DIPLOMARBEIT

Untersuchung der Elektrodynamik im Solitonmodell

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Introduction

Solitons were famously first described by John Scott Russell in 1834 who observed solitary waves in the Union canal in Scotland. The explanation of this phenomenon however was only given in 1895 with help of the Korteweg-de-Vries equation[1] [Kd95].
Roughly speaking localized, nondispersive [2] solutions of a wave equation are called solitary waves. If in addition "interaction" of two solutions leaves them unchanged (except for a possible shift of the phase) after the interaction they are referred to as solitons [Gui04]. In physics the distinction is not always as exact and often the first condition alone is already sufficient to call them solitons.
In general solitons may occur as special solutions of nonlinear differential equations where nonlinear and nondispersive effects manage to cancel each other out such that a nondispersive solution exists [3].
Another large class of solitons, so called topological solitons, emerge from a general set of partial differential equations, not necessarily of the kind described above, that have solutions which are topological non-trivial configurations in their target space. Here the stability of the soliton is a direct consequence of their topological non-trivial nature often as a consequence of the boundary conditions.
In either case the solitonic solutions are in many cases characterized by a "topological charge", i.e., a quantity attributed to the particular solution which can only take discrete values and so makes a smooth decay of the solitonic solution impossible.

The localized and stable nature of solitons naturally leads to an interpretation as particles or quasi-particles. Examples of solitons occurs in different parts of physics amongst others in :

- the Korteweg-de -Vries (KdV) equation [Kd95, GGKM67] : used to describe shallow water waves, hydromagnetic waves in a cold plasma, ion acoustic waves .
- (self-dual) Yang-Mills theories where solitons occur as non trivial ground state configurations (in this setting the solitons are usually referred to as instantons) [Nak03, BPST75].
- 'tHooft - Polyakov monopoles.
- D-branes in string theories.
- Skyrmions as a model for baryons [Sky62].

[1] http://de.wikipedia.org/wiki/Soliton
[2] i.e. wave packets that do not change shape over time.
[3] A large class of models are so called integrable systems where the stability of the solitons is a consequence of the integrability of the equations of motion.

We note that, at least in the physics related literature we have reviewed, "resting" soliton solutions seem to prevail whereas the solution we aim to discuss are moving solitons.

Our discussion will turn to the model given in [Fab01]. The model introduced and discussed by M. Faber posses solitonic solutions that have properties that makes an interpretation as fermions, for example, electrons possible. The interesting characteristics of these solitons are their quantized electromagnetic charge and half-integer spin which both arise naturally as topological charges. The solutions are of finite size (i.e. not pointlike) and thereby have finite (self-) energy which gives rise to a finite mass. Further as a consequence of the non-linearity of the theory two solutions cannot be at the same space-time point, i.e., they behave like fermions. hence the name for the model, Model for Topological Fermions (MFT).
We will in particular study the limit where the "particle size" is sent to zero and one is far away from all charges. In this limit we want to look for solitonic solutions. Interpreting the solitons of the MFT as electrons, solutions in the limit we study may describe electromagnetic waves, i.e., photons.
This has to a certain extent been done and studied in [FBK07] however the solutions found there were not of solitonic character and so some of the nice properties of the original model are lost. We will be looking for solutions travelling with the speed of light and having a spin to energy relationship like the one found in electrodynamics (i.e. $\frac{\hbar}{\hbar\omega}$ for a photon of frequency ω). As we will see there is a possibility for topological solitons in this model.
For a fixed moment in time they will constitute topological nontrivial maps from the 3-sphere to the 2-sphere which are known to be characterized by their Hopf invariant.
Finally we would like to point out that the model studied here as the limit of zero particle size of the MFT also has a connection to YM(-Higgs) theories [Fer06] and might also be of interest in the study of the Skyrme model. Thereby the model studied can naturally be of possible interest for different fields in particle physics.

The work is built up as follows.

- In chapter 1 we give a brief review of the Model of topological fermions (MFT) and discuss the electrodynamic limit.
- In chapter 2 we discuss the electrodynamic limit of the model.
- Chapter 3 reviews conserved quantities in field theory in general and in the MFT in particular.
- In chapter 4 we finally come to the discussion of the solutions to the electromagnetic limit of the MFT. In particular we discuss one of our preferred solutions and also review the solutions given by Ferreira [Fer06].

The appendix includes the following

- In appendix A we briefly repeat the concepts of electrodynamics that are important for our discussions, in particular the properties of photons i.e. electromagnetic waves.
- In appendix B we introduce and briefly discuss the Hopf map and the Hopf invariant.

- In appendix C we list further examples and calculations concerning possible solutions which however either not gave the desired results or simply have not been pursued further due to a lack of time.

The notation and units used in this work can be found after the appendix.

Chapter 1

SOLTION MODEL - Overview

1.1 Model for a charged soliton

The model for topological fermions [1] (MTF) introduced in [Fab01] and further studied in [Fab12, FK02, FBK07] is based on an SU(2) field $Q : \mathbb{R}^1 \times \mathbb{R}^3 \to \mathrm{SU}(2)$ living in Minkowski space-time defined in the following way [2]

$$Q := q_0(x)\,\sigma_0 - i\,\vec{\sigma}\cdot\vec{q} = q_0(x)\,\sigma_0 - i\,\sigma_K q_K(x) = \begin{pmatrix} q_0 - iq_3 & -q_2 - iq_1 \\ q_2 - iq_1 & q_0 + iq_3 \end{pmatrix} \tag{1.1}$$

with $|Q^2| := q_0^2 + \vec{q}^2 = 1$ and σ_K being the Pauli spin matrices. Alternatively we may parameterize $(q_0, \vec{q})$ with help of an angle $\alpha \in [0, 2\pi)$ and a normalised vector $\vec{n}(x)$, $\vec{n}^2 = 1$ [3] through

$$q_0 = \cos\alpha(x) \quad , \quad q_K = n_k(x)\sin\alpha(x)\,. \tag{1.2}$$

We introduce the connection $\vec{\Gamma}_\mu$ through

$$\partial_\mu Q =: -i\left(\vec{\Gamma}_\mu\cdot\vec{\sigma}\right) Q \tag{1.3}$$

which implies

$$\begin{aligned} \vec{\Gamma}_\mu &= \frac{i}{2}\mathrm{Tr}(\vec{\sigma}\partial_\mu Q Q^\dagger) \\ &= \partial_\mu\alpha\,\vec{n} + \sin\alpha\,\cos\alpha\,\partial_\mu\vec{n} + \sin^2\alpha\,\vec{n}\times\partial_\mu\vec{n}. \end{aligned} \tag{1.4}$$

Further we define the curvature $\vec{R}^{\mu\nu}$ by

$$\vec{R}_{\mu\nu} := \vec{\Gamma}_\mu \wedge \vec{\Gamma}_\nu. \tag{1.5}$$

Since the Schwartz integrability condition for Q, $\partial_\mu\partial_\nu Q = \partial_\nu\partial_\mu Q$, implies the Maurer-Cartan relationship for $\vec{\Gamma}_\mu$

$$\partial_\mu\vec{\Gamma}_\nu - \partial_\nu\vec{\Gamma}_\mu = 2\,\vec{\Gamma}_\mu\times\vec{\Gamma}_\nu \tag{1.6}$$

[1] The reason why this model is associated with fermions will become clear in the further discussion below.

[2] We follow the conventions from [Fab12] which are different from but compatible with the ones in [Fab01, FK02, FBK07].

[3] The field $\vec{n}(x)$ is a three dimensional vector in the "internal color space", as we will see in chapter 3, $\vec{n}$ is however also supposed to rotate under spatial rotations of space-time.

we may express the curvature in the following alternative ways

$$\begin{aligned}\vec{R}_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu\vec{\Gamma}_\nu - \partial_\nu\vec{\Gamma}_\mu\right) \\ &= \partial_\mu\vec{\Gamma}_\nu - \partial_\nu\vec{\Gamma}_\mu - \vec{\Gamma}_\mu \times \vec{\Gamma}_\nu\,. \end{aligned} \tag{1.7}$$

The Lagrangian of the MTF which is defined by

$$\mathcal{L} = -\frac{\alpha_f \hbar c}{4\pi}\left(\frac{1}{4}\vec{R}_{\mu\nu}\cdot\vec{R}^{\mu\nu} + \Lambda(q_0)\right)\,, \tag{1.8}$$

where the potential is given by

$$\Lambda(q_0) := \frac{1}{r_0^4}\left(\frac{\mathrm{Tr}Q}{2}\right)^{2m} = \frac{1}{r_0^4}\cos^{2m}\alpha(x)\,,\ \ m = 1,2,3\ldots \tag{1.9}$$

We have introduced a dimensional parameter r_0 having the dimension of a length. So altogether the model contains the two parameters r_0 and the fine structure constant $\alpha_f = \frac{e_0^2}{4\pi\varepsilon_0\hbar c} \approx 1/137$. We note that contrary to the Skyrme model, which is known to describe short ranged baryons, no derivatives occur in the potential. As is discussed in [Fab01, Fab12] this is necessary in order to have particles with long distance interaction.

A relationship to standard electrodynamics is established by defining a vector (in internal space) of (dual) potentials through

$$\vec{C}_\mu = -\frac{e_0}{4\pi\varepsilon_0 c}\vec{\Gamma} \tag{1.10}$$

and a corresponding a vector of (dual) field strength tensors (in Minkowski space) through

$$^*\vec{F}_{\mu\nu} := \frac{1}{2}\partial_\mu\vec{C}_\nu - \partial_\nu\vec{C}_\mu = -\frac{e_0}{4\pi\varepsilon_0 c}\vec{R}_{\mu\nu} = \begin{pmatrix} 0 & \vec{B}_x & \vec{B}_y & \vec{B}_z \\ -\vec{B}_x & 0 & \vec{E}_z/c & -\vec{E}_y/c \\ -\vec{B}_y & -\vec{E}_z/c & 0 & \vec{E}_x/c \\ -\vec{B}_z & \vec{E}_y/c & -\vec{E}_x/c & 0 \end{pmatrix}. \tag{1.11}$$

We finally postulate the following relation between the projection of this dual-field strength vector onto $\vec{n}$ and the dual field strength $^*F_{\mu\nu}$ tensor from standard electrodynamics

$$^*F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix} = {}^*\vec{F}_{\mu\nu}\cdot\vec{n}. \tag{1.12}$$

Hence we may express the Lagrangian in terms of the electromagnetic fields in the following way

$$\mathcal{L} = -\frac{1}{4\mu_0}{}^*\vec{F}_{\mu\nu}{}^*\vec{F}^{\mu\nu} - \frac{\alpha_f\hbar c}{4\pi}\Lambda(q_0) = -\frac{1}{2\mu_0}\left(\frac{\vec{E}\vec{E}}{c^2} - \vec{B}\vec{B}\right) - \frac{\alpha_f\hbar c}{4\pi}\Lambda(q_0). \tag{1.13}$$

1.1.1 Time independent soliton solutions

We will briefly discuss the time independent soliton solutions of (1.8).
For time independent solutions we simply have to minimize the Hamiltonian of our theory. Choosing for $\vec{n}$ the (time-independent) hedgehog solution

$$\vec{n} = \frac{\vec{r}}{r} \tag{1.14}$$

we find the following expression for the Hamiltonian of the MFT

$$H = H_e + H_p = \frac{\alpha_f \hbar c}{r_0} \int_0^\infty d\rho \left[\frac{\sin^4 \alpha}{2\rho^2} + (\partial_\rho \cos\alpha)^2 + \rho^2 \cos^{2m} \alpha \right] . \tag{1.15}$$

where we have defined $\rho := \frac{r}{r_0}$. We note that the scaling behavior of H_e and H_p

$$r \to \lambda r \quad \Rightarrow \quad H_e \to \frac{1}{\lambda} H_e \,, H_p \to \lambda^3 H_p \tag{1.16}$$

are opposing each other , i.e., H_e exerts an "expanding force" while H_p exerts a contracting force on any static soliton solution, so that static soliton solutions become feasible .
Minimizing the Hamiltonian w.r.t. $\alpha(x)$ leads to the following second order differential equation for

$$\partial_\rho^2 \cos\alpha + \frac{(1-\cos^2\alpha)\,\cos\alpha}{\rho^2} - m\,\rho^2 \cos^{2m-1} \alpha = 0 \tag{1.17}$$

still depending on m, i.e., on the exact form of the potential we choose. In order for the solution to have finite energy we obtain the boundary condition

$$\alpha(r) \to \pi/2 \quad \text{for} \quad r \to \infty \,. \tag{1.18}$$

As an example we consider $m = 3$. In this case we obtain

$$\alpha(r) = \arctan\rho \quad , \quad \rho = \frac{r}{r_0} \tag{1.19}$$

as a solution. The energy for this solution is given by

$$H_1 = \frac{\alpha_f \hbar c}{r_0} \frac{\pi}{4} \quad , \quad \text{with } \alpha_f \hbar c = 1.44.\,\text{MeV fm}. \tag{1.20}$$

Comparing this to the electron energy of 0.511 MeV implies $r_0 = 2.21$ fm which is not too far from the classical electron radius $\frac{\alpha_f \hbar c}{m_e c^2} = 2.82$ fm .

1.2 Topological charges

This model allows to define a topological charge. Since Q maps $\mathbb{R}^3$ to $SU(2)$ or equivalently S^3 we may define the number of times the three dimensional sphere is covered by this map as a topological charge. It may be expressed through [Fab12]

$$\mathcal{Q} = \frac{1}{V(S^3)}\int_0^\infty dr \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \Gamma_r(\Gamma_\vartheta \times \Gamma_\varphi) \tag{1.21}$$

$$= \frac{1}{2\pi^2}\int_0^\infty dr \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \sin\vartheta\, \alpha'(r)\sin^2\alpha(r) \tag{1.22}$$

$$= \frac{2}{\pi}\int_{\alpha(0)}^{\alpha(\infty)} d\alpha\, \sin^2\alpha = \frac{1}{\pi}[\alpha(\infty) - \alpha(0)]^4. \tag{1.23}$$

In addition the electric charge carried by a soliton solution may in the standard way be expressed as the integral of the electric field over a surface surrounding the charge, i.e., soliton center [5]. That is we define

$$w_s := \frac{1}{4\pi}\oint_{S^2} d\vartheta d\varphi\, E_r \tag{1.26}$$

$$= \frac{1}{4\pi}\oint_{S^2} d\vartheta d\varphi\, \vec{n}[\partial_\vartheta\vec{n} \times \partial_\varphi\vec{n}] \tag{1.27}$$

which is just the spherical winding number of the map $\vec{n} : S^2 \to S^2$ and is known to take integer vales only for continuous fields $\vec{n}$.
The solution discussed above has topological charge

$$\mathcal{Q} = 1, w_s = 1. \tag{1.28}$$

The following connection to spin s and electric charge Q_{el} is therefore suggested

$$Q_{el} = -e_0\, w_s \quad , \quad s = |\mathcal{Q}|\,. \tag{1.29}$$

As is discussed in [Fab12,Fab01] with this definition of spin and charge it is necessary to identify Q field configurations related by the transformation $\vec{n} \to -\vec{n}$, $\alpha(r) \to \pi - \alpha(r)$ whereby the configuration space is effectively reduced to $SO(3)$ making it possible to interpret the Q-field as spatial rotations.

[4] Conservation of this charge can be seen by introducing the topological current

$$k^\mu = \frac{1}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}\vec{\Gamma}_\nu(\vec{\Gamma}_\rho \times \vec{\Gamma}_\sigma) \tag{1.24}$$

which is conserved and with the help of which $\mathcal{Q}$ may be written as

$$\mathcal{Q} = \int d^3x k^0(x) \tag{1.25}$$

see [Fab01].

