we will consider in the subsequent three different scenarios. In the first one the damage function

$D(.)$ is supposed to be concave-convex while the natural cleaning rate $\alpha(.)$ is constant. In the second scenario, we assume to have a quadratic damage function but a pollution dependent natural cleaning rate with a concave-convex decline, and finally, in the third scenario, we combine these two non-linear feedback effects.

Given this general formulation of the firm's decision problem we now include the possibility to choose between a green (G) and a brown (B) technology. Therefore, the firm not only decides about the height of the production rate, but also which type of technology should be used and at what time a switch to the other type should happen. The vector of optimally determined times is denoted as τ in the subsequent. The multi-stage optimization problem then looks as follows

$$\max_{u,\tau,N} \sum_{i=1}^{N+1} \left(\int_{\tau_{i-1}}^{\tau_i} e^{-rt} U_{\rho(i)}(u, P)\, dt + e^{-r\tau_i} S_{\rho(i)} \right), \quad \rho(i) \in \{B, G\} \tag{10}$$

$$\text{s.t.: } \dot{P} = \begin{cases} \frac{1}{2}\beta_B u^2 - \alpha(P)P & \text{for } \rho(i) = B, \\ \frac{1}{2}\beta_G u^2 - \alpha(P)P & \text{for } \rho(i) = G, \end{cases} \tag{10a}$$

$$U_{\rho(i)} = \begin{cases} a_B u^{\gamma_B} - D(P) & \text{for } \rho(i) = B, \\ a_G u^{\gamma_G} - D(P) & \text{for } \rho(i) = G, \end{cases} \tag{10b}$$

$$S_{\rho(i)} = \begin{cases} S_{BG} & \text{for } \rho(i) = B, \\ S_{GB} & \text{for } \rho(i) = G, \end{cases} \tag{10c}$$

$$\rho(i) \neq \rho(i+1) \tag{10d}$$

where N denotes the optimally determined number of switches, S_{BG} the switching costs from brown to green technology and S_{GB} the switching costs from green to brown technology. For simplicity we assume that the switching costs are fixed and that the costs for implementing brown or green technology at $t = 0$ are the same (or zero). The subscript refers to the kind of technology used in each stage. Note that the differences in efficiency and in pollution of the two technologies are given by the production elasticities, γ_B and γ_G, the factor productivities, a_B and a_G and the pollution intensities, β_B and β_G. Since green technology is supposed to be less efficient in production and thus leads to less profits, we have $a_B u^{\gamma_B} \geq a_G u^{\gamma_G}$. Less pollution is caused by green technology, thus $\beta_G \leq \beta_B$. For the optimally determined switching times it has to hold that

$$0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_{i-1} \leq \tau_i \leq \cdots \leq \tau_N < \tau_{N+1} = \infty.$$

2.2 Solution applying Pontryagin's maximum principle

In order to derive the first order conditions of this optimal control problem we first focus again on the fundamental model in (9) with only one type of technology to add then the necessary matching conditions for the multi-stage approach. The *Lagrangian* of model (9) reads as

$$\mathcal{L} = \lambda_0\big(au^\gamma - D(P)\big) + \lambda_1\left(\frac{1}{2}\beta u^2 - \alpha(P)P\right) + \mu u. \tag{11}$$

If u^* is an optimal solution of problem (9), there exist λ_0, λ_1, not all zero, such that the *first order condition*,

$$\frac{\partial \mathcal{L}}{\partial u} = \lambda_0 a\gamma u^{\gamma-1} + \lambda_1 \beta u + \mu = 0 \quad \Rightarrow \quad u^* = \left(-\frac{\lambda_0 \gamma a + \mu}{\beta \lambda_1}\right)^{\frac{1}{2-\gamma}}, \tag{12}$$

is satisfied where λ_1 is the costate variable. Note that also the *Legendre-Clebsch-Condition*,

$$\frac{\partial^2 \mathcal{L}}{\partial u^2} < 0, \tag{13}$$

holds, if $\lambda_1 < 0$ and $\gamma \le 1$. Further on, the *complementary slackness condition*,

$$\mu u = 0 \quad \text{with } \mu, u \ge 0, \tag{14}$$

has to be fulfilled, where μ is the Lagrange Multiplier for the inequality-condition in (9b). Inserting u^* into the dynamics in (9a) yields together with the costate equation the canonical system in the state-costate space (cf. Grass et al. 2008),

$$\dot{P} = \frac{1}{2}\left(-\frac{\lambda_0 a\gamma + \mu}{\beta\lambda_1}\right)^{\frac{2}{2-\gamma}} - \alpha(P)P, \tag{15}$$

$$\dot{\lambda}_1 = r\lambda_1 - \frac{\partial \mathcal{L}}{\partial P} = \lambda_1\big(r + \alpha'(P)P + \alpha(P)\big) + \lambda_0 D'(P). \tag{16}$$

The considered optimal control problem is *normal*, which means that without loss of generality, $\lambda_0 = 1$. This can be shown as follows.

Assume that $\lambda_0 = 0$. Then, in order to find a steady state of the canonical system, the equations

$$\dot{P} = \frac{1}{2}\left(-\frac{\mu}{\beta\lambda_1}\right)^{\frac{2}{2-\gamma}} - \alpha\big(P^*\big)P^* = 0, \tag{17}$$

$$\dot{\lambda}_1 = \lambda_1\big(r + \alpha'\big(P^*\big)P^* + \alpha\big(P^*\big)\big) = 0, \tag{18}$$

have to be satisfied. Considering Eq. (17), in both, inner as well as boundary solution $\alpha(P^*)P^* = 0$ has to hold. Consequently, this yields two cases:

– $P^* = 0$: Eq. (18) then reads as

$$\lambda_1\big(r + \alpha(0)\big) = 0.$$

Due to the fact that $\alpha(P) \ge 0$, the only solution is $\lambda_1 = 0$ which is contradictory to $(\lambda_0, \lambda_1) \ne (0, 0)$.

– $\alpha(P^*) = 0$: Eq. (18) then reads as

$$\lambda_1\big(r + \alpha'\big(P^*\big)P^*\big) = 0.$$

This however implies that $\alpha'(P^*) = 0$ because $\alpha(.)$ is supposed to be continuously differentiable and greater or equal to zero, which again yields $\lambda_1 = 0$ as the only solution and is contradictory to $(\lambda_0, \lambda_1) \ne (0, 0)$.

Therefore, no solution with $\lambda_0 = 0$ exists and $\lambda_0 > 0$ has to be satisfied. Adequate standardization yields

$$\tilde{\lambda}_0 = \lambda_0 \frac{1}{\lambda_0} = 1,$$

which proofs the assumption. Hence, we set for the following $\lambda_0 = 1$ and use λ instead of λ_1. The canonical system then is given as

$$\dot{P} = \frac{1}{2}\left(-\frac{a\gamma + \mu}{\beta\lambda}\right)^{\frac{2}{2-\gamma}} - \alpha(P)P, \tag{19}$$

$$\dot{\lambda} = \lambda\big(r + \alpha'(P)P + \alpha(P)\big) + D'(P). \tag{20}$$

Searching for a boundary solution, we immediately find that the trivial point

$$(\hat{P}_1, \hat{u}_1, \hat{\lambda}_1) = (0, 0, -\infty) \tag{21}$$

is a steady state as long as $D'(0) = \infty$ holds.