[5] For multi-soliton solutions this is true if the surface is far enough away from all charges [Fab12].

Chapter 2

The electrodynamic limit

2.1 The MFT at big distance from pointlike sources

As we have seen above by solving the e.o.m. explicitly for a fixed potential (i.e., fixed m) or as one might deduct in general from the form of (1.15), we find that the parameter r_0 describes the size of the solutions in the MFT. When we take the limit $r_0 \to 0$ effectively two things happen:

1. The size of the solitons shrink to zero.
2. The potential energy term implies $\alpha(x) = \frac{\pi}{2} + n\,\pi \;\; (n \in \mathbb{N})$ (as it would diverge else), which is equivalent to $r \to \infty$.

So in this limit we are studying the field far away from its now pointlike sources in a theory without potential energy. We will especially be interested in electromagnetic waves. We proceed by studying the equations of motion and topological constraints in this limit.
The SU(2) field Q of equation (1.1) takes the form [Fab12]

$$Q(x) = -i\vec{\sigma}\vec{n}(x) \quad \text{at} \quad r \to \infty. \tag{2.1}$$

This implies that the connection and the curvature can be written as [1]

$$\vec{\Gamma}^{\mu} = \vec{n}(x) \wedge \partial^{\mu}\vec{n}(x), \tag{2.2}$$

$$\vec{R}^{\mu\nu} = \partial^{\mu}\vec{n}(x) \wedge \partial^{\nu}\vec{n}(x). \tag{2.3}$$

The dual electromagnetic field strength takes the form

$$^*f_{\mu\nu} = -\frac{e_0}{4\pi\varepsilon_0 c}[\partial_{\mu}\vec{n}(x) \wedge \partial_{\nu}\vec{n}(x)] \cdot \vec{n}(x). \tag{2.4}$$

The field strength is $f_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}{}^*f^{\rho\sigma}$.
Further the Lagrangian (1.8) reduces to [Fab12]

$$\mathcal{L}_{ED} = -\frac{\alpha_f \hbar c}{4\pi}\frac{1}{4}(\vec{n}\vec{R}_{\mu\nu})(\vec{n}\vec{R}^{\mu\nu}) = -\frac{1}{4\mu_0}{}^*f_{\mu\nu}(x){}^*f^{\mu\nu}(x). \tag{2.5}$$

[1] Notice that $\vec{\Gamma}^{\mu} \cdot \vec{n} = 0$.

The electric and magnetic field can be expressed through the $\vec{n}$ field and its derivatives in the following way [FBK07]

$$\mathbf{E} = -\frac{e_0}{4\pi\varepsilon_0}\begin{pmatrix} (\partial_y\vec{n}\wedge\partial_z\vec{n})\cdot\vec{n} \\ (\partial_z\vec{n}\wedge\partial_x\vec{n})\cdot\vec{n} \\ (\partial_x\vec{n}\wedge\partial_y\vec{n})\cdot\vec{n} \end{pmatrix} \quad , \quad \mathbf{B} = -\frac{1}{c^2}\frac{e_0}{4\pi\varepsilon_0}\begin{pmatrix} (\partial_t\vec{n}\wedge\partial_x\vec{n})\cdot\vec{n} \\ (\partial_t\vec{n}\wedge\partial_y\vec{n})\cdot\vec{n} \\ (\partial_t\vec{n}\wedge\partial_z\vec{n})\cdot\vec{n} \end{pmatrix}. \tag{2.6}$$

We may also write this with the help of $\vec{\Gamma}^\mu$ in the following form

$$\mathbf{E} = -\frac{e_0}{4\pi\varepsilon_0}\frac{1}{2}\left(\vec{n}\cdot\nabla\times\vec{\mathbf{\Gamma}}\right) \quad , \quad \mathbf{B} = -\frac{e_0}{4\pi\varepsilon_0 c}\frac{1}{2}\left(\vec{n}\cdot(\partial_0\vec{\mathbf{\Gamma}} - \nabla\vec{\Gamma}_0)\right). \tag{2.7}$$

To make further notation a bit simpler we define $\kappa := -\frac{e_0}{4\pi\varepsilon_0}$.
The equations of motion in this limit take the form [FBK07] $\partial^\mu\vec{n}\partial^\nu(\vec{R}_{\mu\nu}\cdot\vec{n}) = 0$ or more explicit

$$\partial^\mu\vec{n}\,\partial^\nu\{[\partial_\mu\vec{n}(x)\wedge\partial_\nu\vec{n}(x)]\cdot\vec{n}(x)\} = 0. \tag{2.8}$$

Due to the identity $\vec{n}\cdot\partial_\mu\vec{n} = 0$ (i.e., since $\vec{n}$ is a normalized vector) these are two independent equations only.
Using the definition of the electromagnetic field strength one may rewrite (2.8) in the following form [FBK07]

$$\partial^\mu\vec{n}\partial^{\nu *}f_{\mu\nu} = \partial^\mu\vec{n}\begin{pmatrix} \nabla\cdot\mathbf{B} \\ \frac{1}{c}(-\partial_t\mathbf{B} - \nabla\times\mathbf{E}) \end{pmatrix}_\mu = 0. \tag{2.9}$$

The quantity G^μ defined through

$$-\frac{e_0}{4\pi\epsilon_0}G^\mu := g^\mu := \begin{pmatrix} c\rho_{mag} \\ \vec{g} \end{pmatrix} := c\partial_\nu{}^* f^{\mu\nu} = \begin{pmatrix} c\nabla\cdot\mathbf{B} \\ \left(-\partial_t\mathbf{B} - \vec{\nabla}\times\mathbf{E}\right) \end{pmatrix} \tag{2.10}$$

can be viewed as a (conserved) magnetic current.
For topological reasons also the inhomogeneous Maxwell equations have to hold [FK02]. Since we assume to be far away from all sources we get

$$\nabla\cdot\mathbf{E} = 0 \quad , \quad c^2\nabla\times\mathbf{B} - \partial_t\mathbf{E} = 0. \tag{2.11}$$

Inserting (2.6) we may express these conditions through the $\vec{n}$ field

$$\begin{aligned} \nabla\cdot\mathbf{E} &= (\partial_y\vec{n}\wedge\partial_z\vec{n})\cdot\partial_x\vec{n} + (\partial_z\vec{n}\wedge\partial_x\vec{n})\cdot\partial_y\vec{n} + (\partial_x\vec{n}\wedge\partial_y\vec{n})\cdot\partial_z\vec{n} \\ &= 3(\partial_y\vec{n}\wedge\partial_z\vec{n})\cdot\partial_x\vec{n} = 0\,, \qquad (2.12) \\ \vec{\nabla}\times\mathbf{B} - \partial_t\mathbf{E} &= -3\begin{pmatrix} \partial_y\vec{n}\wedge\partial_z\vec{n} \\ \partial_z\vec{n}\wedge\partial_x\vec{n} \\ \partial_x\vec{n}\wedge\partial_y\vec{n} \end{pmatrix}\cdot\partial_t\vec{n} = 0\,, \qquad (2.13) \end{aligned}$$

where in the last formula the inner product is that of the internal space and is to be taken componentwise. Obviously these conditions are "trivially" fulfilled by the fact that $\vec{n}\in S^2$ which has a two dimensional tangent space.
Next we specialize the expressions for the energy (3.24) and the spin (3.29, 3.31) for the electrodynamic limit. The energy is given by $P^0 = \frac{1}{c}E$

$$\begin{aligned} E &= \int d^3x\,\Theta^{00} = \int d^3x\frac{\varepsilon_0}{2}[\mathbf{E}^2 + c^2\mathbf{B}^2] \\ &= \frac{\varepsilon_0\kappa^2}{2}\int d^3x\left(([\partial_y\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n})^2 + ([\partial_z\vec{n}\wedge\partial_x\vec{n}]\cdot\vec{n})^2 + ([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n})^2\right. \\ &+ \left.([\partial_t\vec{n}\wedge\partial_x\vec{n}]\cdot\vec{n})^2 + ([\partial_t\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n})^2 + ([\partial_t\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n})^2\right)\,, \qquad (2.14) \end{aligned}$$

and spin can be expressed as

$$\begin{aligned}
\vec{S} &= \frac{1}{c}\int d^3x\,\vec{\pi}^0 = -\frac{\alpha_f\hbar}{4\pi}\int d^3x(\vec{n}\cdot\vec{R}^{0\nu})\partial_\nu\vec{n} \\
&= +\frac{\kappa^2\varepsilon_0}{c}\int d^3x\,\frac{1}{c}\left(\left([\partial_t\vec{n}\wedge\partial_x\vec{n}]\cdot\vec{n}\right)\partial_x\vec{n} + \left([\partial_t\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n}\right)\partial_y\vec{n} + \left([\partial_t\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}\right)\partial_z\vec{n}\right) \\
&= \kappa\varepsilon_0\int d^3x\,\left(B_x\,\partial_x\vec{n} + B_y\,\partial_y\vec{n} + B_z\,\partial_z\vec{n}\right) = \kappa\varepsilon_0\int d^3x\,\mathbf{B}\cdot\nabla\vec{n}\,, \qquad (2.15)
\end{aligned}$$

which may be seen as a sum over the linear combination of the vectors $\partial_x\vec{n}, \partial_y\vec{n}, \partial_z\vec{n}$ weighted by the area they form with $\partial_t\vec{n}$, i.e., by the respective component of the magnetic field.

2.2 Waves in the the electrodynamic limit

2.2.1 General considerations

We want to look for solitonic solutions of the e.o.m. (2.8) for $\vec{n} : \mathbb{R}^1 \times \mathbb{R}^3 \to S^2$. In order for the energy to be finite we assume [2] $\vec{n}$ to converge to some constant value $\vec{n}_0$ at infinity, i.e.,

$$\vec{n} \to \vec{n}_0 \quad \text{for} \quad r := \sqrt{x^2+y^2+z^2} \to \infty\,. \qquad (2.16)$$

and also $\vec{n}(x, y, z, t \to \infty) = \vec{n}_0$.
Thereby we essentially compactify $\mathbb{R}^3$ to the 3-sphere S^3 and so obtain a map $\vec{n} : \mathbb{R}^1 \times S^3 \to S^2$. Since we no longer have a potential as in the original MFT Lagrangian and since the Lagrangian is not scale invariant, Derriks theorem eliminates the possibility of time independent solutions. Lorentz invariance of our theory therefore then implies that any solitonic solution has to “move” with the speed of light.
For some fixed time t_0, a solution $\vec{n}$ is a map from S^3 to S^2. In order to have a solitonic solution we expect $\vec{n}(t_0, \vec{x}) : S^3 \to S^2$ to be homotopically nontrivial. Maps of this kind can be characterized by their (non-vanishing) Hopf invariant.
We note that since we lose one dimension by going from S^3 to S^2 there should be a constant direction in every space time point, i.e., a line of constant $\vec{n}$ field passes through every space time point. Since the value of $\vec{n}$ is fixed for $r \to \infty$ not all of these lines can extend to spatial infinity, since their value would be fixed to the (constant) value at infinity and we would obtain a constant $\vec{n}$ field. So at least some of these lines need to be closed. Indeed something much stricter holds and the condition of $\vec{n}(t_0, \vec{x}) : S^3 \to S^2$ being homotopically nontrivial can be characterized through the lines of constant $\vec{n}$.
Given any two curves c_1, c_2

$$c_1 := \{\vec{x} \in \mathbb{R}^3 | \vec{n}(t_0, \vec{x}) = \vec{n}_1\} \quad , \quad c_2 := \{\vec{x} \in \mathbb{R}^3 | \vec{n}(t_0, \vec{x}) = \vec{n}_2\} \qquad (2.17)$$

of constant $\vec{n}$ field, $\vec{n}_1 \neq \vec{n}_2$, their linking number is exactly the Hopf invariant of the map $\vec{n}(t_0, \vec{x})$ [3] [Sch94, RB82].
As a typical example we might think of the Hopf map/ fibration which has Hopf invariant 1.

[2] Note that the form of (2.14) seems to leave open the possibility for different boundary conditions with finite energy as well. Take for example the case where $\vec{n}$ only depends on z at infinity. We will however not pursue this issue further here.

[3] Lines of constant $\vec{n}$ field might not need to be simply connected. Also they may be knotted. [Fad01]

2.2.2 $\vec{n}(x,y,z,t) = \vec{n}(\zeta(z,t), \eta(x,y,\zeta(z,t)))$

In [FBK07] solutions of the e.o.m. (2.8) of the form

$$\vec{n} = \vec{n}(\zeta(z,t), \eta(x,y,\zeta(z,t))) \tag{2.18}$$

were studied. This ansatz implies that $\partial_x \vec{n}$ and $\partial_y \vec{n}$ are parallel to each other and the same holds true for $\partial_t \vec{n}$ and $\partial_z \vec{n}$. Therefore the z-components of the electric and magnetic field vanish and as can be seen through (2.6). This is what we would expect from standard electrodynamics for waves moving in z-direction. More explicitly we get [FBK07]

$$\mathbf{E} = \kappa\, \vec{n} \cdot (\partial_\zeta \vec{n} \wedge \partial_\eta \vec{n})\, \partial_z \zeta \begin{pmatrix} -\partial_y \eta \\ \partial_x \eta \\ 0 \end{pmatrix} \;,\; c\,\mathbf{B} = \kappa\, \frac{1}{c}\, \vec{n} \cdot (\partial_\zeta \vec{n} \wedge \partial_\eta \vec{n})\, \partial_t \zeta \begin{pmatrix} \partial_x \eta \\ \partial_y \eta \\ 0 \end{pmatrix} .^{4} \tag{2.19}$$

We note that $\vec{E} \cdot \vec{B} = 0$ holds.
With this ansatz and using (1.11) we find that the curvature tensor takes the form

$$R_{\mu\nu} = (\vec{n} \cdot (\partial_\zeta \vec{n} \wedge \partial_\eta \vec{n})) \cdot \begin{pmatrix} 0 & \frac{1}{c}\partial_t\zeta\partial_x\eta & \frac{1}{c}\partial_t\zeta\partial_y\eta & 0 \\ -\frac{1}{c}\partial_t\zeta\partial_x\eta & 0 & 0 & -\partial_z\zeta\partial_x\eta \\ -\frac{1}{c}\partial_t\zeta\partial_y\eta & 0 & 0 & -\partial_z\zeta\partial_y\eta \\ 0 & \partial_z\zeta\partial_x\eta & \partial_z\zeta\partial_y\eta & 0 \end{pmatrix} . \tag{2.20}$$

Assuming that the factor in front can be made equal to 1 by appropriate choice of η and ζ (see [FBK07]) we would like to formulate the e.o.m. (2.8) for our ansatz. We define the two dimensional versions of the gradient and the laplace operator as

$$\nabla_{x,y} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \quad , \quad \Delta_{x,y} = \partial_x^2 + \partial_y^2 \,. \tag{2.21}$$

and start with calculating

$$-\partial^\nu R_{\nu\mu} = (\partial_0, -\partial_x, -\partial_y, -\partial_z) \cdot \begin{pmatrix} 0 & \frac{1}{c}\partial_t\zeta\partial_x\eta & \frac{1}{c}\partial_t\zeta\partial_y\eta & 0 \\ -\frac{1}{c}\partial_t\zeta\partial_x\eta & 0 & 0 & -\partial_z\zeta\partial_x\eta \\ -\frac{1}{c}\partial_t\zeta\partial_y\eta & 0 & 0 & -\partial_z\zeta\partial_y\eta \\ 0 & \partial_z\zeta\partial_x\eta & \partial_z\zeta\partial_y\eta & 0 \end{pmatrix} \tag{2.22}$$

$$= \begin{pmatrix} \partial_0\zeta\Delta_{xy}\eta \\ \partial_0(\partial_0\zeta\partial_x\eta) - \partial_z(\partial_z\zeta\partial_x\eta) \\ \partial_0(\partial_0\zeta\partial_y\eta) - \partial_z(\partial_z\zeta\partial_y\eta) \\ \partial_z\zeta\Delta_{xy}\eta \end{pmatrix}^T = \begin{pmatrix} \partial_0\zeta\Delta_{xy}\eta \\ (\partial_0^2 - \partial_z^2)\zeta\,\partial_x\eta + ((\partial_0\zeta)^2 - (\partial_z\zeta)^2)\partial_x\partial_\zeta\eta \\ (\partial_0^2 - \partial_z^2)\zeta\,\partial_y\eta + ((\partial_0\zeta)^2 - (\partial_z\zeta)^2)\partial_y\partial_\zeta\eta \\ \partial_z\zeta\Delta_{xy}\eta \end{pmatrix}^T$$

Now we continue with

$$\begin{aligned} &\partial^\nu R_{\nu\mu}\partial^\mu \vec{n} = \\ &= \begin{pmatrix} \partial_0\zeta\Delta_{xy}\eta \\ (\partial_0^2 - \partial_z^2)\zeta\,\partial_x\eta + ((\partial_0\zeta)^2 - (\partial_z\zeta)^2)\partial_x\partial_\zeta\eta \\ (\partial_0^2 - \partial_z^2)\zeta\,\partial_y\eta + ((\partial_0\zeta)^2 - (\partial_z\zeta)^2)\partial_y\partial_\zeta\eta \\ \partial_z\zeta\Delta_{xy}\eta \end{pmatrix}^T \cdot \left(\begin{pmatrix} \partial_0\zeta \\ 0 \\ 0 \\ -\partial_z\zeta \end{pmatrix} \partial_\zeta\vec{n} + \begin{pmatrix} \partial_\zeta\eta\partial_0\zeta \\ -\partial_x\eta \\ -\partial_y\eta \\ -\partial_\zeta\eta\partial_z\zeta \end{pmatrix} \partial_\eta\vec{n} \right) \\ &= \gamma\partial_\zeta\vec{n} + \lambda\partial_\eta\vec{n}, \end{aligned} \tag{2.23}$$

[4]It is sufficient to calculate the $\partial_\zeta \vec{n}$ derivative w.r.t. the explicit ζ dependence of the $\vec{n}$ field only since the dependence on ζ through η gives no contribution in the wedge product.

where

$$\gamma = \Delta_{xy}\eta\,((\partial_0\zeta)^2 - (\partial_z\zeta)^2) \qquad \text{and} \tag{2.24}$$

$$\lambda = \gamma\partial_\zeta\eta - \left[(\nabla_{xy}\eta)^2(\partial_0^2\zeta - \partial_z^2\zeta) + \frac{1}{2}((\partial_0\zeta)^2 - (\partial_z\zeta)^2)\partial_\zeta(\nabla_{xy}\eta)^2\right]. \tag{2.25}$$

Assuming that $\partial_\zeta\vec{n}$ and $\partial_\eta\vec{n}$ are linear independent [5] we finally arrive at the

$$0 = \Delta_{xy}\eta\,((\partial_0\zeta)^2 - (\partial_z\zeta)^2), \tag{2.26}$$

$$0 = 2(\nabla_{xy}\eta)^2(\partial_0^2\zeta - \partial_z^2\zeta) + ((\partial_0\zeta)^2 - (\partial_z\zeta)^2)\partial_\zeta(\nabla_{xy}\eta)^2 \tag{2.27}$$

which are the form the e.o.m. take for this ansatz and coincides with the formula (18) in [FBK07]. In particular $\vec{n}$ fields only depending on either $z_+ := z + ct$ or $z_- := z - ct$ are solutions to these equations. In [FBK07] it is shown that the set of solutions include linear polarized plane waves of the form

$$\mathbf{E} = \kappa\frac{k}{d}\sin kz_\pm(\sin\varepsilon, -\cos\varepsilon, 0))\,, \tag{2.28}$$

$$c\mathbf{B} = \kappa\frac{k}{d}\sin kz_\pm(\cos\varepsilon, \sin\varepsilon, 0) \tag{2.29}$$

where ε is a constant angle, as well as circular polarized waves of the form

$$\mathbf{E} = \frac{\kappa}{dl}(\sin kz_\pm, -\cos kz_\pm, 0))\,, \tag{2.30}$$

$$c\mathbf{B} = \pm\frac{\kappa}{dl}(\cos kz_\pm, \sin kz_\pm, 0) \tag{2.31}$$

where d and l are arbitrary length parameters. For this solution the constraint $-1 \le \eta \le 1$ implies that the wave exists only on a strip of width d see [FBK07].
Since $\vec{n}$ in this ansatz depends on x, y only through $\eta(x, y)$ there will be lines of constat $\vec{n}$ lying in the $x - y$ plane. Two such curves have zero linking number which implies that the $\vec{n}$ field arising from this ansatz has zero Hopf invariant (see appendix B).
Assuming the special dependence on $\vec{n}(x, y, z, t) = \vec{n}(\eta(x, y, \zeta(z - ct)), \zeta(z - ct))$ of the $\vec{n}$ field we may use (2.19) and (2.46) to give a more explicit expressions of the field energy (3.22 , 3.24) and the spin (3.31) of such a state

$$\begin{aligned} E = c\,P^0 &= \int d^3x\,\frac{\varepsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) \\ &= \kappa^2\varepsilon_0\int d^3x\,(\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n}))^2(\partial_z\zeta)^2((\partial_x\eta)^2 + (\partial_y\eta)^2) \end{aligned} \tag{2.32}$$

$$\begin{aligned} \vec{S} &= \frac{\kappa\,\varepsilon_0}{c}\int d^3x\,(E_x(\partial_y\vec{n}) - E_y(\partial_x\vec{n})) \\ &= -\kappa^2\,\varepsilon_0\int d^3x\,(\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n}))\partial_z\zeta((\partial_x\eta)^2 + (\partial_y\eta)^2)\,\partial_\eta\vec{n}. \end{aligned} \tag{2.33}$$

[5] We have already assumed $\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n}) = 1$ and at points where $\partial_\zeta\vec{n}$, $\partial_\eta\vec{n}$ are parallel to each other the electric and the magnetic field both vanish.

2.2.3 $\vec{n}(x,y,z,t) = \vec{n}(x,y,z-ct)$

We will assume that $\vec{n}$ depends on z,t only through $z-ct$, i.e., what should amount to a particle moving with the speed of light in positive z-direction. In particular this implies $\partial_t = -c\,\partial_z\vec{n}$ (equivalently $\partial_0\vec{n} = -\partial_z\vec{n}$).
In this situation the curvature tensor takes the form

$$\partial_\mu\vec{n}\wedge\partial_\nu\vec{n} = \begin{pmatrix} 0 & \partial_x\vec{n}\wedge\partial_z\vec{n} & \partial_y\vec{n}\wedge\partial_z\vec{n} & 0 \\ -\partial_x\vec{n}\wedge\partial_z\vec{n} & 0 & \partial_x\vec{n}\wedge\partial_y\vec{n} & \partial_x\vec{n}\wedge\partial_z\vec{n} \\ -\partial_y\vec{n}\wedge\partial_z\vec{n} & -\partial_x\vec{n}\wedge\partial_y\vec{n} & 0 & \partial_y\vec{n}\wedge\partial_z\vec{n} \\ 0 & -\partial_x\vec{n}\wedge\partial_z\vec{n} & -\partial_y\vec{n}\wedge\partial_z\vec{n} & 0 \end{pmatrix}. \tag{2.34}$$

From (2.6) we see that the magnetic field can be expressed through the components of the electric field in the following way $\vec{B} = \frac{1}{c}\kappa(-E_y, E_x, 0)$. This also implies that the Lagrangian density of the $\vec{n}$ field takes the form[6]

$$\mathcal{L} \propto ((\partial_x\vec{n}\wedge\partial_y\vec{n})\cdot\vec{n})^2 = E_z^2. \tag{2.35}$$

For the e.om. we first calculate

$$\partial^\nu([\partial_\nu\vec{n}\wedge\partial_\mu\vec{n}]\cdot\vec{n}) = \begin{pmatrix} -\partial_x([\partial_x\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) - \partial_y([\partial_y\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) \\ -\partial_y([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n}) \\ \partial_x([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n}) \\ \partial_x([\partial_x\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) + \partial_y([\partial_y\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) \end{pmatrix}^T, \tag{2.36}$$

which furthermore gives

$$\begin{aligned} \partial^\nu([\partial_\mu\vec{n}\wedge\partial_\nu\vec{n}])\partial^\mu\vec{n} &= \begin{pmatrix} -\partial_x([\partial_x\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) - \partial_y([\partial_y\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) \\ -\partial_y([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n}) \\ \partial_x([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n}) \\ \partial_x([\partial_x\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) + \partial_y([\partial_y\vec{n}\wedge\partial_z\vec{n}]\cdot\vec{n}) \end{pmatrix}^T \cdot \begin{pmatrix} -\partial_z\vec{n} \\ -\partial_x\vec{n} \\ -\partial_y\vec{n} \\ -\partial_z\vec{n} \end{pmatrix} \\ &= \partial_y([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n})\,\partial_x\vec{n} - \partial_x([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n})\partial_y\vec{n}. \end{aligned} \tag{2.37}$$

The e.o.m. state that this expression has to be zero everywhere. We may distinguish two cases:

1. In a point where $\partial_x\vec{n}$,$\partial_y\vec{n}$ are not parallel to each other the e.o.m. imply that

   $$\partial_x([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n}) = 0 = \partial_y([\partial_x\vec{n}\wedge\partial_y\vec{n}]\cdot\vec{n}), \tag{2.38}$$

   i.e. ,

   $$\nabla E_z = 0 \tag{2.39}$$

   that is that we are in a local extremum of the E_z in the x,y-plane. If this conditions hold in an open subset of $\mathbb{R}^3$ we find that E_z is only a function of z in such an open subset and in particular constant in a corresponding open subset of the x,y-plane with $z = const.$[7].
   Since $[\partial_x\vec{n}\wedge\partial_y\vec{n}]$ is parallel to $\vec{n}$ and $||\vec{n}|| = 1$ we have

   $$\partial_x\vec{n}\wedge\partial_y\vec{n} = \sin\theta\,||\partial_x\vec{n}||\,||\partial_y\vec{n}||\,\vec{n}\,, \tag{2.40}$$

   where θ is the angle between $\partial_x\vec{n}$ and $\partial_y\vec{n}$. So in this case we may further interpret the e.o.m. that the length of $[\partial_x\vec{n}\wedge\partial_y\vec{n}]$ (equivalently the size of the parallelogram spanned by the two vectors $\partial_x\vec{n}$, $\partial_y\vec{n}$) does not change in a plane with z =const..

[6] The Lagrangian density in terms of electric and magnetic field is proportional to $\mathcal{L} \propto (\vec{E}^2 - \vec{B}^2)$.
[7] These considerations might have to be modified in the situation where E_z is not smooth or differentiable

2. The other case consists of points where that $\partial_x\vec{n}$, $\partial_y\vec{n}$ are parallel to each other. In this case we have $E_z = 0$ and the e.o.m. tell us that

$$\partial_x E_z\, \partial_y\vec{n} - \partial_y E_z \partial_x\vec{n} = 0\,. \tag{2.41}$$

Considering all of $\mathbb{R}^3$ now we want to reflect upon which combination of these two cases are possible [8]. If we demand E_z to be a continuous (differentiable) function of x, y, z it seems that we are only able to have the situation $E_z = 0$.
If we allow E_z to have jumps of finite size (and hence its derivative to be proportional to a Kronecker-δ function) we may envision a situations with constant non-zero E_z field in some finite [9] region and zero everywhere outside that region.
The **inhomogeneous Maxwell equations** (2.12) reduce to the single equation

$$(\partial_x\vec{n} \wedge \partial_y\vec{n}) \cdot \partial_z\vec{n} = 0. \tag{2.42}$$

and is trivially fulfilled since the tangent space of S^2 is only two dimensional.

The electric and magnetic field are of the form

$$\mathbf{E} = \kappa \begin{pmatrix} (\partial_y\vec{n} \wedge \partial_z\vec{n}) \cdot \vec{n} \\ -(\partial_x\vec{n} \wedge \partial_z\vec{n}) \cdot \vec{n} \\ (\partial_x\vec{n} \wedge \partial_y\vec{n}) \cdot \vec{n} \end{pmatrix} \quad , \quad \mathbf{B} = \frac{\kappa}{c^2} \begin{pmatrix} (\partial_t\vec{n} \wedge \partial_t\vec{n}) \cdot \vec{n} \\ (\partial_t\vec{n} \wedge \partial_y\vec{n}) \cdot \vec{n} \\ (\partial_t\vec{n} \wedge \partial_z\vec{n}) \cdot \vec{n} \end{pmatrix} = \frac{1}{c} \begin{pmatrix} -E_x \\ E_y \\ 0 \end{pmatrix} \tag{2.43}$$

The energy takes the following form

$$\begin{aligned} E &= \varepsilon_0 \int d^3x (E_x^2 + E_y^2 + \frac{1}{2} E_z^2) \\ &= \varepsilon_0 \kappa^2 \int d^3x \left(([\partial_y\vec{n} \wedge \partial_z\vec{n}] \cdot \vec{n})^2 + ([\partial_z\vec{n} \wedge \partial_x\vec{n}] \cdot \vec{n})^2 + \frac{1}{2} ([\partial_x\vec{n} \wedge \partial_y\vec{n}] \cdot \vec{n})^2 \right) \end{aligned} \tag{2.44}$$

For $\vec{\pi}^\mu$ we find

$$\begin{aligned} \vec{\pi}^\mu &= -\frac{\alpha_f \hbar c}{4\pi} (\vec{n} \cdot \vec{R}^{\mu\nu}) \partial_\nu \vec{n} \\ &= -\frac{\alpha_f \hbar c}{4\pi}\, \vec{n} \cdot \begin{pmatrix} 0 & \partial_x\vec{n} \wedge \partial_z\vec{n} & \partial_y\vec{n} \wedge \partial_z\vec{n} & 0 \\ -\partial_x\vec{n} \wedge \partial_z\vec{n} & 0 & \partial_x\vec{n} \wedge \partial_y\vec{n} & \partial_x\vec{n} \wedge \partial_z\vec{n} \\ -\partial_y\vec{n} \wedge \partial_z\vec{n} & -\partial_x\vec{n} \wedge \partial_y\vec{n} & 0 & \partial_y\vec{n} \wedge \partial_z\vec{n} \\ 0 & -\partial_x\vec{n} \wedge \partial_z\vec{n} & -\partial_y\vec{n} \wedge \partial_z\vec{n} & 0 \end{pmatrix} \begin{pmatrix} -\partial_z\vec{n} \\ \partial_x\vec{n} \\ \partial_y\vec{n} \\ \partial_z\vec{n} \end{pmatrix} \\ &= -\frac{\alpha_f \hbar c}{4\pi} \begin{pmatrix} -\vec{n} \cdot (\partial_x\vec{n} \wedge \partial_z\vec{n}) \partial_x\vec{n} - \vec{n} \cdot (\partial_y\vec{n} \wedge \partial_z\vec{n}) \partial_y\vec{n} \\ \vec{n} \cdot (\partial_x\vec{n} \wedge \partial_y\vec{n}) \partial_y\vec{n} \\ -\vec{n} \cdot (\partial_x\vec{n} \wedge \partial_y\vec{n}) \partial_x\vec{n} \\ -\vec{n} \cdot (\partial_x\vec{n} \wedge \partial_z\vec{n}) \partial_x\vec{n} - \vec{n} \cdot (\partial_y\vec{n} \wedge \partial_z\vec{n}) \partial_y\vec{n} \end{pmatrix}. \end{aligned} \tag{2.45}$$

In particular

$$\begin{aligned} (\vec{\pi}^0)_i &= \frac{\alpha_f \hbar c}{4\pi} \left(\vec{n} \cdot (\partial_x\vec{n} \wedge \partial_z\vec{n}) (\partial_x\vec{n})_i + \vec{n} \cdot (\partial_y\vec{n} \wedge \partial_z\vec{n}) (\partial_y\vec{n})_i \right) \\ &= \kappa\, \varepsilon_0 (E_x (\partial_y\vec{n})_i - E_y (\partial_x\vec{n})_i). \end{aligned} \tag{2.46}$$

[8] We should keep in mind that it is not possible to have a nowhere vanishing vector field on S^2 so that we can never only have this second case but at least at some point $\partial_x\vec{n}$ will be zero and at some point $\partial_y\vec{n}$ will be zero. The behavior at such points may have to be studied in more detail.