Looking for further steady states within the admissible region ($\mu = 0$), the calculation of the instantaneous equilibrium points of Eq. (19) yields

$$\lambda = \pm \frac{a}{\beta\sqrt{2\alpha(P)P}}, \tag{22}$$

but we can exclude the solution with positive λ since this steady state is not admissible due to the non-negativity condition for u. Together with (20), this leads the equation

$$-\frac{a}{\beta\sqrt{2\alpha(P)P}} = \frac{-D'(P)}{r + \alpha'(P)P + \alpha(P)}, \tag{23}$$

which finally has to be solved to obtain the steady states.

If the natural cleaning rate is assumed to be constant, $\alpha(P) := \alpha$, an equilibrium condition can be derived, as in Forster (1977), which states that the marginal utility of the production rate u in the equilibrium has to be equal to the loss in utility due to an increment in the production rate. This utility is lost due to the generation of an additional unit of pollution as a result of the temporary increase in production. Note that this loss would last forever if pollution would not decay. But in the current approach the loss in utility in any period is less due to the natural cleaning rate α and the present value with respect to the discount rate r is given as

$$a = \frac{D'(P^*)u^*}{r + \alpha}. \tag{24}$$

For the multi-stage approach the derived conditions apply for both technology types, but with the technology specific parameters instead. This means that $\mathcal{L}_B$ is equal to the Lagrangian $\mathcal{L}$ in (11) but with $\gamma = \gamma_B$, $a = a_B$ and $\beta = \beta_B$, for $\mathcal{L}_G$ vice versa. At switching points additionally some matching conditions have to hold (see Tomiyama 1985; Tomiyama and Rossana 1989; Makris 2001 and Sect. 8.1 in Grass et al. 2008), which for a switch from brown to green technology are given as

$$\mathcal{L}_B(\tau_i) + rS_{BG} = \mathcal{L}_G(\tau_i), \qquad \lambda_B(\tau_i) = \lambda_G(\tau_i), \tag{25}$$

and for a switch from green to brown technology read as

$$\mathcal{L}_G(\tau_i) + rS_{GB} = \mathcal{L}_B(\tau_i), \qquad \lambda_G(\tau_i) = \lambda_B(\tau_i). \tag{26}$$

For these two cases the conditions reflect whether it is optimal to switch. The first condition, respectively, serves to determine the optimal switching time while the second condition means that the marginal utility of the state must be continuous at the switching point. Note that in the case of one stage being clearly preferable the firm must not switch at all. At the switching points it has to hold that

$$-rS_{BG} = \mathcal{L}_B(\tau_i) - \mathcal{L}_G(\tau_i) = a_B u_B^{\gamma_B} - a_G u_G^{\gamma_G} + \frac{1}{2}\lambda\big(\beta_B u_B^2 - \beta_G u_G^2\big), \tag{27}$$

$$-rS_{GB} = \mathcal{L}_G(\tau_i) - \mathcal{L}_B(\tau_i) = a_G u_G^{\gamma_G} - a_B u_B^{\gamma_B} + \frac{1}{2}\lambda\big(\beta_G u_G^2 - \beta_B u_B^2\big). \tag{28}$$

Since we assume separability in state and control variables the matching conditions do not explicitly depend on the state variable, only on the current level of production and the costate variable. Note that the matching conditions are only necessary conditions. In order to check whether it is really optimal to switch from brown to green technology or vice versa we have to proceed to numerical calculations.

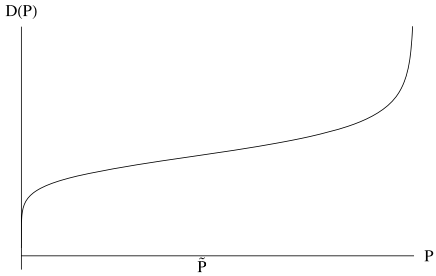


Fig. 1 Concave-convex damage function

3 Production-pollution model with a concave-convex damage function

For the first approach we assume that the damage function $D(P)$ is a concave-convex function in P, which is given for the following as

$$D(P) = \frac{bP^c}{(1-P)^c}, \tag{29}$$

with parameter c defining the curvature of the function, and is shown in Fig. 1. It is assumed that there exists an average level of pollution, $\tilde{P}$, which is considered as standard level of pollution by the public. At this standard level, the public does exert a negative pressure on the firm but the marginal damage is very low. This means that a change upward or downward in the pollution stock in some neighborhood close to this average pollution level $\tilde{P}$ is not considered as a remarkable worsening or improvement by the public. To attract the public's attention the firm either has to put an outstanding effort into being environmentally friendly, which is rewarded with a high increase in reputation and hence a reduced damage (concave part), or utterly pollutive leading to a high reputation loss (convex part).

Further on, we assume in this model version that the natural cleaning rate is constant and therefore independent of the current stock of pollution,

$$\alpha(P) = \alpha. \tag{30}$$

Before we start with the analysis of the multi-stage framework for this approach we first investigate the problem of the representative firm with only one type of technology to see what is the optimal choice of the production rate given these concave-convex damage function.

3.1 Optimal solution with only one type of technology

The model we consider in this subsection is given as

$$\max_u \int_0^\infty e^{-rt}\left(au^\gamma - \frac{bP^c}{(1-P)^c}\right)dt \tag{31}$$

$$\text{s.t.: } \dot{P} = \frac{1}{2}\beta u^2 - \alpha P, \tag{31a}$$

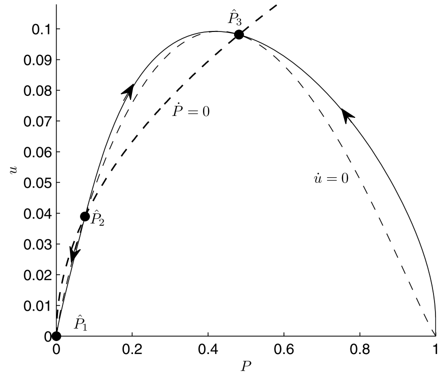
$$u \geq 0, \tag{31b}$$

which yields, according to the derivation in Sect. 2.2, the canonical system

$$\dot{P} = \frac{1}{2}\left(-\frac{a\gamma + \mu}{\beta\lambda}\right)^{\frac{2}{2-\gamma}} - \alpha P, \tag{32}$$

$$\dot{\lambda} = \lambda(r+\alpha) + \frac{cbP^{c-1}}{(1-P)^{c+1}}. \tag{33}$$

Fig. 2 Phase portrait for $r = 0.04$, $a = 1$, $\alpha = 0.01$, $\beta = 1$, $\gamma = 1$, $b = 0.9$, $c = \frac{1}{7}$ (weak Skiba case)



3.1.1 Steady states

Because $D'(0) = \infty$, the boundary solution derived in (21) is admissible for this model approach. Solving Eq. (32) for λ in order to find inner solutions ($\mu = 0$) yields

$$\lambda = -\frac{a\gamma}{\beta(2\alpha P)^{(1-\gamma/2)}}, \tag{34}$$

which leads together with Eq. (33) to the equation

$$\frac{a\gamma}{\beta(2\alpha P)^{1-\gamma/2}} = \frac{1}{r+\alpha}\frac{cbP^{c-1}}{(1-P)^{c+1}}, \tag{35}$$

that has to be solved in order to find steady states. Numerical solution for the parameters $r = 0.04$, $a = 1$, $\alpha = 0.01$, $\beta = 1$, $\gamma = 1$, $b = 0.9$, $c = 1/7$ then yields two inner solutions so that, together with the boundary solution, one has three steady states, which are

$$(\hat{P}_1, \hat{u}_1, \hat{\lambda}_1) = (0, 0, -\infty), \tag{36}$$

$$(\hat{P}_2, \hat{u}_2, \hat{\lambda}_2) = (0.0758, 0.0389, -25.6906), \tag{37}$$

$$(\hat{P}_3, \hat{u}_3, \hat{\lambda}_3) = (0.4814, 0.0981, -10.1911). \tag{38}$$

Note, that here the equilibrium condition, as derived in (24), holds.