[9] If we demand finite action and energy.

and

$$\vec{S} = \frac{\kappa\,\varepsilon_0}{c} \int d^3x \,(E_x(\partial_y \vec{n}) - E_y(\partial_x \vec{n})). \tag{2.47}$$

Chapter 3

Conserved Quantities

3.1 Symmetries and conserved quantities

For a general field theoretic Lagrangian

$$\mathcal{L} = \mathcal{L}(\Phi, \partial_\mu \Phi, x^\mu) \tag{3.1}$$

invariance under a symmetry transformations leads to the existence of conserved currents. For the special case of a invariance under spatial translations and rotations the respective currents are the energy momentum tensor and the tensor of angular momentum.

We first recall some important steps and formulas from [Fab12] Appendix G.4. We want to study the change in coordinates and fields under a passive coordinate transformation. Assuming some continuous symmetry of the lagrangian that is parameterized by α we write the change of the coordinates that arises from a small change $\delta\alpha$ in α in the form

$$x'^\mu(\delta\alpha) = x^\mu + \delta x^\mu(\delta\alpha) = x^\mu + \dot{x}^\mu \delta\alpha \,, \tag{3.2}$$

where the dot indicates the derivative w.r.t. $\delta\alpha$. The associated change of a field Φ is assumed to take the form

$$\delta'\Phi(x) := \Phi'(x') - \Phi(x) = D(\delta\alpha)\Phi(x) - \Phi(x) = \dot{D}\Phi(x)\delta\alpha \tag{3.3}$$

where $D(\delta\alpha)$ denotes the transformation matrix of the field in an appropriate representation. The change in the derivative of the field then becomes

$$\delta'\partial_\mu\Phi := \partial'_\mu\Phi'(x') - \partial_\mu\Phi(x) = \partial_\mu\delta'\Phi - \partial_\nu\partial_\mu\delta x^\nu. \tag{3.4}$$

This leads to the following conserved current (G.26)

$$j^\mu\delta\alpha = \left\{ \left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi}\partial_\nu\Phi - \mathcal{L}\delta^\mu_\nu \right) \dot{x}^\nu - \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi}\dot{D}\Phi \right\}\delta\alpha. \tag{3.5}$$

3.2 Energy Momentum and Angular Momentum Tensor

We start by considering invariance under translations in the x^λ direction

$$\delta\alpha = \delta x^\lambda \quad , \quad \dot{x}^\nu = \delta^\nu_\lambda \quad , \quad \dot{D} = 0. \tag{3.6}$$

This lead to the conserved **energy-momentum tensor**

$$\Theta^\mu{}_\lambda = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\lambda \Phi - \mathcal{L}\, \delta^\mu_\lambda. \tag{3.7}$$

Assuming that the fields Φ and their derivatives vanish fast enough at infinity the integral over the zero component yield the conserved energy momentum four vector

$$P^\mu = \frac{1}{c}\int d^3x\, \Theta^0_\nu g^{\nu\mu}\,. \tag{3.8}$$

Next we study the invariance under spatial rotations around the x^i axis

$$\begin{aligned} & x^{0\prime} = x^0\,,\ \vec{x}' = e^{\delta\Omega_i \varepsilon^i}\vec{x} = [1 + \delta\Omega\varepsilon_i + \mathcal{O}(\delta\Omega^2)]\vec{x}\,, \\ & \delta\alpha = \delta\Omega_i\,,\ \dot{x}^\nu = \delta^\nu_j\, \varepsilon_{ijk}\, x^k\,. \end{aligned} \tag{3.9}$$

Under these transformations the change in a three vector field $\Phi \to \vec{A}$ takes the form

$$(\dot{D}A)^j = (\partial_{\Omega_i} e^{\Omega_i \varepsilon_i}|_{\Omega=0})_{jk} A^k = \varepsilon_{ijk}\, A^k. \tag{3.10}$$

We split the conserved current associated to invariance under these transformations into an orbit part (associated with the transformation of the coordinates)

$$\left(\frac{\partial \mathcal{L}}{\partial\partial_\mu \Phi}\partial_\nu \Phi - \mathcal{L}\delta^\mu_\nu\right)\dot{x}^\nu = \Theta^\mu_\nu \delta^\nu_j \varepsilon_{ijk} x^k = -\Theta^{\mu j}\varepsilon_{ijk} x^k \tag{3.11}$$

and a spin part (associated with the transformation of the field)

$$-\frac{\partial \mathcal{L}}{\partial\partial_\mu \Phi}\dot{D}\Phi = -\frac{\partial \mathcal{L}}{\partial\partial_\mu \vec{A}^j}(\dot{D}\vec{A})^j = \frac{\partial \mathcal{L}}{\partial\partial_\mu A^j}\varepsilon_{ijk} A^k\,. \tag{3.12}$$

Again the spatial integrals of the zero components give rise to the conserved orbital angular momentum

$$L_i = \frac{1}{c}\int d^3x \varepsilon_{ijk} x^j \Theta^{0k} \tag{3.13}$$

and the conserved spin

$$S_i = -\frac{1}{c}\int d^3x (\frac{\partial \mathcal{L}}{\partial\partial_0 A^j}\varepsilon_{ijk} A^k). \tag{3.14}$$

The sum of these two two quantities is the conserved angular momentum $\mathbf{M} = \mathbf{L} + \mathbf{S}$.
As a consistency check we calculate if the dimensions of these expressions are correct

$$\begin{aligned} [L_i] &= [\frac{1}{c}][\int d^3x \Theta^{0k}]\varepsilon_{ijk}[x^j] && (3.15) \\ &= \frac{[t]}{[l]}[\text{Energy}][l] = [\text{action}] = [\text{angular momentum}]\,, \\ [S_i] &= \frac{[t]}{[l]}[\text{Energy}][l]\varepsilon_{ijk}\frac{[A^k]}{[A^j]} = [\text{action}] = [\text{angular momentum}]\,. && (3.16) \end{aligned}$$

3.3 Conserved quantities in the MFT

We will be interested in the expressions for conserved energy and the conserved spin part of the angular momentum in the setting of the model of topological fermions.
We will specify the expressions introduced above for the model of topological fermions.
We start by noting that from (1.3) it follows that [Fab12]

$$\frac{\partial \vec{\Gamma}_\lambda}{\partial \partial_\mu \alpha_K} \partial_\nu \alpha_K = \delta_{\lambda\mu} \vec{\Gamma}_\nu. \tag{3.17}$$

This immediately yields the useful relation

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \alpha_K} \partial_\nu \alpha_K = \frac{\partial \mathcal{L}}{\partial \Gamma_{\lambda L}} \frac{\partial \Gamma_{\lambda L}}{\partial \partial_\mu \alpha_K} \partial_\nu \alpha_K = \frac{\partial \mathcal{L}}{\partial \Gamma_{\mu K}} \Gamma_{\nu K}. \tag{3.18}$$

The canonical energy momentum tensor can now be written in the following form

$$\Theta^\mu_\nu(x) = \frac{\partial \mathcal{L}(x)}{\partial(\partial_\mu \alpha_k)} \partial_\nu \alpha_k - \mathcal{L}(x)\delta^\mu_\nu = \frac{\partial \mathcal{L}(x)}{\partial \vec{\Gamma}_\mu} \vec{\Gamma}_\nu - \mathcal{L}(x)\delta^\mu_\nu. \tag{3.19}$$

We define the generalized momentum density

$$\pi^\mu := \frac{\partial \mathcal{L}}{\partial \vec{\Gamma}_\mu} = -\frac{\alpha_f \hbar c}{4\pi} \vec{\Gamma}_\nu \times \vec{R}^{\mu\nu}. \tag{3.20}$$

From this and the explicit form of the lagrangian (1.8) we get

$$\Theta^\mu{}_\nu = -\frac{\alpha_f \hbar c}{4\pi} \left\{ \left(\vec{\Gamma}_\nu \times \vec{\Gamma}_\sigma \right) \left(\vec{\Gamma}^\mu \times \vec{\Gamma}^\sigma \right) \right\} - \mathcal{L}(x)\delta^\mu_\nu. \tag{3.21}$$

The conserved quantity is the four-momentum

$$P^\mu = \frac{1}{c} \int \Theta^{0\mu} \mathrm{d}^3 x \tag{3.22}$$

where the integration has to be taken over a volume-element where no quantities are flowing in or out.
We can further express $\Theta^\mu{}_\nu$ in terms of the electric and magnetic field strength ($\vec{E}$ and $\vec{B}$) in the following form

$$\Theta^\mu{}_\nu = \frac{1}{\mu_0} \left({}^*\vec{F}_{\nu\sigma} \cdot {}^*\vec{F}^{\sigma\mu} + \frac{1}{4} \delta^{\mu *}_\nu \vec{F}_{\rho\lambda} \cdot {}^*\vec{F}^{\rho\lambda} \right) + \mathcal{H}_p. \tag{3.23}$$

Its components are, with $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$:

$$\begin{aligned}
\Theta^{00} &= \mathcal{H} = \frac{1}{2}\left[\mathbf{ED} + \mathbf{BH}\right] + \mathcal{H}_p = \frac{\varepsilon_0}{2}\left[\mathbf{E}^2 + c^2\mathbf{B}^2\right] + \mathcal{H}_p, && (3.24)\\
\Theta^{i0} &= \Theta^{0i} = c\varepsilon_0 \epsilon_{ijk} \mathbf{E}_j \mathbf{B}_k, && (3.25)\\
\Theta^{ij} &= \left[\frac{1}{2}\mathbf{ED} + \frac{1}{2}\mathbf{BH} - \mathcal{H}_p\right]\delta_{ij} - \mathbf{E}_i\mathbf{D}_j - \mathbf{B}_i\mathbf{H}_j \\
&= \left[\frac{\varepsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) - \mathcal{H}_p\right]\delta_{ij} - \varepsilon_0(\mathbf{E}_i\mathbf{E}_j + c^2\mathbf{B}_i\mathbf{B}_j) && (3.26)
\end{aligned}$$

Now we turn to the angular momentum. Under the rotation

$$x^{0\prime} = x^0 \; , \; \vec{x}' = [1 + \delta\Omega_i \varepsilon_i + \mathcal{O}(\delta\Omega^2)]\vec{x} \tag{3.27}$$

i.e. $\dot{x}^\nu = \delta^\nu_j \varepsilon_{ijk} x^k$ we assume that $\Phi \to \vec{\alpha} := \alpha\,\vec{n}$ transforms like a 3-vector, i.e.,

$$(\dot{D}\alpha)_j = \varepsilon_{ijk}\,\alpha_k. \tag{3.28}$$

Again we decompose the total angular momentum **M** into an orbital part **L** and a spin part **S**. We will mostly be interested in the spin part which can be can be written in the following way

$$S_i = -\frac{1}{c}\int d^3x \vec{\pi}^0 \sin\alpha[\cos\alpha \vec{n} \times \vec{e}_i + \sin\alpha \vec{n} \times (\vec{n} \times \vec{e}_i)], \tag{3.29}$$

e_i being the i-th canonical basis vector. Note that if we had assumed $\vec{n}$ to be a vector not sensitive to space-time rotations we would not obtain a spin component. Non vanishing spin thereby implies the transformation property of $\vec{n}$.
In the **electrodynamic limit** we get

$$\begin{aligned} \vec{\pi}^\mu = \frac{\partial \mathcal{L}_{ED}}{\partial \vec{\Gamma}_\mu} &= \left(-\frac{\alpha_f \hbar c}{4\pi}\left(\vec{n} \times \vec{\Gamma}_\nu\right)\left(\vec{n} \cdot \vec{R}^{\nu\mu}\right)\right) \\ &= -\frac{\alpha_f \hbar c}{4\pi}\vec{\Gamma}_\nu \times \vec{R}^{\mu\nu} \\ &= \frac{\alpha_f \hbar c}{4\pi}(\vec{n} \cdot (\partial^\mu \vec{n} \times \partial^\nu \vec{n}))\partial_\nu \vec{n} \\ &= -\frac{\alpha_f \hbar c}{4\pi}(\vec{n} \cdot \vec{R}^{\mu\nu})\partial_\nu \vec{n} \end{aligned} \tag{3.30}$$

which is conserved (i.e., $\partial_\mu \pi^\mu = 0$) as a consequence of the e.o.m. and $\vec{\pi}^\mu \cdot \vec{n} = 0$ (since $\vec{n} \cdot \partial_\mu \vec{n} = 0$). Further for the spin part of the angular momentum

$$\begin{aligned} S_i &= -\frac{1}{c}\int d^3x \vec{\pi}^0 \cdot \vec{n} \times (\vec{n} \times \vec{e}_i) \\ &= -\frac{1}{c}\int d^3x (\vec{\pi}^0 \cdot \vec{n})\vec{n}_i - (\vec{\pi}^0 \cdot \vec{e}_i) \\ &= \frac{1}{c}\int d^3x (\pi^0_i). \end{aligned} \tag{3.31}$$

Chapter 4

Solutions and Examples

We now turn to finding soliton solutions (2.8). We start by discussing a particular ansatz that seems to lead to promising results. Thereafter we briefly comment on the solutions found in [Fer06].