3.1.2 Stability and phase portraits

The *Jacobian* of the system (32)–(33) is given as

$$J = \begin{pmatrix} -\alpha & \frac{1}{\gamma-2}\left(-\frac{a\gamma+\mu}{\beta}\right)^{\frac{2}{2-\gamma}}\lambda^{\frac{\gamma-4}{2-\gamma}} \\ \frac{bc(c+2P-1)}{P^{2-c}(1-P)^{2-c}} & r+\alpha \end{pmatrix}. \tag{39}$$

The calculation of the eigenvalues of the Jacobian at the three steady states for $c = 1/7$ shows, that the first and the third steady state are saddle points while the one in between, $\hat{P}_2$, is an unstable node. Hence, $\hat{P}_2$ separates the basins of attraction as a threshold but not as an indifference point because the optimal trajectories are not disjunct. $\hat{P}_2$ therefore is a so-called weak Skiba point (for more detail see Grass et al. 2008, Chap. 5). Figure 2 shows the phase portrait for this weak Skiba case, where the dashed lines are the isoclines and the solid lines are the solution paths. In this case the optimal solution is purely history-dependent. Starting exactly at this steady state the optimal strategy is to keep production and pollution at its

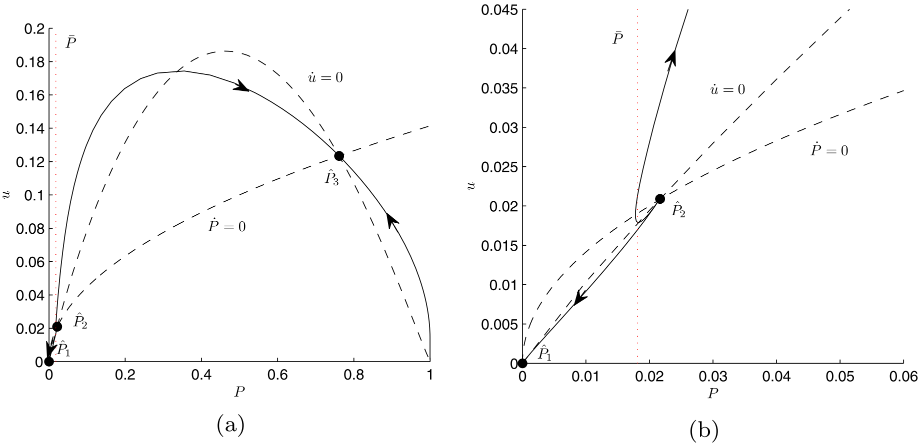


Fig. 3 (**a**) Phase portrait for $r = 0.04$, $a = 1$, $\alpha = 0.01$, $\beta = 1$, $\gamma = 1$, $b = 0.9$ and $c = 0.075$ (Skiba case), (**b**) Zoom

initial level. However, on the left hand side of the unstable node, the lower equilibrium in the origin is approached, while on the right hand side, the system converges towards the high steady state. This means that in case of an initially very low level of pollution, where additional pollutants cause a strong damage in reputation, the shutdown of production is preferable to the increasing damage if production would be continued. The result to stop production completely in the long run might seem odd at the first glance. In this setup, however, it makes sense as we assume that the firm faces negative pressure from the public as long as the pollution is positive, even when nothing at all is produced anymore, but also that, starting from zero pollution, a slight step closer to the average pollution level causes a comparatively high damage. If, however, the initial pollution level is already so high that it is close to the average pollution level $\tilde{P}$ where the marginal impact of additional pollution is not that strong anymore, the further increase of production is optimal. In case of a pollution level beyond the high equilibrium value $\hat{P}_3$, which is actually quite close to the average pollution level $\tilde{P}$, the marginal impact increases again, the pollution level therefore has to decline and the system converges towards the upper steady state.

3.1.3 *Bifurcation analysis with respect to c*

The value of the bifurcation parameter c is crucial for the form of the damage function shown in Fig. 1. Doing the same analysis as in the previous subsection for $c = 0.075$, one obtain again three steady states,

$$(\hat{P}_1, \hat{u}_1, \hat{\lambda}_1) = (0, 0, -\infty), \tag{40}$$

$$(\hat{P}_2, \hat{u}_2, \hat{\lambda}_2) = (0.0215, 0.0207, -48.2645), \tag{41}$$

$$(\hat{P}_3, \hat{u}_3, \hat{\lambda}_3) = (0.7613, 0.1234, -8.1041), \tag{42}$$

but this time, the stability analysis shows, that the unstable steady state is not a node as in the previous case, but a focus. Therefore, one can find a point $\bar{P}$ which separates the basins of attraction both as a threshold as well as an indifference point and hence is a so-called Skiba point. This Skiba case can be seen in the phase portrait in Fig. 3(a). At $\bar{P}$, which does not coincide with the steady state $\hat{P}_2$ and which can be seen in Fig. 3(b), a firm is indifferent

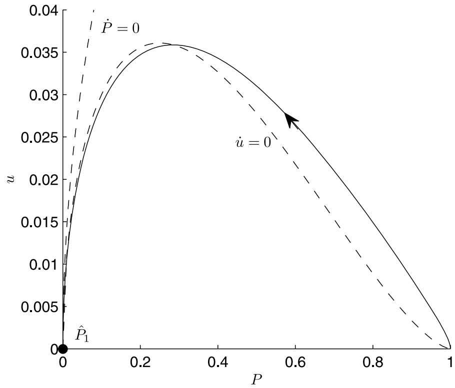


Fig. 4 Phase portrait for $r = 0.04$, $a = 1$, $\alpha = 0.01$, $\beta = 1$, $\gamma = 1$, $b = 0.9$ and $c = 0.5$

between producing little and ending up in a clean environment and producing much, but having to deal with a dirty environment.

If one, however, sets $c = 0.5$, no inner solution can be found anymore and the only equilibrium is given by the boundary solution,

$$(\hat{P}_1, \hat{u}_1, \hat{\lambda}_1) = (0, 0, -\infty). \tag{43}$$

In this case, the impact of an increase in pollution due to a higher production rate makes any production greater than zero too costly and therefore, the only optimal solution is the shut down of production, see Fig. 4.

To generalize these results, Fig. 5(a) shows a bifurcation diagram with c varying between 0 and 0.2. For high c the damage caused by the negative reputation due to pollution is too severe, thus it is always optimal to approach the steady state with no pollution. For intermediate c there are three admissible steady states—two saddle points and one unstable node. If the initial state value is above the unstable node $\hat{P}_2$, the optimal solution is to approach the steady state with a high level of pollution. Starting below the unstable node, one would choose a production strategy so that one would end up in the steady state with no pollution. For small c the steady state $\hat{P}_2$ becomes an unstable focus and we can find a Skiba point, depicted as dashed line in Fig. 5(b), where one is indifferent between approaching either the steady state with high or no pollution. If c becomes even smaller the state value of the unstable steady state in the middle becomes closer and closer to zero, so unless starting in an absolutely clean environment it is always optimal to go for the high pollution-high production steady state.

3.2 Multi-stage approach

Considering this approach in the multi-stage framework, of course it is pretty much clear that if we consider parameters such that brown technology is much more efficient than green technology or causes not much more pollution, it is always optimal to use brown technology and vice versa. However, we want to focus on scenarios where it is not so clear which technology is better. We will see that history-dependence plays a role to decide which steady state is optimal to approach to in the long run but also which technology should be adapted at what time and for how long.

In order to gain some basic understanding about the system behavior we first assume that there are no switching costs. Figure 6(a) depicts such a scenario. For the parameters used for Fig. 6 we can find six admissible steady states—three where one would use brown

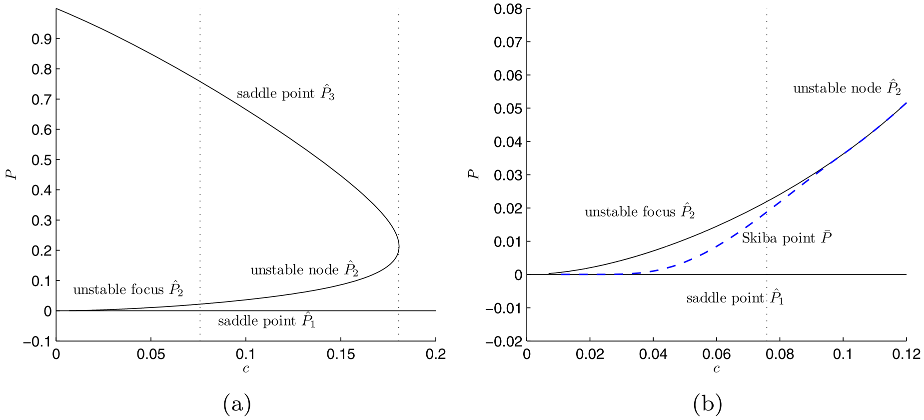


Fig. 5 (**a**) Bifurcation diagram for $r = 0.04, a = 1, \alpha = 0.01, \beta = 1, \gamma = 1, b = 0.9$ showing the steady states and Skiba points for different values of parameter c, (**b**) Zoom

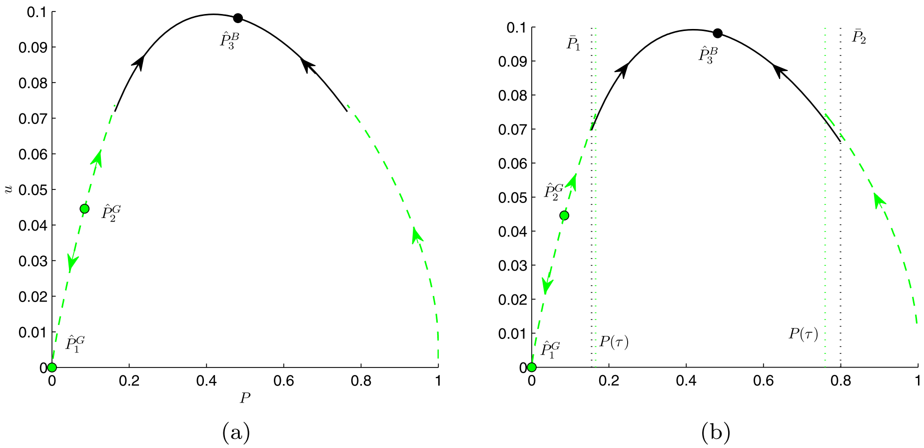
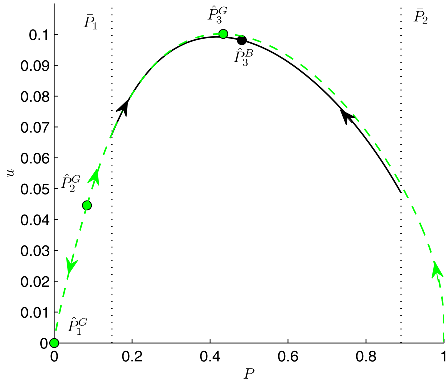


Fig. 6 Phase portrait for $r = 0.04$, $a_B = 1$, $a_G = 0.8$, $b = 0.9$, $c = 0.1429$, $\alpha = 0.01$, $\beta_B = 1$, $\beta_G = 0.85$, $\gamma_B = 1$, $\gamma_G = 0.945$ and (**a**) $S_{GB} = S_{BG} = 0$, (**b**) $S_{GB} = S_{BG} = 0.001$

technology and three where one would use green technology in the long run. We have four saddle points and two unstable nodes. However, we find that even though the steady states are admissible, not every one of them is relevant for the optimal solution. If the initial pollution is rather small, i.e. $P_0 < \hat{P}_2^G$ (the superscript of the steady state shows which stage is optimal there, the subscript refers to the corresponding steady state in the one-stage version of the model), it is optimal to use green technology (dashed line) and reduce production, so that there is no pollution in the long run. The reason for this is that, starting at an initially low pollution level, an increase in pollution generates a comparatively high negative impact on the reputation of the firm which hence is rather costly. On the other hand if the initial pollution level exceeds the threshold $\hat{P}_2^G$, it is optimal to increase production, first using green technology, but then switching to brown technology (solid line) due to its higher efficiency. Here, an increase in pollution is not that costly anymore, because the initial level is already quite close to the average pollution level $\tilde{P}$ which has a low marginal impact. Also

Fig. 7 Phase portrait for $r = 0.04$, $a_B = 1$, $a_G = 0.8$, $b = 0.9$, $c = 0.1429$, $\alpha = 0.01$, $\beta_B = 1$, $\beta_G = 0.85$, $\gamma_B = 1$, $\gamma_G = 0.945$ and $S_{GB} = S_{BG} = 1$



for high initial pollution it is optimal to use green technology first because of the severity of the damage caused by pollution. The firm then would first produce only a little in order to reduce pollution and to improve its reputation. As the pollution stock decreases, it slowly can increase its production again until finally it is optimal to switch to the more efficient brown technology where the production rate than is increased until pollution input and pollution decay are equally high.

In Fig. 6(b) we consider the case of intermediate switching costs. Since costs for implementing any kind of technology very much depend on the specific technology used, we assume for simplicity that the costs for adopting brown technology are the same as for green technology. Note, however, that asymmetric switching costs only means in this model that the switch to the cheaper technology happens sooner, if it is optimal at all. Like before, if the initial pollution level is below the weak Skiba point it is always optimal to use green technology. Further on, one would use green technology first if the initial state value is slightly above the threshold point or very large. However, the time period in which one remains in the first, green stage is longer due to the switching costs. Looking at the matching conditions, we see that because of the switching costs, the value of the Lagrangian is greater for the brown technology at the optimal switching point $P(\tau)$. Thus, starting at this point, it is optimal to use brown technology and not switch at all. However, we can find two Skiba points $\bar{P}_1$, $\bar{P}_2$, where a firm is indifferent between always using brown technology and using green technology first and switching to brown technology at $P(\tau)$. As we increase the switching costs, $\bar{P}_1$ shifts to the left and $\bar{P}_2$ to the right while $P(\tau)$ gets closer to the brown technology steady state. Thus, the optimal switching time increases, until it is not optimal anymore to switch at all. The impact of high switching costs can be seen in Fig. 7. For the given parameters it is not optimal at all anymore to switch between the two technologies. For small and for high initial pollution it is optimal to use green technology for reasons discussed above, while for intermediate pollution levels the best strategy is to rely on brown technology.