4.1 Cylindrical coordinates

For our ansatz we start by introducing cylindrical coordinates in the xy - plane in the standard way

$$\begin{aligned} x = r\cos\varphi\ ,\ y = r\sin\varphi\ , \\ r = (x^2+y^2)^{\frac{1}{2}}, \varphi = \arctan\frac{y}{x}. \end{aligned} \tag{4.1}$$

We look for solutions of (2.8) in these coordinates moving in z direction with speed of light, i.e., of the form $\vec{n}(x,y,z,t) = \vec{n}(r,\varphi,z-ct)$.
We start by transforming the e.o.m. into cylindrical coordinates. The derivatives of the new coordinates w.r.t. the old ones are

$$\partial_x r = \frac{x}{r}\ ,\ \partial_x\varphi = -\frac{y}{r^2}\ ,\ \partial_y\varphi = \frac{x}{r^2}\ . \tag{4.2}$$

The x and y derivatives of the $\vec{n}$ field transform as

$$\partial_x\vec{n} = \partial_r\vec{n}\,\frac{x}{r} - \partial_\varphi\vec{n}\,\frac{y}{r^2}\ , \tag{4.3}$$

$$\partial_y\vec{n} = \partial_r\vec{n}\,\frac{y}{r} + \partial_\varphi\vec{n}\,\frac{x}{r^2}. \tag{4.4}$$

Thereby we find

$$\begin{aligned} (\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n} &= (\partial_r\vec{n}\times\partial_\varphi\vec{n})\cdot\vec{n}\,\frac{1}{r} \\ &=: f(r,\varphi,z)\,\frac{1}{r}\ . \end{aligned} \tag{4.5}$$

Next we transform

$$\partial_x((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) = \partial_x f(r,\varphi,z)\,\frac{1}{r} - f(r,\varphi,z)\,\frac{x}{r^3}$$

$$= (\partial_r f\,\frac{x}{r} - \partial_\varphi f\,\frac{y}{r^2})\frac{1}{r} - f(r,\varphi,z)\frac{x}{r^3}\,, \tag{4.6}$$

$$\partial_y((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) = (\partial_r f\,\frac{y}{r} + \partial_\varphi f\,\frac{x}{r^2})\frac{1}{r} - f(r,\varphi)\frac{y}{r^3} \tag{4.7}$$

and further

$$\begin{aligned}
& \partial_x((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n})\,\partial_y\vec{n} - \partial_y((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n})\,\partial_x\vec{n} \\
= \; & \partial_x((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n})\,(\partial_r\vec{n}\,\frac{y}{r} + \partial_\varphi\vec{n}\,\frac{x}{r^2}) - \partial_y((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n})\,(\partial_r\vec{n}\,\frac{x}{r} - \partial_\varphi\vec{n}\,\frac{y}{r^2}) \\
= \; & \partial_r\vec{n}\,[\frac{y}{r}\,\partial_x((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) - \frac{x}{r}\,\partial_y((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n})] \\
+ \; & \partial_\varphi\vec{n}\,[\frac{x}{r^2}\,\partial_x((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) + \frac{y}{r^2}\,\partial_y((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n})].
\end{aligned} \tag{4.8}$$

The square brackets in above expression are evaluated to give

$$\begin{aligned}
& \frac{y}{r}\,\partial_x((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) - \frac{x}{r}\,\partial_y((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) \\
= \; & \frac{y}{r}\,\left((\partial_r f\,\frac{x}{r} - \partial_\varphi f\,\frac{y}{r^2})\frac{1}{r} - f(r,\varphi,z)\frac{x}{r^3}\right) - \frac{x}{r}\,\left((\partial_r f\,\frac{y}{r} + \partial_\varphi f\,\frac{x}{r^2})\frac{1}{r} - f(r,\varphi)\frac{y}{r^3}\right) \\
= \; & (-1)\left[\partial_r f(\frac{xy}{r^3} - \frac{xy}{r^3}) - \partial_\varphi f(\frac{y^2}{r^4} + \frac{x^2}{r^4}) + f\,(\frac{xy}{r^4} - \frac{xy}{r^4})\right] \\
= \; & -\partial_\varphi f\frac{1}{r^2}
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
& \frac{x}{r^2}\,\partial_x((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) + \frac{y}{r^2}\,\partial_y((\partial_x\vec{n}\times\partial_y\vec{n})\cdot\vec{n}) \\
= \; & \frac{x}{r^2}\,\left((\partial_r f\,\frac{x}{r} - \partial_\varphi f\,\frac{y}{r^2})\frac{1}{r} - f(r,\varphi,z)\frac{x}{r^3}\right) + \frac{y}{r^2}\,\left((\partial_r f\,\frac{y}{r} + \partial_\varphi f\,\frac{x}{r^2})\frac{1}{r} - f(r,\varphi)\frac{y}{r^3}\right) \\
= \; & \left[\partial_r f(\frac{x^2}{r^4} + \frac{y^2}{r^4}) + \partial_\varphi f(\frac{x\,y}{r^5} - \frac{x\,y}{r^5}) - f(\frac{x^2}{r^5} + \frac{y^2}{r^5})\right] \\
= \; & [\partial_r f\,\frac{1}{r^2} - f\frac{1}{r^3}]
\end{aligned} \tag{4.10}$$

Inserting this into (2.8) we find the following expressions for the e.o.m in cylindrical coordinates

$$\left[(\partial_\varphi f\,\frac{1}{r^2})\,\partial_r\vec{n} - (\partial_r f\frac{1}{r^2} - f\,\frac{1}{r^3})\partial_\varphi\vec{n}\right] = 0 \tag{4.11}$$

or after inserting the expression for f

$$\left[\big(\frac{1}{r^2}\,\partial_\varphi[(\partial_r\vec{n}\times\partial_\varphi\vec{n})\cdot\vec{n}]\big)\,\partial_r\vec{n} - \big(\frac{1}{r^2}\,\partial_r[(\partial_r\vec{n}\times\partial_\varphi\vec{n})\cdot\vec{n}] - \frac{1}{r^3}\,[(\partial_r\vec{n}\times\partial_\varphi\vec{n})\cdot\vec{n}]\big)\partial_\varphi\vec{n}\right] = 0\,.$$

Using $E_z = f(r,\varphi,z-ct)\,\frac{\kappa}{r}$ we may finally rewrite this as

$$\frac{1}{r}\,[(\partial_\varphi E_z)\,\partial_r\vec{n} - (\partial_r E_z)\partial_\varphi\vec{n}] = 0 \tag{4.12}$$

which indeed bears close resemblance to the e.o.m. in standard cartesian coordinates.
In points where $\partial_r \vec{n}\,,\, \partial_\varphi \vec{n}$ are linear independent this amounts to

$$(\partial_\varphi E_z) = 0 \text{ and } (\partial_r E_z) = 0. \tag{4.13}$$

Points where $\partial_r \vec{n}\,,\, \partial_\varphi \vec{n}$ are linear dependent need to be studied independently. Note however that E_z vanishes in such points. A special case of the last situation is the case where either $\partial_r \vec{n}$ or $\partial_\varphi \vec{n}$ (or both) are zero.
In a situation where we impose E_z only to depend on z a possible solution is given by $f = g(z) r$ hence $E_z = \kappa\,(\partial_x \vec{n} \times \partial_y \vec{n}) \cdot \vec{n} = f(r,\varphi)\,\frac{\kappa}{r} = \kappa\, g(z)$.

4.2 An ansatz

We now come to a special ansatz for solving (4.12) displayed in fig. 4.1 . We construct our $\vec{n}$ field by intersecting S^2 with a plane in every space time point and imagine the $\vec{n}$ field to point to somewhere along the resulting intersection circle.
More specifically we imagine the $S^2 \subset \mathbb{R}^3$ to have its center in the origin. We start by laying a plane tangential to the sphere at the north-pole (i.e., where the positive z-axis intersects the sphere). Now we rotate this plane by Ω around one of the straight lines that lie in it and pass through the north pole and whose projection to the xy-plane is the image of the y-axis under a counterclockwise rotation by χ. The intersection of this rotated plane with the sphere is a circle with center

$$\vec{M} = \begin{pmatrix} 0 \\ 0 \\ \cos^2\Omega \end{pmatrix} + \sin\Omega\cos\Omega \begin{pmatrix} \cos\chi \\ \sin\chi \\ 0 \end{pmatrix}. \tag{4.14}$$

Naturally Ω may vary from 0 to π and χ may take values between 0 and 2π.
The radius vector (from the center to a point on the intersection circle) lies in the plane spanned by the two vectors

$$\vec{b}_1 = \sin\Omega \begin{pmatrix} -\cos\chi\cos\Omega \\ -\sin\chi\cos\Omega \\ \sin\Omega \end{pmatrix} \quad , \quad \vec{b}_2 = \sin\Omega \begin{pmatrix} \sin\chi \\ -\cos\chi \\ 0 \end{pmatrix}, \tag{4.15}$$

$$\vec{b}_1 \cdot \vec{M} = \vec{b}_1 \cdot \vec{M} = 0 \quad , \quad \vec{b}_1 \cdot \vec{b}_2 = 0 \quad , \quad ||\vec{b}_i|| = \sin\Omega \;,\; ||\vec{M}|| = \cos\Omega.$$

and may be parameterized by the angle ψ, $0 \leq \psi < 2\pi$, (see figure 4.1) in the following way

$$\vec{\rho} = (\cos\psi\,\vec{b}_1 + \sin\psi\,\vec{b}_2) \tag{4.16}$$

$$= \sin\Omega(\,\cos\psi \begin{pmatrix} -\cos\chi\cos\Omega \\ -\sin\chi\cos\Omega \\ \sin\Omega \end{pmatrix} + \sin\psi \begin{pmatrix} \sin\chi \\ -\cos\chi \\ 0 \end{pmatrix}). \tag{4.17}$$

For a fixed Ω the circle and hence our $\vec{n}$ field is defined by

$$\vec{n} = \vec{M} + \vec{\rho} \tag{4.18}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 - \sin^2\Omega\,(1-\cos\psi) \end{pmatrix} + \sin\Omega\,\cos\Omega\,(1-\cos\psi) \begin{pmatrix} \cos\chi \\ \sin\chi \\ 0 \end{pmatrix} + \sin\Omega\,\sin\psi \begin{pmatrix} \sin\chi \\ -\cos\chi \\ 0 \end{pmatrix}.$$

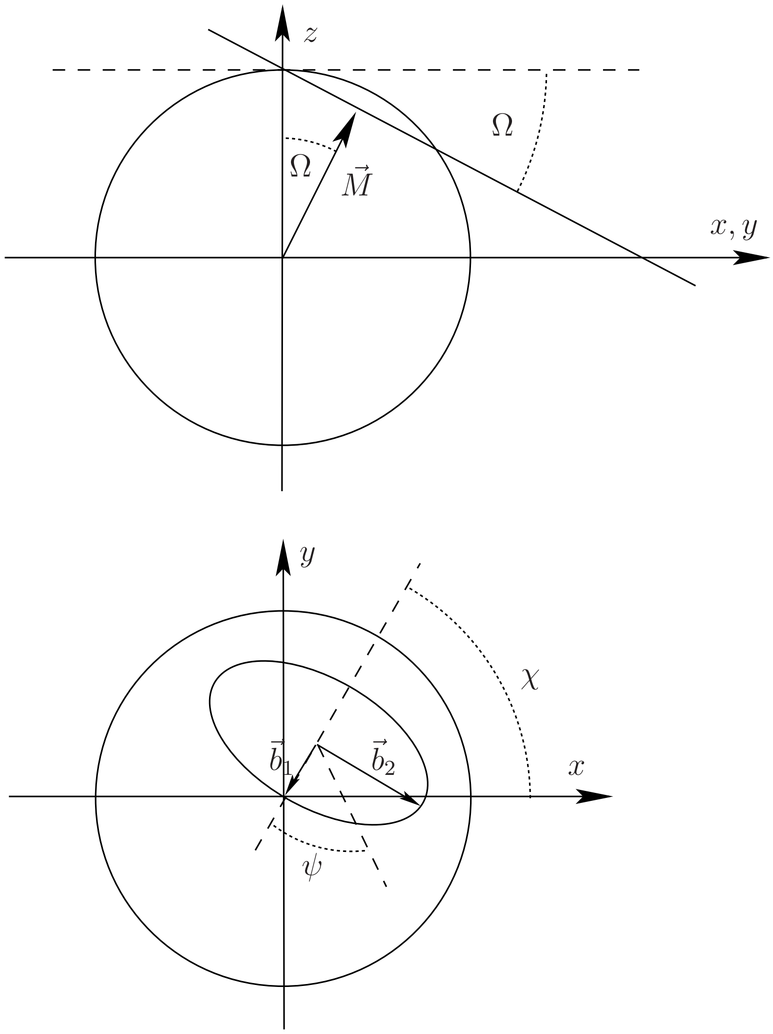

Figure 4.1: Intersecting S^2 with a plane

For our ansatz we will assume the following dependence on $r, \varphi, z - ct$

$$\Omega = \Omega(r, z)\,,\ \psi = \psi(r, z)\,,\ \chi = \chi(z, r, \varphi). \tag{4.19}$$

As boundary conditions for our configuration we choose

$$\Omega(z_0) = 0\,,\ \Omega(z_1) = \frac{\pi}{2},\ z_0 \leq z_1 \tag{4.20}$$
$$\psi(0, z) = 0 \text{ or } \pi\,,\ \psi(r \to \infty, z) = 0 \text{ or } \pi \text{ or } 2\pi \tag{4.21}$$
$$\chi(z_0, r, \varphi) = 0,\ \chi(z_1, r, \varphi) = \pi \text{ or } 2\pi, \tag{4.22}$$

Useful vector products for further calculations are

$$\partial_\Omega \vec{n} \wedge \partial_\chi \vec{n} = 2\,\sin\Omega\,\cos\Omega\,(1 - \cos\psi)\,\vec{n} \tag{4.23}$$
$$\partial_\psi \vec{n} \wedge \partial_\chi \vec{n} = \sin^2\Omega\,\sin\psi\,\vec{n} \tag{4.24}$$
$$\partial_\psi \vec{n} \wedge \partial_\Omega \vec{n} = (-1)\,\sin\Omega\,(1 - \cos\psi)\,\vec{n}\,. \tag{4.25}$$

Thereby we find

$$\begin{aligned}
(\partial_r\vec{n}\wedge\partial_\varphi\vec{n})\cdot\vec{n} &= [(\partial_r\Omega\,\partial_\Omega\vec{n}+\,\partial_r\psi\,\partial_\psi\vec{n})\wedge(\partial_\varphi\chi\,\partial_\chi\vec{n})]\cdot\vec{n} \\
&= \partial_\varphi\chi(\partial_r\Omega\,2\,\sin\Omega\,\cos\Omega\,(1-\cos\psi)+\partial_r\psi\,\sin^2\Omega\,\sin\psi)) \\
&= \partial_\varphi\chi((\partial_r\sin^2\Omega)(1-\cos\psi)+\sin^2\Omega\partial_r(1-\cos\psi)) \\
&= \partial_\varphi\chi\partial_r(\sin^2\Omega(1-\cos\psi)) \\
&= f(r,z,\varphi).
\end{aligned} \tag{4.26}$$

In the case where we assume χ to depend on φ linearly we see that $f(r,z,\varphi)$ and so E_z is independent of φ. This in particular implies that the e.o.m. take the form

$$\frac{1}{r}\,(\partial_r E_z)\partial_\varphi\vec{n}=0 \tag{4.27}$$

In the following we will study this ansatz in the situations where we either have or do not have a dependence on the cylinder coordinate angle φ. Also we will consider the two situations where Ω is allowed to vary or where $\Omega\equiv\pi/2$.

4.3 No φ dependence

In a first step we will consider the case where $\vec{n}$ does not have any φ dependence at all. In this situation the e.o.m. are (trivially) fulfilled since $\partial_\varphi\vec{n}$ (and so E_z) vanishes. However from our discussion of the Hopf invariant we do not expect solitonic solutions in this approach.

4.3.1 No φ dependence and variable Ω

Without φ dependence our field is of the form $\vec{n}=\vec{n}(\eta,\zeta)$ (see section 2.2.2) with $\eta=r$ and $\zeta=z-ct$. In this case we can use equation (2.19) to calculate the electric field.
We start by calculating

$$\begin{aligned}
\vec{n}(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n}) &= \vec{n}(\partial_z\vec{n}\wedge\partial_r\vec{n}) = \\
&= (\partial_z\Omega\,\partial_r\psi-\partial_z\psi\,\partial_r\Omega)(\partial_\Omega\vec{n}\wedge\partial_\psi\vec{n}) \\
&+ (\partial_z\Omega\,\partial_r\chi-\partial_z\chi\,\partial_r\Omega)(\partial_\Omega\vec{n}\wedge\partial_\chi\vec{n}) \\
&+ (\partial_z\psi\,\partial_r\chi-\partial_z\chi\,\partial_r\psi)(\partial_\psi\vec{n}\wedge\partial_\chi\vec{n}) \\
&= (\partial_z\Omega\,\partial_r\psi-\partial_z\psi\,\partial_r\Omega)\,\sin\Omega\,(1-\cos\psi) \\
&+ (\partial_z\Omega\,\partial_r\chi-\partial_z\chi\,\partial_r\Omega)\,2\,\sin\Omega\,\cos\Omega\,(1-\cos\psi) \\
&+ (\partial_z\psi\,\partial_r\chi-\partial_z\chi\,\partial_r\psi)\;\sin^2\Omega\,\sin\psi\,.
\end{aligned} \tag{4.28}$$

Specializing this expression for the situation where $\psi=\psi(r)$ we obtain

$$\begin{aligned}
\vec{n}(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n}) &= \partial_z\Omega\sin\Omega\,(1-\cos\psi)(\,\partial_r\psi\,+2\,\partial_r\chi\,\cos\Omega) \\
&- \partial_z\chi\partial_r(\sin^2\Omega\,(1-\cos\psi))\,.
\end{aligned} \tag{4.29}$$

The electric field is then given by

$$\vec{E} = \kappa\, \vec{n} \cdot (\partial_\zeta \vec{n} \wedge \partial_\eta \vec{n})\, \partial_z \zeta \begin{pmatrix} -\partial_y \eta \\ \partial_x \eta \\ 0 \end{pmatrix} \tag{4.30}$$
$$= \kappa[\partial_z \Omega \sin\Omega\, (1-\cos\psi)(\, \partial_r \psi \, + 2\, \partial_r \chi \, \cos\Omega) - \partial_z \chi \partial_r (\sin^2\Omega\, (1-\cos\psi))]\, \vec{e}_\varphi .$$

If we specialize this further to the case where χ and Ω only depend on z i.e. $\Omega = \Omega(z), \chi = \chi(z)$, the electric field takes the slightly simpler form

$$\vec{E} = \kappa[\partial_z \Omega \sin\Omega\, (1-\cos\psi)(\, \partial_r \psi) + \partial_z \chi \sin^2\Omega\, \partial_r \cos\psi))]\, \vec{e}_\varphi$$
$$= \kappa[-\partial_z \cos\Omega\, \partial_r(\psi - \sin\psi) + \partial_z \chi\, \sin^2\Omega\, \partial_r \cos\psi]\, \vec{e}_\varphi . \tag{4.31}$$

The square of the electric field is

$$\vec{E}^2 = \kappa^2[(\partial_z \cos\Omega)^2\, (\partial_r(\psi - \sin\psi))^2 + (\partial_z \chi)^2\, \sin^4\Omega\, (\partial_r \cos\psi)^2$$
$$- \; 2\, \partial_z \cos\Omega\, \sin^2\Omega\, \partial_r(\psi - \sin\psi)\, \partial_r \cos\psi\, \partial_z \chi] \tag{4.32}$$

and the energy reads

$$E = \varepsilon_0 \int d^3x (\vec{E}_x^2 + E_y^2) \tag{4.33}$$
$$= \varepsilon_0 \kappa^2 \int d^3x [(\partial_z \cos\Omega)^2\, (\partial_r(\psi - \sin\psi))^2 + (\partial_z \chi)^2\, \sin^4\Omega\, (\partial_r \cos\psi)^2 \tag{4.34}$$
$$- \; 2\, \partial_z \cos\Omega\, \sin^2\Omega\, \partial_r(\psi - \sin\psi)\, \partial_r \cos\psi\, \partial_z \chi] \, . \tag{4.35}$$

For the spin we obtain

$$\vec{S} = -\kappa^2\, \varepsilon_0 \int \quad d^3x \quad (\partial_z \Omega\, \sin\Omega\, (1-\cos\psi)(\, \partial_r \psi \, + 2\, \partial_r \chi \, \cos\Omega)$$
$$- \quad \partial_z \chi\, \partial_r (\sin^2\Omega\, (1-\cos\psi)))\partial_r \vec{n} \, . \tag{4.36}$$

In the case that χ is constant only the first term remains and we obtain the relative simple expressions

$$E = \varepsilon_0 \kappa^2 \int d^3x [(\partial_z \cos\Omega)^2\, (\partial_r(\psi - \sin\psi))^2]$$
$$= \varepsilon_0 \kappa^2 \int dz \int d\varphi \int r dr [(\partial_z \cos\Omega)^2\, (\partial_r(\psi - \sin\psi))^2] \, , \tag{4.37}$$
$$\vec{S} = \kappa^2\, \varepsilon_0 \int d^3x (\partial_z \cos\Omega\, \partial_r(\psi - \sin\psi))\partial_r \vec{n} \, . \tag{4.38}$$

4.3.2 No φ dependence and $\Omega \equiv \frac{\pi}{2}$

As a special case of our ansatz (4.18) we consider $\Omega \equiv \frac{\pi}{2}$ i.e. the situation where we intersect S^2 with a plane that contains the z-axis.
The $\vec{n}$ field defined above takes the form

$$\vec{n} = \begin{pmatrix} 0 \\ 0 \\ \cos\psi \end{pmatrix} + \sin\psi \begin{pmatrix} \sin\chi \\ -\cos\chi \\ 0 \end{pmatrix} . \tag{4.39}$$

Setting

$$\chi = -\gamma(r, z) \text{ and } \psi = -\theta(r, z) \tag{4.40}$$

we arrive at

$$\vec{n} = \begin{pmatrix} \sin(\gamma(r, z)) \sin(\theta(r, z)) \\ \cos(\gamma(r, z)) \sin(\theta(r, z)) \\ \cos(\theta(r, z)) \end{pmatrix}. \tag{4.41}$$

In this situation (4.28) takes the form

$$\begin{aligned} \vec{n}(\partial_\varsigma \vec{n} \wedge \partial_\eta \vec{n}) &= \vec{n}(\partial_z \vec{n} \wedge \partial_r \vec{n}) = \\ &= (\partial_z \theta \, \partial_r \gamma - \partial_z \theta \, \partial_r \gamma)(-1) \cdot \sin\theta \\ &= (\partial_z \cos\theta \, \partial_r \gamma - \partial_z \cos\theta \, \partial_r \gamma) \,. \end{aligned} \tag{4.42}$$

and so the electric field is given by

$$\vec{E} = -\kappa[\partial_r \cos\theta \, \partial_z \gamma - \partial_z \cos\theta \, \partial_r \gamma] \, \vec{e}_\varphi \tag{4.43}$$

If we similar to before specialize $\gamma = \gamma(z)$ and $\theta = \theta(r)$ we find the following expressions for the energy and the spin

$$E = \varepsilon_0 \kappa^2 \int d^3x \, (\partial_r \cos\theta)^2 \, (\partial_z \gamma)^2 \,, \tag{4.44}$$

$$\vec{S} = \frac{\kappa^2 \varepsilon_0}{c} \int d^3x \, \partial_z \gamma \begin{pmatrix} \sin\gamma \, \partial_r \sin\theta \, \partial_r \cos\theta \\ \cos\gamma \, \partial_r \sin\theta \, \partial_r \cos\theta \\ (\partial_r \cos\theta)^2 \end{pmatrix}. \tag{4.45}$$

We will assume that the z-integration extends from z_0 to z_1. Then we see that if we assume the boundary conditions $\gamma(z_0) = 0$ and $\gamma(z_1) = n\,\pi$ the y-component of the spin vanishes whereas the x-component is proportional to $(1 - (-1)^n)$. In order to have a spin that only has a z-component we will demand $\gamma(z_1) = 2n\,\pi$.
Specifying further $\gamma = \frac{\omega}{c} z$ and integrating from z_0 to z_1 with $z_1 - z_0 = \frac{2n\pi\, c}{\omega} z$ we find the following expressions for energy and spin.

$$E = \alpha_f \, n \, \pi \, \hbar \, \omega \int dr \, r \, (\partial_r \cos\theta)^2 \,, \tag{4.46}$$

$$S_z = \alpha_f \, n \, \pi \, \hbar \int dr \, r \, (\partial_r \cos\theta)^2 \,. \tag{4.47}$$

In this situation we obtain

$$\frac{S_z}{E} = \frac{\hbar}{\hbar\omega}. \tag{4.48}$$

4.3.3 Combining variable Ω with $\Omega \equiv \frac{\pi}{2}$, no φ dependence

We would like to combine the situation of variable Ω with that of fixed Ω in the following way:

a) For $z < z_0$ we have a constant $\vec{n}$ field, $\vec{n} \equiv \vec{n}_0$. $z_0 = -\infty$ is also be possible here.

b) For $z_0 \leq z \leq z_1$ we are in the situation of variable Ω. We impose the boundary conditions

$$\Omega(z_0) = 0\ ,\ \Omega(z_1) = \frac{\pi}{2} \quad , \quad \vec{n}(\Omega, r=0) = \vec{n}_0 = \vec{n}(\Omega, r=\infty). \tag{4.49}$$

In addition we will assume χ to be constant, $\chi \equiv 0$.

c) For $z_1 < z \leq z_2$ we are in the situation of $\Omega \equiv \frac{\pi}{2}$. In order for the solution to be more easily readable we will also use χ and γ here, writing χ for $-\gamma$ and ψ for $-\theta$. We will assume that $\chi = \chi(z)$ and $\psi = \psi(r)$. The boundary conditions in z are $\chi(z_1) = 0, \chi(z_2) = 2n\pi$. The boundary conditions for $r = 0$ and $r = \infty$ are the same as above.

d) For $z_2 < z \leq z_3$ we are again in the situation of variable Ω. This time we impose the boundary conditions

$$\Omega(z_2) = \frac{\pi}{2}\ ,\ \Omega(z_3) = 0 \quad , \quad \vec{n}(\Omega, r=0) = \vec{n}_0 = \vec{n}(\Omega, r=\infty). \tag{4.50}$$

Again we assume χ to be constant, $\chi = 2n\pi$.

e) Finally for $z > z_3$ we assume a constant $\vec{n}$ field, $\vec{n} \equiv \vec{n}_0$. Again the case $z_3 = \infty$ is possible.

We will in particular choose $\vec{n}_0 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and in the cases of variable Ω assume that Ω only depends on z.

Naturally we will have to ensure that the $\vec{n}$ field as well as the electric and magnetic field are continuous at the transition points. In fact because of the connection between the $\vec{E}$ and $\vec{B}$ field it is enough to check continuity for, e.g., the $\vec{E}$ field. We will turn to this subject now.

- At z_0 continuity is trivially fulfilled by our ansatz since $\vec{n}(0, r) = \vec{n}_0$ and the electric field (4.30) is zero for $\Omega = 0$.
- At $z = z_1$ we have the transition from variable Ω to fixed Ω. The $\vec{n}$ field has already been chosen to be continuous. We denote the electric field coming from the $\vec{n}$-field with variable Ω as $\vec{E}_{var}$ and the electric field in the situation $\Omega = \frac{\pi}{2}$ as $\vec{E}_{fix}$. We find

$$\vec{E}_{var} = -\kappa[\partial_z\Omega\, \partial_r(\sin\psi - \psi)]|_{z=z_1}\, \vec{e}_\varphi \stackrel{!}{=} \kappa[\partial_r \cos\psi\, \partial_z\chi]|_{z=z_1}\, \vec{e}_\varphi = \vec{E}_{fix} \tag{4.51}$$

  where the exclamation mark indicates that this equivalence has to be fulfilled. We see that we can achieve this by demanding $\partial_z\Omega|_{z=z_1} = 0$ and $\partial_z\chi|_{z=z_1} = 0$.
- For $z = z_2$ we have the opposite transition, i.e., from $\Omega \equiv \frac{\pi}{2}$ to variable Ω. Obviously this leads to similar transition conditions.
- The final transition to a constant $\vec{n}$-field is again trivially fulfilled.

The boundary conditions for $r = 0, \infty$ are

$$\cos\psi(0) = 1 = \cos\psi(\infty)\ ,\ \text{i.e.,}\ \psi(0) = 2n\pi\ ,\ \psi(\infty) = 2m\pi. \tag{4.52}$$

The following table gives an overview over the conditions.

z	$\vec{n}$	boundary conditions in z	boundary condition in r	transition condition at the end of the interval
$z < z_0$	$\vec{n}_0$			
$z_0 \leq z \leq z_1$	$\vec{n}_{var}$	$\Omega(z_0) = 0,$ $\Omega(z_1) = \frac{\pi}{2}$	$\psi(0) = 2n_1\pi,$ $\psi(\infty) = 2m_1\pi$	$\partial_z\Omega\|_{z=z_1} = 0,$ $\partial_z\chi\|_{z=z_1} = 0,$
$z_1 \leq z \leq z_2$	$\vec{n}_{fix}$	$\chi(z_1) = 0,$ $\chi(z_2) = 2n\pi$	$\psi(0) = 2k_1\pi,$ $\psi(\infty) = 2k_2\pi$	$\partial_z\Omega\|_{z=z_2} = 0,$ $\partial_z\chi\|_{z=z_2} = 0$
$z_2 \leq z \leq z_3$	$\vec{n}_{var}$	$\Omega(z_2) = \frac{\pi}{2},$ $\Omega(z_3) = 0$	$\psi(0) = 2n_2\pi,$ $\psi(\infty) = 2m_2\pi$	
$z_3 < z$	$\vec{n}_0$			

We give a final overview of this combined ansatz on the next page.