4 Production-pollution model with a pollution-dependent cleaning rate

In the previous section, we assumed that the natural cleaning rate is constant and does not depend on the pollution stock. However, as also explained in Xepapadeas (2005), the use of linear dynamics like the one implied by exponential decay, maybe is not a good approximation because they might obscure important characteristics such as irreversibility or hysteresis

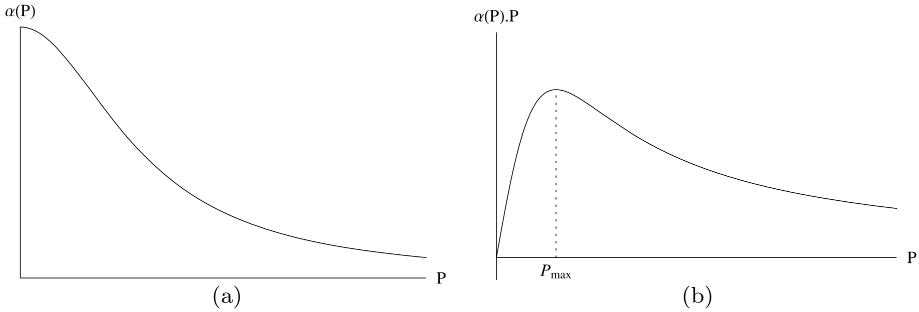


Fig. 8 (**a**) Concave-convex natural cleaning rate, (**b**) total pollution decay

effects. In Xepapadeas (2005), in addition to a constant natural cleaning rate, an S-shaped, non-linear feedback is considered which is called *internal loadings*. However, instead of considering the cleaning process and the internal loadings separately, we assume that the natural cleaning rate depends non-linearly on the pollution stock representing the hysteresis effect. Hence, the higher the stock of pollution, the lower is the self-cleaning capacity of the environment. While in many papers the cleaning rate is considered to be a concave function of the pollution stock, cf. Forster (1975), Tahvonen and Withagen (1996), Hediger (2009), Prieur (2009), we follow Tahvonen and Salo (1996) by assuming that this non-linear cleaning rate is even concave-convex,

$$\alpha(P) = 1 - \frac{P^2}{P^2 + n}, \tag{44}$$

as one can also see in Fig. 8(a). This means that, as long as pollution is low, the environment is in a quite 'healthy' state and its capacity to recover is very high so that a marginal unit of pollution hardly has an effect. However, if pollution gets higher, the 'health status' of the environment gets worse and the capacity to recover declines with every marginal unit until finally, the damage is so high that the cleaning rate $\alpha(.)$ converges towards zero. Considering the total pollution decay, this yields a hump-shaped function as one can see in Fig. 8(b). One can see that the decay is concave-convex as well and reaches a maximum at $P_{\max}$. In order to underline the extend of such a hysteresis effect we simplify the assumption of a concave-convex damage function from Sect. 3 by using a quadratic one instead, $D(P) = P^2$, which means that for this approach we focus only on the negative impact on reputation on pollution and not on the positive impact of keeping pollution low. We again first solve the simpler model with only one type of technology before proceeding to the multi-stage problem.

4.1 Optimal solution with only one type of technology

The model we investigate in this subsection is

$$\max_u \int_0^\infty e^{-rt}\big(au^\gamma - P^2\big)\,dt \tag{45}$$

$$\text{s.t.: } \dot{P} = \frac{1}{2}\beta u^2 - \left(1 - \frac{P^2}{P^2 + n}\right)P \tag{45a}$$

$$u \geq 0 \tag{45b}$$

and the canonical system, as derived in Sect. 2.2, reads

$$\dot{P} = \frac{1}{2}\left(-\frac{a\gamma + \mu}{\beta\lambda}\right)^{\frac{2}{2-\gamma}} - \left(1 - \frac{P^2}{P^2+n}\right)P, \tag{46}$$

$$\dot{\lambda} = \lambda\left(r - \frac{2nP}{(n+P^2)^2}P + \left(1 - \frac{P^2}{P^2+n}\right)\right) + 2P. \tag{47}$$

4.1.1 *Steady states*

As in the previous approach no boundary solution can be found here. But solving Eq. (47) for λ in order to find inner solutions yields

$$\lambda = -\frac{2P(P^2+n)^2}{(P^2+n)(r(P^2+n)+n) - 2nP}, \tag{48}$$

which finally leads together with Eq. (46) to the equation

$$\frac{1}{2}\left(\frac{a\gamma((P^2+n)(r(P^2+n)+n) - 2nP)}{\beta 2P(P^2+n)^2}\right)^{\frac{2}{2-\gamma}} = \left(1 - \frac{P^2}{P^2+n}\right)P, \tag{49}$$

that has to be solved. Numerical solution for $a = 1$, $r = 0.04$, $\beta = 1$, $\gamma = 1$, and $n = 0.00025$ yields the three steady states

$$(\hat{P}_1, \hat{u}_1, \hat{\lambda}_1) = (0.0171, 0.1256, -7.9647), \tag{50}$$

$$(\hat{P}_2, \hat{u}_2, \hat{\lambda}_2) = (0.0929, 0.0723, -13.8276), \tag{51}$$

$$(\hat{P}_3, \hat{u}_3, \hat{\lambda}_3) = (0.7842, 0.0252, -39.61). \tag{52}$$

4.1.2 *Stability and phase portraits*

The Jacobian is given as

$$J = \begin{pmatrix} \frac{n(P^2-n)}{(n+P^2)^2} & \frac{1}{\gamma-2}\left(-\frac{a\gamma+\mu}{\beta}\right)^{\frac{2}{2-\gamma}}\lambda^{\frac{4-\gamma}{\gamma-2}} \\ -\lambda\frac{nP(4(n+P^2)-8P^2-2P(n+P^2))}{(n+P^2)^3} + 2 & r + \left(1 - \frac{p^2}{p^2+n}\right) \end{pmatrix} \tag{53}$$

and the corresponding eigenvalues show, that the first and the third steady state are saddle points while the second one is an instable focus. This yields again a Skiba case as one can see in the phase portrait in Fig. 9. In contrast to Sect. 3, where the indifference was given between a high production rate with a high level of pollution or zero production with no pollution in the Skiba point, here the firm is indifferent between a lower production rate with a high pollution level or high production rate with a low pollution level. Note that the pollution level of the lower saddle point exactly coincides with the pollution level $P_{\max} > 0$, where the pollution decay is maximal. This explains why zero pollution here is not optimal anymore. If pollution is lower than $P_{\max}$ the production rate can be increased without affecting pollution too much because the increasing pollution decay is compensating the additional pollution to some degree. This increase is worthwhile until finally the marginal utility of a further increase of the production rate equals the marginal disutility of an increase in pollution, which is exactly at $P_{\max}$. Figure 9 further shows the mentioned hysteresis effect due to the dependent cleaning rate. If the initial pollution level is beyond the Skiba point, a long-run level of high pollution cannot be avoided, even if the production rate goes down. If the initial pollution level is beneath the Skiba point, the production rate can increase without affecting the environment too much due to the high regeneration capacity of the environment.

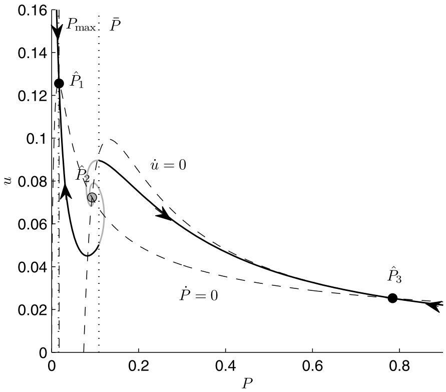


Fig. 9 Phase portrait for $a=1$, $r=0.04$, $\beta=1$, $\gamma=1$ and $n=0.00025$ (Skiba case)

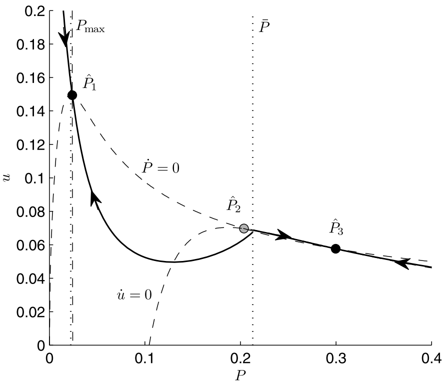


Fig. 10 Phase portrait for $a=1$, $r=0.04$, $\beta=1$, $\gamma=1$, and $n=0.0005$

4.1.3 Bifurcation analysis with respect to n

The same analysis for $n=0.0005$ also yields three steady states, which are

$$(\hat{P}_1, \hat{u}_1, \hat{\lambda}_1) = (0.024, 0.1493, -6.6958) \tag{54}$$

$$(\hat{P}_2, \hat{u}_2, \hat{\lambda}_2) = (0.2036, 0.0697, -14.3557) \tag{55}$$

$$(\hat{P}_3, \hat{u}_3, \hat{\lambda}_3) = (0.2996, 0.0576, -17.357), \tag{56}$$

but this time, the stability analysis shows that the second steady state turns out to be an instable node, while the two others remain saddle points. In Sect. 3 we saw a scenario where the instable node served as threshold. Here, however, we have a nice example that having an instable node does not always correspond to the weak Skiba case. In Fig. 10, one can see that even though the steady state in the middle is an instable node, there is an overlap of the trajectories which are the candidates for the optimal solution. Evaluating the Lagrangian along these solution paths, we find that there is a (strong) Skiba point at $\bar{P}=0.2129$ at which the firm has the choice between two options: slightly reduce production in the beginning so that pollution can decrease to a level where the environment's capacity to recover is high and hence production can be increased again or produce initially more accepting an increase in pollution which leads, however, to a lower production in the long run due to the high pollution stock. It appears that the choice of the parameter n is crucial for the stability of the instable steady state. To investigate this interrelationship bifurcation analysis is used with

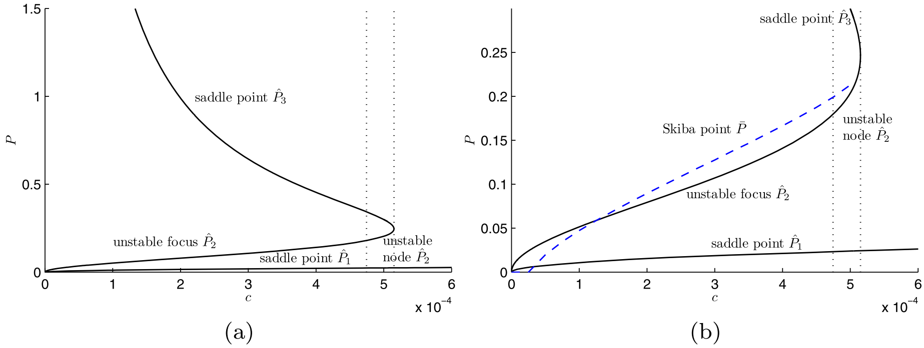
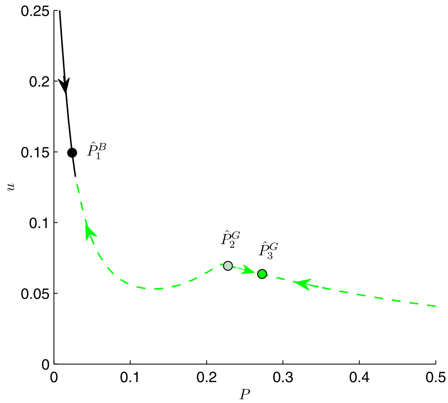


Fig. 11 (**a**) Bifurcation diagram with respect to n, (**b**) Zoom with Skiba point

Fig. 12 Phase portrait for $r = 0.04$, $n = 0.0005$, $a_B = 1$, $a_G = 0.85$, $\beta_B = 1$, $\beta_G = 0.9$, $\gamma_B = 1$, $\gamma_G = 0.945$ and $S_{GB} = S_{BG} = 0$. The *black solid line* depicts green technology, the *gray dotted line* brown technology



respect to n. In Fig. 11(a) one can see bifurcation diagrams that shows how the position and the stability of the three steady states change with increasing n. Starting from $n = 0$, the instable steady state remains a focus, until $n = 0.00047$, where the complex eigenvalues become real and the instable focus turns into an instable node. Note, that an increasing value of n means a higher capacity of the environment to recover. This is the reason why the high, catastrophic equilibrium decreases in P and increases in u with increasing value of n. In Fig. 11(b), the zooming shows the position of the Skiba point, which lies for some values of n below, for some others above the unstable steady state.

4.2 Multi-stage approach

While production is more efficient for brown technology, less pollution is caused by green technology. In order to gain some basic understanding about what can happen in the current approach if there is more than just one stage, let us assume that there are no switching costs. If initial pollution is high one would end up in a steady state with green technology but high pollution, as one can see in Fig. 12. The unstable green steady states is a weak Skiba point, starting with a smaller pollution level ending up in the brown technology steady state. For high initial pollution one would always use green technology in order to keep the damage caused by pollution within some limit. On the other hand starting just below the weak Skiba point it is optimal to first use green technology and produce little so pollution decreases.