$\underline{z < z_0}$:

$$\vec{n} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{E} = 0,\ E = 0,\ \vec{S} = 0. \tag{4.53}$$

$\underline{z_0 \leq z \leq z_1}$:

$$\begin{aligned}
\vec{n} &= \begin{pmatrix} \sin\Omega\,\cos\Omega\,(1-\cos\psi) \\ -\sin\Omega\,\sin\psi \\ 1-\sin^2\Omega\,(1-\cos\psi) \end{pmatrix},\ \Omega = \Omega(z),\ \psi = \psi(r)\,, \\
& \Omega(z_0) = 0,\ \Omega(z_1) = \frac{\pi}{2}, \psi(0) = 0, \psi(\infty) = 2\pi, \\
\vec{E} &= \kappa[-\partial_z \cos\Omega\,\partial_r(\psi - \sin\psi)]\,\vec{e}_\varphi\,, \\
E &= \varepsilon_0 \kappa^2\, 2\pi \int_{z_0}^{z_1} dz (\partial_z \cos\Omega)^2 \int_0^\infty drr\,(\partial_r(\psi - \sin\psi))^2\,, \\
\vec{S} &= \varepsilon_0\,\kappa^2\,2\pi \int_{z_0}^{z_1} dz \int_0^\infty drr(\partial_z \cos\Omega\,\partial_r(\psi - \sin\psi))\partial_r \vec{n}\,.
\end{aligned} \tag{4.54}$$

$\underline{z_1 < z \leq z_2}$: $\gamma = -\chi\,,\ \theta = -\psi$

$$\begin{aligned}
\vec{n} &= \begin{pmatrix} \sin(\chi(z))\sin(\psi(r)) \\ -\cos(\chi(z))\sin(\psi(r)) \\ \cos(\psi(r)) \end{pmatrix}, \chi = \chi(z),\ \psi = \psi(r), \\
& \chi(z_1) = 0,\ \chi(z_2) = 2\pi,\ \psi(0) = 0,\ \psi(\infty) = 2\pi\,, \\
\vec{E} &= \kappa[\partial_r \cos\psi\,\partial_z \chi]\,\vec{e}_\varphi\,, \\
E &= 2\pi\,\varepsilon_0 \kappa^2 \int_{z_1}^{z_2} dz\,(\partial_z \chi)^2 \int_0^\infty dr\,r\,(\partial_r \cos\psi)^2\,, \\
\vec{S} &= (2\pi)^2\,\frac{\varepsilon_0\,\kappa^2}{c} \int_0^\infty drr\ (\partial_r \cos\psi)^2 \vec{e}_z\,.
\end{aligned} \tag{4.55}$$

$\underline{z_2 < z \leq z_3}$:

$$\begin{aligned}
\vec{n} &= \begin{pmatrix} \sin\Omega\,\cos\Omega\,(1-\cos\psi) \\ -\sin\Omega\,\sin\psi \\ 1-\sin^2\Omega\,(1-\cos\psi), \end{pmatrix},\ \Omega = \Omega(z),\ \psi = \psi(r)\,, \\
& \Omega(z_2) = \frac{\pi}{2},\ \Omega(z_3) = 0, \psi(0) = 0, \psi(\infty) = 2\pi\,, \\
\vec{E} &= \kappa[-\partial_z \cos\Omega\,\partial_r(\psi - \sin\psi)]\,\vec{e}_\varphi\,, \\
E &= \varepsilon_0 \kappa^2\, 2\pi \int_{z_2}^{z_3} dz (\partial_z \cos\Omega)^2 \int_0^\infty drr\,(\partial_r(\psi - \sin\psi))^2\,, \\
\vec{S} &= \varepsilon_0\,\kappa^2\,2\pi \int_{z_2}^{z_3} dz \int_0^\infty drr(\partial_z \cos\Omega\,\partial_r(\psi - \sin\psi))\partial_r \vec{n}\,.
\end{aligned} \tag{4.56}$$

$\underline{z > z_3}$:

$$\vec{n} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{E} = 0,\ E = 0,\ \vec{S} = 0. \tag{4.57}$$

We consider the energy and the spin of this combined solution.
First in the situations with variable Ω we see from, e.g, (4.54) that the energy is proportional to $\int_{z_0}^{z_1} dz\,(\partial_z \cos\Omega)^2$. This integral is minimal when $\cos\Omega(z)$ is a linear function and takes the value $\frac{1}{(z_1-z_0)}$. From this we deduce that the energy of the parts with variable Ω can be made arbitrary small by letting $z_0 \to -\infty$, $z_3 \to \infty$.
The spin of the parts of variable Ω also decomposes into an integral over z and an integral over r. Using the explicit form of $\partial_r \vec{n}$ it reads

$$\vec{S} = \varepsilon_0\,\kappa^2\,2\pi \int_{z_0}^{z_1} dz \int_0^\infty drr(\partial_z \cos\Omega\,\partial_r(\psi - \sin\psi)) \begin{pmatrix} \sin\Omega\,\cos\Omega\,\partial_r\cos\psi \\ -\sin\Omega\,\partial_r\sin\psi \\ \sin^2\Omega\,\partial_r\cos\psi \end{pmatrix}. \tag{4.58}$$

We see that if we assume ψ to be identical in (4.54) and (4.56) the r-integration is identical for those two parts. The z-integration on the other hand has the opposite sign assuming $\Omega(z)$ in (4.56) to be the mirror image of (4.54). So taken together these two parts cancel each other.
Thereby the energy E_{comb} and spin $\vec{S}_{comb}$ of the combined solution is simply the energy and spin of (4.55).
The conditions $\partial_z\chi|_{z_1} = \partial_z\chi|_{z_2} = 0$ coming from demanding a smooth electric field do not allow χ to depend on z linearly. Since the z integral of the spin in (4.44) only depends on the boundary conditions for χ one should still be able to choose χ such that the correct relation between spin and energy is obtained.

4.4 φ dependence

We now turn to the slightly more complicated situation where $\partial_\varphi \vec{n}$ (and also E_z) is non vanishing.

4.4.1 φ dependence , variable Ω

As we have already found in (4.26) the z-component of the electric field is

$$E_z = \kappa\frac{f}{r} = \kappa\;\frac{1}{r}\,\partial_\varphi\chi\partial_r(\sin^2\Omega(1-\cos\psi)). \tag{4.59}$$

In order to fulfill the e.o.m. we set

$$\partial_\varphi\chi\partial_r(\sin^2\Omega(1-\cos\psi)) = r\,\tilde{g}(z). \tag{4.60}$$

$E_z = 0$:

In case we want E_z to vanish we need to set $\tilde{g}(z) = 0$. Demanding a φ dependence this then leads to

$$\sin^2\Omega(1-\cos\psi) = h(z) \tag{4.61}$$

for some function $h(z)$. $h(z)$ has to vanish whenever there exists some $\tilde{r}$ so that either $\sin^2\Omega(\tilde{r},z) = 0$ or $(1-\cos\psi(\tilde{r},z)) = 0$ for all z. The further discussion therefore depends very much on our boundary conditions.
In case we demand that for any z $\vec{n} \to (0,0,1)^T$, for $r \to \infty$ i.e. $(1-\cos\psi) \to 0$ for $r \to \infty$ this implies $h(z) \equiv 0$. As we can see this will not lead to any interesting solutions so we go over to $E_z \neq 0$.
Else if we impose $\vec{n} = (0,0,1)^T$ on the z- axis (i.e for $r = 0$) we get $(1-\cos\psi(0,z)) = 0$ for all z equally leading to no further interesting results.
Another option is to let $\vec{n}$ vary along the z-axis and demanding $\vec{n} \to (0,0,-1)$ for $r \to \infty$ i.e. $(1-\cos\psi) \to 2$ and $\sin^2\Omega \to 1$ for $r \to \infty$. This implies $h \equiv 2$, i.e.,

$$\sin\Omega = \sqrt{\frac{2}{1-\cos\psi}} \tag{4.62}$$

which in turn implies $\vec{n} \equiv (0,0,-1)$ so also no interesting solution.

$E_z \neq 0$:

In this section we restrain Ω to depend on z only.
In particular the z-component of the electric field is

$$E_z = -\kappa\,\frac{1}{r}\,\sin^2\Omega\,\partial_r\cos\psi\,\partial_\varphi\chi. \tag{4.63}$$

In order to fulfill the e.o.m. we set

$$\sin^2\Omega\,\sin\psi\,\partial_r\psi\,\partial_\varphi\chi = r\,\tilde{g}(z). \tag{4.64}$$

Together with the condition $\vec{n}(r=0,\varphi,z) = (0,0,1)^T$ and assuming $\chi = k\,\varphi$ this yields

$$\cos\psi(r,z) = \frac{r^2}{k}\,g(z) + 1 \quad , \quad r^2 \leq -\frac{2\,k}{g} \tag{4.65}$$

We may try to extend this solution by setting

$$\cos\psi = \frac{r^2}{k}\,\hat{g}(z) + \hat{h}(z) \tag{4.66}$$

for $r^2 \geq -\frac{2\,k}{g}$ and demanding

$$\cos\psi(\sqrt{-2\,k/g}, z) = -1\,. \tag{4.67}$$

This implies

$$\hat{h}(z) = -1 + 2\,\frac{\hat{g}(z)}{g(z)}. \tag{4.68}$$

As a simple solution we set $\hat{g}(z) = -g(z)$ which gives $\hat{h}(z) = -3$ and so

$$\cos\psi = -\frac{r^2}{k}\,g(z) - 3 \tag{4.69}$$

for $-\frac{2k}{g} \leq r^2 \leq -\frac{4k}{g}$. So in all we have

$$\begin{aligned}\cos\psi(r,z) &= (\frac{r^2}{k}\,g(z)+1)\Theta(\sqrt{-\frac{2\,k}{g}}-r)+(-\frac{r^2}{k}\,g(z)-3)\Theta(r-\sqrt{-\frac{2\,k}{g}}) \\ &= 2\,(\frac{r^2}{k}\,g(z)+2)\Theta(\sqrt{-\frac{2\,k}{g}}-r)-(\frac{r^2}{k}\,g(z)+3) \qquad (4.70)\end{aligned}$$

for $r^2 \leq -\frac{4k}{g}$.
We check if this can fulfill the e.o.m.. First

$$\partial_r\cos\psi(r,z)=\frac{4r}{k}\,g(z)\,\Theta(\sqrt{-\frac{2\,k}{g}}-r)-\frac{2r}{k}\,g(z)-2\,\underbrace{(\frac{r^2}{k}\,g(z)+2)\,\delta(\sqrt{-\frac{2\,k}{g}}-r)}_{=0}. \qquad (4.71)$$

The last term containing the Dirac-delta is obviously the contribution that comes from extending the original solution. However it is zero since it is multiplied with a function that vanishes on its support.
For the e.o.m. we also need the second derivative, leaving out the last term from above

$$\partial_r^2\cos\psi=\frac{4}{k}\,g(z)\,\Theta(\sqrt{-\frac{2\,k}{g}}-r)-\frac{2}{k}\,g(z)-\frac{4r}{k}\,g(z)\,\delta(\sqrt{-\frac{2\,k}{g}}-r)\,. \qquad (4.72)$$

The e.o.m. amount to

$$0=\left[\partial_r^2\cos\psi-\partial_r\cos\psi\frac{1}{r}\right]\,\partial_\varphi\vec{n}=\left[-\frac{4r}{k}\,g(z)\,\delta(\sqrt{-\frac{2\,k}{g}}-r)\right]\,\partial_\varphi\vec{n}\,. \qquad (4.73)$$

Since

$$\partial_\varphi\vec{n}=\left[\sin\Omega\,\cos\Omega\,(1-\cos\psi)\begin{pmatrix}-\sin\chi\\ \cos\chi\\ 0\end{pmatrix}+\sin\Omega\sin\psi\begin{pmatrix}\cos\chi\\ \sin\chi\\ 0\end{pmatrix}\right]\partial_\varphi\chi \qquad (4.74)$$

is non-vanishing at $r=\sqrt{-\frac{2k}{g}}$, i.e., $\psi=\pi$ we see that extending the solution further in this fashion does not seem possible.

The action of this field is

$$
\begin{aligned}
S &= \int dt \int d^3x \mathcal{L} = -\frac{1}{2\,\mu_0} \int dt \int d^3x (\frac{\vec{E}^2}{c^2} - \vec{B}^2) \\
&= -\frac{1}{2\,\mu_0\,c^2} \int dt \int d^3x\, E_z^2 \\
&= -\frac{\kappa^2\,\varepsilon_0}{2} \int dt \int d^3x \frac{1}{r^2} \,\sin^4\Omega (\partial_r \cos\psi)^2\,(\partial_\varphi \chi)^2 \\
&= -\frac{\kappa^2\,\varepsilon_0}{2} \int dt \int d^3x \sin^4\Omega\, g^2\, k^2 \\
&= -\frac{\kappa^2\,\varepsilon_0}{2} \int dt \int dz \int_0^{2\pi} d\varphi \int_0^{\frac{2}{\sqrt{g}}} dr\, r \sin^4\Omega\, g^2\, k^2 \\
&= -\frac{\kappa^2\,\varepsilon_0}{2} \int dt \int dz\, 2\pi\, \frac{2\,g^2\,k^2}{g} \sin^4\Omega \\
&= \alpha_f \frac{\hbar\,c}{2} g\, k^2 \int dt \int dz \sin^4\Omega\,. \qquad (4.75)
\end{aligned}
$$

4.4.2 φ dependence , $\Omega \equiv \frac{\pi}{2}$

For $\Omega = \frac{\pi}{2}$ the $\vec{n}$ field defined above takes the form

$$
\vec{n} = \begin{pmatrix} 0 \\ 0 \\ \cos\psi \end{pmatrix} + \sin\psi \begin{pmatrix} \sin\chi \\ -\cos\chi \\ 0 \end{pmatrix}. \qquad (4.76)
$$

Setting

$$
\chi = -(\gamma(r,z) + k\varphi) \text{ and } \psi = -\theta(r,z) \qquad (4.77)
$$

we arrive at

$$
\vec{n} = \begin{pmatrix} \sin(\gamma(r,z) + k\varphi)\sin(\theta(r,z)) \\ \cos(\gamma(r,z) + k\varphi)\sin(\theta(r,z)) \\ \cos(\theta(r,z)) \end{pmatrix}. \qquad (4.78)
$$

Derivatives are

$$
\begin{aligned}
\partial_\phi \vec{n} &= \begin{pmatrix} \cos(\gamma(r,z) + k\varphi)\sin(\theta(r,z)) \\ -\sin(\gamma(r,z) + k\varphi)\sin(\theta(r,z)) \\ 0 \end{pmatrix}, \partial_\theta \vec{n} = \begin{pmatrix} \sin(\gamma(r,z) + k\varphi)\cos(\theta(r,z)) \\ \cos(\gamma(r,z) + k\varphi)\cos(\theta(r,z)) \\ -\sin(\theta(r,z)) \end{pmatrix} \\
\partial_r \vec{n} &= \partial_r \gamma\, \partial_\gamma \vec{n} + \partial_r \theta\, \partial_\theta \vec{n}\,, \qquad (4.79) \\
\partial_\varphi \vec{n} &= k\, \partial_\phi \vec{n}\,, \qquad (4.80) \\
\partial_z \vec{n} &= \partial_z \gamma\, \partial_\gamma \vec{n} \,+\, \partial_z \theta\, \partial_\theta \vec{n} \qquad (4.81) \\
\partial_x \vec{n} &= (\partial_x \gamma + \partial_x(k\varphi))\partial_\gamma \vec{n} + \partial_x \theta\, \partial_\theta \vec{n} = (\partial_r \gamma\, \frac{x}{r} - k\, \frac{y}{r^2})\, \partial_\phi \vec{n} + \partial_r \theta\, \frac{x}{r} \partial_\theta \vec{n} \qquad (4.82) \\
\partial_y \vec{n} &= (\partial_r \gamma\, \frac{y}{r} + k\, \frac{x}{r^2}) \partial_\phi \vec{n} + \partial_r \theta\, \frac{y}{r} \partial_\theta \vec{n} \qquad (4.83)
\end{aligned}
$$

Also

$$
\begin{aligned}
\partial_\phi \vec{n} \times \partial_\theta \vec{n} &= \sin\theta\, \vec{n} \quad , && (4.84)\\
\partial_r \vec{n} \times \partial_\varphi \vec{n} &= k\, \partial_r\theta\, (\partial_\theta \vec{n} \times \partial_\gamma \vec{n}) = -k\, \partial_r\theta \sin\theta\, \vec{n} = k\, \partial_r \cos\theta\, \vec{n}. && (4.85)
\end{aligned}
$$

The ansatz (4.78) gives the following general expression for the electric field

$$
\begin{aligned}
\vec{E} &= -\kappa \left[\partial_z \cos\theta \begin{pmatrix} (\partial_r\gamma\, \frac{y}{r} + k\, \frac{x}{r^2}) \\ (-\,\partial_r\gamma\, \frac{x}{r} + k\, \frac{y}{r^2}) \\ 0 \end{pmatrix} - \partial_r \cos\theta \begin{pmatrix} \frac{y}{r}\partial_z\gamma \\ -\frac{x}{r}\partial_z\gamma \\ \frac{k}{r} \end{pmatrix} \right] \\
&= -\kappa[\, \vec{e}_\varphi (\partial_r \cos\theta\, \partial_z\gamma - \partial_z \cos\theta\, \partial_r\gamma)) + \frac{k}{r}(\partial_z \cos\theta\, \vec{e}_r - \partial_r \cos\theta\, \vec{e}_z)] \qquad (4.86)
\end{aligned}
$$