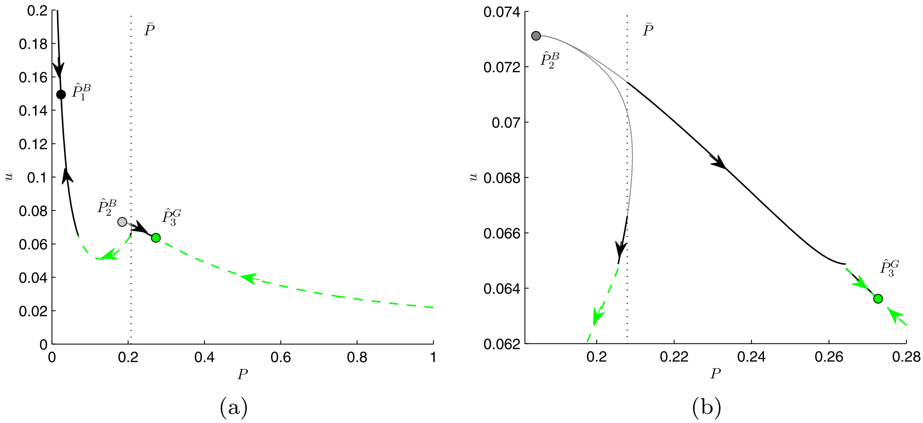


Fig. 13 (**a**) Phase portrait for $r = 0.04$, $n = 0.0005$, $a_B = 1.04$, $a_G = 0.85$, $\beta_B = 1$, $\beta_G = 0.9$, $\gamma_B = 1$, $\gamma_G = 0.945$ and $S_{GB} = S_{BG} = 0$, (**b**) Zoom. The *black solid line* depicts green technology, the *gray dotted line* brown technology

Since we assume that the pollution rate is more efficient when pollution is low, one can increase production and pollution would still decline. After some time one can even switch to the more efficient brown technology and end up in a relatively low polluted steady state.

Note that whether it is optimal to use green or brown technology in the long run depends entirely on the used parameters. If the efficiency in production of green technology is not much below that of brown technology or causes much less harmful pollution, it is always optimal to use green technology. If brown technology leads to much higher profits, it is—unfortunately for nature—optimal to always use brown technology. Also, the location and number of switches between the two technologies depends on the trade-off between production and pollution that a firm faces when confronted with green and brown technology. This can nicely be seen in Fig. 13 where we assume that brown technology is slightly more efficient in production than in the previous case.

If we assume that there are no switching costs, the steady states relevant for the optimal solutions are the low pollution brown saddle point, the intermediate pollution brown unstable node and the high pollution green saddle point. In this scenario we find a (strong) Skiba point at which one is indifferent between two options: The first is to use brown technology and produce so much that pollution would increase. However, due to the damage caused by the negative reputation due to pollution one has to gradually decrease production. After some time one would switch to green technology and end up in the green steady state. The second option at the Skiba point is to use brown technology first, but produce so little that nature can recover and pollution decrease. At this part of the solution path one would exploit the higher efficiency of production of brown technology. However, after some time one would want pollution to decrease faster, thus, it is optimal to switch to green technology. As in the scenario described before, due to the more efficient cleaning rate for low pollution, one can start to produce more after some time and even switch to brown technology ending up in the brown steady state.

In Fig. 14 we can see what happens if we have high switching costs. Then it does not pay off at all to switch between the technologies for any $\tau > 0$. If the initial pollution level is low, one would always use brown technology and end up in the steady state with low pollution but high production. For intermediate initial pollution, one would also always use brown technology, but one would initially produce more leading to a highly polluted steady

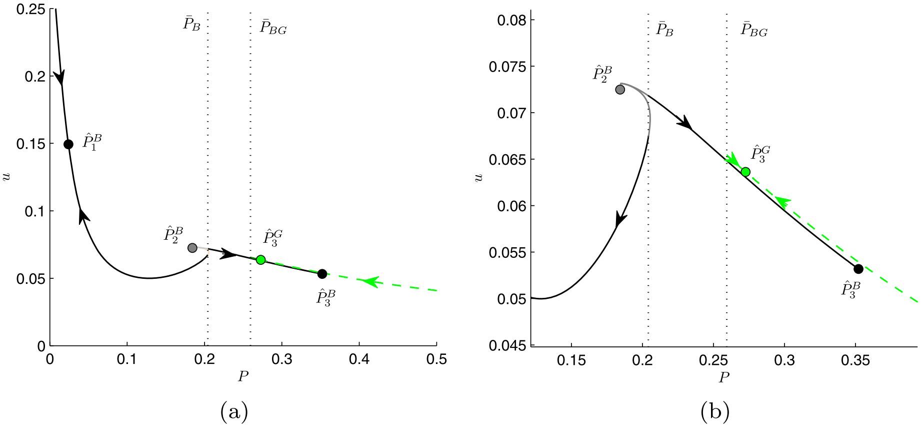


Fig. 14 (**a**) Phase portrait for $r = 0.04$, $n = 0.0005$, $a_B = 1.04$, $a_G = 0.85$, $\beta_B = 1$, $\beta_G = 0.9$, $\gamma_B = 1$, $\gamma_G = 0.945$ and $S_{GB} = S_{BG} = 1$, (**b**) Zoom. The *black solid line* depicts green technology, the *gray dotted line* brown technology

state with brown technology. If the initial pollution level is high, it is optimal to use green technology from the very beginning in order to keep the damage caused by pollution low. There are two indifference points, one where the firm has the choice between the high and the low pollution brown steady state and one where it has the choice between green and brown technology with a high steady state level of pollution.

5 Pollution dependent natural cleaning rate with a concave-convex damage function

In the previous sections we assumed that either the damage function is concave-convex or that the cleaning rate itself depends on the pollution. We saw that both of these mechanisms can lead to history-dependence. In the first case we saw that if the pollution stock is below the Skiba level one would always reduce and even stop production in the long run due to the damage caused even by a low, non-zero level of pollution. On the other hand when we considered that the natural cleaning rate depends on the current stock of pollution and assumed quadratic damage of pollution, we saw that for a small level of pollution it is optimal to produce a lot. This raises of course the question about what happens if both of the described effects occur.

The damage function and natural cleaning rate thus are

$$D(P) = \frac{bP^c}{(1-P)^c}, \qquad \alpha(P) = 1 - \frac{P^2}{P^2 + n}.$$

5.1 Optimal solution with only one type of technology

In order not to be too repetitive we will not discuss this case in detail as those before. Figure 15 depicts an example for this setup. It is not surprising that also in this case one can find Skiba points. We can see that in a certain sense the impact of the concave-convex damage function is stronger at least for the used parameters. At the Skiba point one has the choice between increasing production for some time and ending up in a highly polluted environment with a lower degree of production, and reducing or even stopping production in the

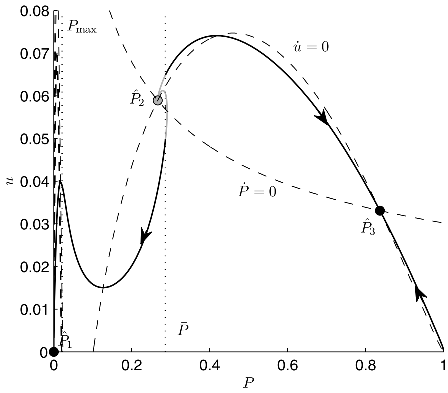
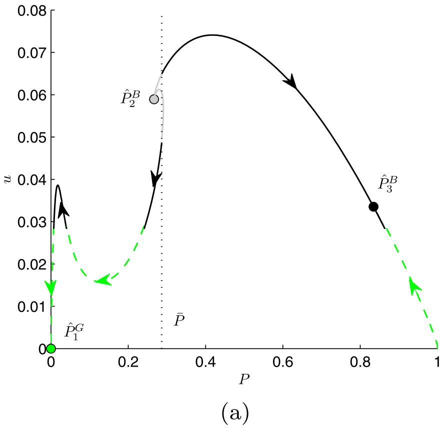


Fig. 15 Phase portrait for $r = 0.04$, $a = 1$, $\alpha = 0.01$, $\beta = 1$, $\gamma = 1$, $b = 0.9$, $c = 1/7$ and $n = 0.00047$

long run. In the latter case, however, the non-constant natural cleaning rate has some interesting effect on the solution path. Unlike before, where one would always monotonically decrease production, here one would first decrease production to reduce pollution. Then, after some time, the natural cleaning rate becomes more effective due to the lower level of pollution and one can increase production at least for some time and pollution would still decrease. However, in the long run one is still better of by stopping production completely and end up in a non-polluted environment.

5.2 Multi-stage approach

Probably the most interesting case with respect to the number of optimal switches can be seen in Fig. 16. For a high initial pollution the optimal strategy is pretty similar to those considered in the previous cases: Assuming that there are no switching costs Fig. 16(a) shows that then it is optimal to use green technology first to reduce pollution and improve reputation and then gradually increase production so that for a certain pollution level it makes sense to exploit the higher efficiency and switch to brown technology. For medium initial pollution levels we have a Skiba point. While it is always optimal there to use brown technology, the firm has the choice between increasing production and ending up in a highly polluted environment with a very bad reputation or decrease production. For the latter one, brown technology is used first to exploit the higher efficiency. However, very soon it is optimal to switch to green technology so that pollution can decrease faster. After some time the higher efficiency of the cleaning rate comes into play, because then a firm can start to produce more while pollution still is decreasing. The high cleaning rate for pollution becomes even so strong that it is optimal to switch to brown technology again and produce even more. However, when pollution has further decreased, at some point of time the pollution level is reached where a marginal further decline leads to a very high improvement in reputation. Because in this area the loss of less production is lower than the gain of an improved reputation, it is optimal for the firm to switch again to the green technology and decrease production even further until there is no pollution at all. Thus, on this particular solution path it is optimal to switch three times—but of course such a result is only obtainable if we assume that switching costs are low or non-existent. Note also that if green technology is less inefficient it would be optimal to switch once at most from brown to green early on. However, if brown technology causes less pollution one would switch from green to brown technology later and also at most once.



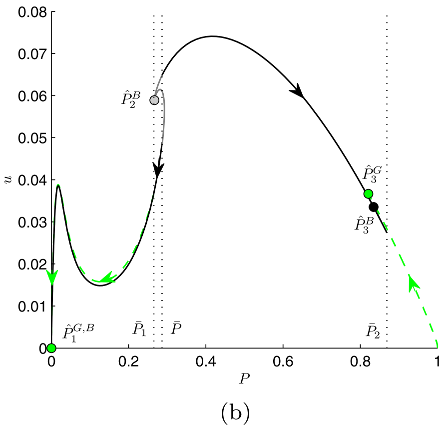


Fig. 16 Phase portrait for $r = 0.04$, $a_B = 1$, $a_G = 0.76$, $b = 0.9$, $c = 0.1429$, $n = 0.00047$, $\beta_B = 1$, $\beta_G = 0.85$, $\gamma_B = 1$, $\gamma_G = 0.945$ and (**a**) $S_{GB} = S_{BG} = 0$, (**b**) $S_{GB} = S_{BG} = 1$. The *black solid line* depicts green technology, the *gray dotted line* brown technology

Considering the case with high switching costs as can be seen in Fig. 16(b), we find that it is never optimal to switch from one stage to another. Yet, we find three Skiba points. At $\bar{P}$ a firm has the choice not between two technologies, as brown is always the best there, but between increasing production and ending up in a highly polluted environment or decreasing production and ending up in a clean environment. At $\bar{P}_1$ the firm is indifferent between brown and green technology, however, it is always optimal to reduce pollution and approach the clean steady state. For initial pollution levels smaller than $\bar{P}_1$ one would always use green, for intermediate initial levels between $\bar{P}_1$ and $\bar{P}$ one would always use brown technology. At the third indifference point $\bar{P}_2$ the firm can again choose between green and brown technology, but would always increase production and end up in a polluted environment. If the initial pollution is very high then the optimal strategy is to use green technology, if the initial level is below $\bar{P}_2$, brown technology is used.

6 Conclusion

We investigate a simple two-stage production-pollution model in which a representative firm has the choice to switch between a pollutive and an environmental friendly technology in order to produce its output and to simultaneously take care about its pollution level which affects the firm's reputation in the public. With this model we show that under some circumstances, there can exist solutions with a different number of switches leading into different long-run outcomes which are, however, very sensitive to the height of the switching costs.

To have a profound understanding about the occurring multiple equilibria, we always consider the one-state version of the model first, which shows that history-dependence plays an important role when it comes to the question whether to end up in a clean environment when confronted with a trade-off between production and pollution. In the present paper we consider different extensions of a model described in Forster (1977) and see that these mechanisms can lead to Skiba points.

If the damage of a firm due to negative reputation is best approximated by a concave-convex damage function, we see that the optimal strategy delicately depends on the initial pollution level. If pollution already is high enough the additional damage is not too severe and one would approach a steady state where production, but unfortunately also pollution is at a high level. For a small initial pollution level the best solution is to decrease production and end up in a clean environment.

When we consider a scenario where the cleaning rate depends on the pollution, we see that in contrast to the first case, one would produce most if the initial pollution is low as the natural cleaning works most efficient then. If the initial pollution level is high, the capacity of the environment to recover is already affected too much so that the high damage of the pollution only enables a low production level.

If both of these effects, a concave-convex damage and a pollution dependent cleaning rate occur, we see that the impact of the damage function dominates in a certain sense, as the optimal strategy resembles more the one where we only have this kind of damage function. However, the pollution dependent cleaning rate affects the optimal strategy for some, rather small initial state values. Instead of monotonously decreasing the production, one would exploit this cleaning effect and start to produce more at least for some time.

Back to the multi-stage approach it turns out that the decision which technology to use does not only depend on the assumptions concerning the damage function as well as the natural cleaning rate but also on the initial state values. If switching costs between the two technologies are low or even zero, it is optimal in certain scenarios to change the used technology even several times. Higher costs for implementing the other technology means that—again depending on the initial pollution level—a firm should either wait longer to do the change or abstain from it completely. We found indifference points where a firm can choose either between always sticking to one technology, and using one technology first and switching to another after some time. If the switching costs are high, it is of course never optimal to use another technology than the one that one started with. At certain initial state values a firm even has the choice between always using green and always using brown technology.

Summing up, we have investigated several scenarios in which the firm maximizes its profit subject to its polluting impact on the environment. However, the only incentive for this so far has been given by the disutility of the firm due to a bad reputation if it pollutes too much. Especially in the approach with the concave-convex damage function, we further on found out that the only possibility to have a lower pollution is given by zero production. This of course is a rather pessimistic outcome, but it underlines the urgency of policy intervention by the government to create more incentives in order to make green technology attractive. Therefore, we think about extension of the model for future work in which we include policy instruments like pollution restrictions (cf. Helfand 1991) or pollution permits (cf. Chevallier et al. 2011). Another possible instrument would be the taxation of pollution, like in Xepapadeas (1992). Considering the two different technologies, directed technical change driven by R&D efforts towards a cleaner production would be of special interest, like in Rauscher (2009) and Acemoglu et al. (2012). In particular, it would be interesting to consider more than two stages, i.e. not the switch between two entirely different technologies, but the gradual transit from a brown to a green technology. An important issue also would be to study the impact of the choice of technology on the reputation of the firm in more detail.

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