Before turning towards the e.o.m. we consider the special case where $\theta = \theta(r)$ and $\gamma = \gamma(z)$. This yields

$$
\vec{E} = -\kappa[\, \vec{e}_\varphi (\partial_r \cos\theta\, \partial_z\gamma) - \frac{k}{r}(\partial_r \cos\theta)\, \vec{e}_z]. \qquad (4.87)
$$

In this case the energy takes the form

$$
\begin{aligned}
E &= \varepsilon_0 \int d^3x [(E_x^2 + E_y^2) + \frac{1}{2}E_z^2] \\
&= \varepsilon_0 \kappa^2 \int d^3x (\partial_r \cos\theta)^2 [(\partial_z\gamma)^2 + \frac{k^2}{2\, r^2}] \\
&= |\gamma = \frac{\omega}{c}\, ,\, z_1 - z_0 = \frac{2\pi c}{\omega}| = \\
&= \varepsilon_0\, \kappa^2 \frac{4\pi^2\, \omega}{c} \int dr\, r (\partial_r \cos\theta^2)(1 + \frac{1}{2}\frac{k^2}{r^2}\frac{c^2}{\omega^2})) \\
&= \alpha_f\, \pi\, \hbar\, \omega \int dr\, r (\partial_r \cos\theta^2)(1 + \frac{1}{2}\frac{k^2}{r^2}\frac{c^2}{\omega^2})). \qquad (4.88)
\end{aligned}
$$

To calculate the spin in this situation we start by evaluating

$$
\begin{aligned}
E_x\, \partial_y \vec{n} - E_y \partial_x \vec{n} &= -\frac{\kappa}{r}\, \partial_r \cos\theta\, \partial_z\gamma (-y\, (k\, \frac{x}{r^2}\partial_\Phi \vec{n} + \partial_r\theta\, \frac{y}{r}\, \partial_\theta \vec{n}) - x(-k\, \frac{y}{r^2}\, \partial_\Phi \vec{n} + \partial_r\theta\, \frac{x}{r}\, \partial_\theta \vec{n})) \\
&= \kappa\, \partial_z\gamma\, \partial_r \cos\theta\, \partial_r\theta\, \partial_\theta \vec{n}. \qquad (4.89)
\end{aligned}
$$

The spin then reads

$$
\begin{aligned}
\vec{S} &= \frac{\kappa^2 \varepsilon_0}{c} \int d^3x (E_x\, \partial_y \vec{n} - E_y \partial_x \vec{n}) \\
&= \frac{\kappa^2 \varepsilon_0}{c} \int d^3x (\kappa\, \partial_z\gamma\, \partial_r \cos\theta\, \partial_r\theta\, \partial_\theta \vec{n}) \\
&= \frac{\kappa^2 \varepsilon_0}{c} \int d^3x\, \partial_z\gamma \begin{pmatrix} \sin(\gamma\, +\, k\, \varphi)\, \partial_r \cos\theta\, \partial_r \sin\theta \\ \cos(\gamma\, +\, k\, \varphi)\, \partial_r \cos\theta\, \partial_r \sin\theta \\ (\partial_r \cos\theta)^2 \end{pmatrix} \qquad (4.90)
\end{aligned}
$$

In particular the z-component of the spin reads

$$
S_z = \frac{\kappa^2 \varepsilon_0}{c} \int d^3x\, \partial_z\gamma\, (\partial_r \cos\theta)^2. \qquad (4.91)
$$

Now we turn to solving the e.o.m.. From our discussion above we set $\partial_r \cos\theta = \tilde{g}(z)\, r$ which leads to

$$\cos\theta = \tilde{g}(z)\frac{r^2}{2} + h(z)\ ,\ |\tilde{g}(z)\frac{r^2}{2} + h(z)| \leq 1. \tag{4.92}$$

The boundary condition $\theta(0,z) = 0 \bmod 2\pi$ implies $h(z) = 1$. In order to have a nonvanishing solution we require $\tilde{g}(z) \leq 0$. Writing $g(z) = -\tilde{g}(z)$ we set

$$\cos\theta = -g(z)\,\frac{r^2}{2} + 1 \quad \text{for} \quad r \leq \frac{2}{\sqrt{g(z)}}. \tag{4.93}$$

This implies

$$\sin\theta = (\pm 1) r\,\sqrt{g(z)}\sqrt{1 - g(z)\frac{r^2}{4}}. \tag{4.94}$$

Thereby $\vec{n}$ takes the form

$$\vec{n} = \begin{pmatrix} \sin(\gamma + k\varphi)\, r\,\sqrt{g(z)}\sqrt{1 - g(z)\frac{r^2}{4}} \\ \cos(\gamma + k\varphi)\, r\,\sqrt{g(z)}\sqrt{1 - g(z)\frac{r^2}{4}} \\ -g(z)\frac{r^2}{2} + 1 \end{pmatrix} \quad , \quad r \leq \frac{2}{\sqrt{g(z)}}\,. \tag{4.95}$$

We may continue constant outside this domain i.e. $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ for $r \geq \frac{2}{\sqrt{g(z)}}$.

Next we will calculate the electric field. We start by calculating the necessary vector-products

$$\begin{aligned} \partial_x\vec{n} \times \partial_z\vec{n} &= \big((\partial_r\gamma\,\partial_z\theta - \partial_r\theta\partial_z\gamma)\frac{x}{r} - \partial_z\theta\, k\,\frac{y}{r^2}\big)\sin\theta\,\vec{n} \\ &= (-1)\cdot\big((\partial_r\gamma\,\partial_z\cos\theta - \partial_r\cos\theta\,\partial_z\gamma)\frac{x}{r} - \partial_z\cos\theta\, k\,\frac{y}{r^2}\big)\,\vec{n} \\ &= \big((\partial_r\gamma\, g'\,\frac{r}{2} - g\,\partial_z\gamma)\, x - g'\, k\,\frac{y}{2}\big)\,\vec{n}\ , \end{aligned} \tag{4.96}$$

$$\partial_y\vec{n} \times \partial_z\vec{n} = \big((\partial_r\gamma\, g'\,\frac{r}{2} - g\,\partial_z\gamma)\, y + k\, g'\,\frac{x}{2}\big)\,\vec{n}\,, \tag{4.97}$$

$$\partial_x\vec{n} \times \partial_y\vec{n} = -k\, g(z)\,\vec{n}. \tag{4.98}$$

Hence the electric field reads

$$\begin{aligned} \vec{E} &= \kappa \begin{pmatrix} (\partial_y\vec{n} \times \partial_z\vec{n})\cdot\vec{n} \\ -(\partial_x\vec{n} \times \partial_z\vec{n})\cdot\vec{n} \\ (\partial_x\vec{n} \times \partial_y\vec{n})\cdot\vec{n} \end{pmatrix} = \kappa \left[\frac{g'}{2}\begin{pmatrix} y\, r\,\partial_r\gamma + k\, x \\ -x\, r\partial_r\gamma + k\, y \\ 0 \end{pmatrix} + g \begin{pmatrix} -y\,\partial_z\gamma \\ x\,\partial_z\gamma \\ -k \end{pmatrix}\right] \\ &= -\kappa\left[(\frac{g'}{2}\,\partial_r\gamma\, r - g\partial_z\gamma)\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} - \frac{g'\, k}{2}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + g\, k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] \\ &= -\kappa\left[(\frac{g'}{2}\,\partial_r\gamma\, r - g\partial_z\gamma)\,\vec{e}_\phi - \frac{g'\, k}{2}\, r\,\vec{e}_r + g\, k\,\vec{e}_z\right] \end{aligned} \tag{4.99}$$

for $r \leq \frac{2}{\sqrt{g(z)}}$ and is zero otherwise.

For our further calculations we assume g to be constant and γ only to depend on z, .i.e.,

$$g' = 0 \quad , \quad \gamma = \gamma(z)\,. \tag{4.100}$$

In this case the electric field takes the form

$$\vec{E} = -\kappa\, g \begin{pmatrix} \partial_z \gamma\, y \\ -\partial_z \gamma\, x \\ k \end{pmatrix}. \tag{4.101}$$

The energy then takes the form

$$\begin{aligned}
E &= \varepsilon_0\, \kappa^2 \int d^3x\, g^2\, (r^2 (\partial_z \gamma)^2 + \frac{1}{2} k^2)\, \Theta(\frac{2}{\sqrt{g(z)}} - r) \\
&= \varepsilon_0\, \kappa^2 \int dz \int_0^{2\pi} d\varphi \int_0^{\frac{2}{\sqrt{g}}} dr\, r g^2\, (r^2 (\partial_z \gamma)^2 + \frac{1}{2} k^2) \\
&= \varepsilon_0\, \kappa^2 \int dz \int_0^{2\pi} d\varphi\, g^2\, (\frac{r^4}{4}\, (\partial_z \gamma)^2 + \frac{r^2}{4} k^2)_0^{\frac{2}{\sqrt{g}}} \\
&= \varepsilon_0\, \kappa^2 \int dz \int_0^{2\pi} d\varphi\, (4\, (\partial_z \gamma)^2 + g\, k^2)\,.
\end{aligned} \tag{4.102}$$

We further assume γ to be of the form

$$\gamma = \frac{\omega}{c}\, z \tag{4.103}$$

and restrict the z-integration to "one period" from z_0 to z_1 with $z_1 - z_0 = \lambda = \frac{2\pi}{\omega}\, c$. The expression for the energy then becomes

$$\begin{aligned}
E &= \varepsilon_0\, \kappa^2\, 2\pi\, \left(z_1 - z_0\right) (4\, \frac{\omega^2}{c^2} + g\, k^2) \\
&= \varepsilon_0\, \kappa^2\, 2\pi\, \lambda\, (4\, \frac{\omega^2}{c^2} + g\, k^2) \\
&= |\varepsilon_0\, \kappa^2 = \alpha_f\, \frac{\hbar\, c}{4\, \pi}| = 4\pi\, \alpha_f\, \left(\hbar\omega + \frac{1}{4}\, \hbar\, k^2\, \frac{c^2 g}{\omega}\right).
\end{aligned} \tag{4.104}$$

Next we calculate the spin

$$\vec{S} = \frac{\kappa\, \varepsilon_0}{c} \int d^3x (E_x\, \partial_y \vec{n} - E_y\, \partial_x \vec{n}). \tag{4.105}$$

We start by evaluating the equations in the brackets

$$E_x\, \partial_y \vec{n} = \kappa \left[(\frac{g'}{2}\, \partial_r \gamma\, r - g \partial_z \gamma)(-y)\, \partial_y \vec{n} - \frac{g' k}{2}\, x\, \partial_y \vec{n} \right] \tag{4.106}$$

$$E_y\, \partial_x \vec{n} = \kappa \left[(\frac{g'}{2}\, \partial_r \gamma\, r - g \partial_z \gamma)(x)\, \partial_x \vec{n} - \frac{g' k}{2}\, y\, \partial_x \vec{n} \right] \tag{4.107}$$

$$\begin{aligned}
(E_x\, \partial_y \vec{n} - E_y\, \partial_x \vec{n}) &= \kappa [(\frac{g'}{2}\, \partial_r \gamma\, r - g \partial_z \gamma)(-1)(y \partial_y \vec{n} + x \partial_x \vec{n}) - \frac{g' k}{2} (x \partial_y \vec{n} - y \partial_x \vec{n})] \\
&= \kappa [(\frac{g'}{2}\, \partial_r \gamma\, r - g \partial_z \gamma)(-1)\, r\, \partial_r \vec{n} - \frac{g' k}{2} \partial_\varphi \vec{n}] \\
&= \kappa [(\frac{g'}{2}\, \partial_r \gamma\, r - g \partial_z \gamma)(-1)\, r\, (\partial_r \gamma \partial_\phi \vec{n} + \partial_r \theta\, \partial_\theta \vec{n}) - \frac{g' k^2}{2} \partial_\phi \vec{n}]
\end{aligned} \tag{4.108}$$

Specifying g to be constant and $\gamma(r,z) = \frac{w}{c}\, z$ we obtain

$$\begin{aligned}(E_x\, \partial_y \vec{n} - E_y\, \partial_x \vec{n}) &= \kappa\, g \frac{\omega}{c}\, r\, \partial_r \theta \partial_\theta \vec{n} \\ &= \kappa\, g \frac{\omega}{c}\, r \begin{pmatrix} \sin(\gamma + k\,\varphi)\, \partial_r \sin\theta \\ \cos(\gamma + k\,\varphi)\, \partial_r \sin\theta \\ \partial_r \cos\theta \end{pmatrix}\end{aligned} \tag{4.109}$$

For the spin we so obtain

$$\begin{aligned}\vec{S} &= \frac{\kappa\, \varepsilon_0}{c} \int dz \int d\varphi \int_0^{\frac{2}{\sqrt{g}}} r\, dr \kappa\, g \frac{\omega}{c}\, r \begin{pmatrix} \sin(\gamma + k\,\varphi)\, \partial_r \sin\theta \\ \cos(\gamma + k\,\varphi)\, \partial_r \sin\theta \\ \partial_r \cos\theta \end{pmatrix} \\ &= g \frac{\kappa^2\, \varepsilon_0}{c} \frac{\omega}{c} \int dz \int_0^{\frac{2}{\sqrt{g}}} dr r^2 \begin{pmatrix} 0 \\ 0 \\ 2\,\pi\, \partial_r \cos\theta \end{pmatrix} = |\partial_r \cos\theta = -g\, r| \\ &= \frac{\kappa^2\, \varepsilon_0}{c} \frac{\omega}{c} \int dz\, (-g^2)\, 2\,\pi \int_0^{\frac{2}{\sqrt{g}}} dr r^3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{\kappa^2\, \varepsilon_0}{c} \frac{\omega}{c} \int dz\, (-1)\, 8\,\pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{\kappa^2\, \varepsilon_0}{c} \frac{\omega}{c}\, (z_1 - z_0)\, 8\,\pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= -2\,\alpha_f \frac{\hbar\omega}{c}\, (z_1 - z_0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= |z_1 - z_0 = \frac{2\pi}{\omega}\, c| = -4\pi\, \alpha_f\, \hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.\end{aligned} \tag{4.110}$$

For the proportion of spin to energy we find

$$\frac{\vec{S}}{E} = \frac{\hbar}{(\hbar\omega + \frac{1}{4}\, \hbar\, k^2\, \frac{c^2 g}{\omega})} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.111}$$

We may also rescale the lagrangian by the factor $\mathcal{L} \rightarrow \frac{1}{4\pi\alpha_f} \mathcal{L}$ to obtain

$$E = \hbar\omega + \frac{1}{4}\, \hbar\, k^2\, \frac{c^2 g}{\omega} \quad , \quad \vec{S} = \begin{pmatrix} 0 \\ 0 \\ -\hbar \end{pmatrix} \tag{4.112}$$

Extended solution:

Making use of the fact that

$$\partial_\varphi \vec{n}|_{\theta=\pi} = \partial_\varphi \vec{n}|_{\theta=2\pi} = 0 \tag{4.113}$$

we can extend the solution from above beyond $r \leq \frac{2}{\sqrt{g(z)}}$ and still fulfill the e.o.m.. As a first try /natural extension we set

$$\cos\theta = g(z)\,\tfrac{r^2}{2} - 3 \quad , \quad \text{for } \frac{2}{\sqrt{g(z)}} < r \leq \frac{2\sqrt{2}}{\sqrt{g(z)}} \tag{4.114}$$

$$\cos\theta = 1 \qquad\quad , \quad \text{for } r > \frac{2\sqrt{2}}{\sqrt{g(z)}}\,. \tag{4.115}$$

Writing this with Theta-Heavyside-functions we obtain

$$\cos\theta = 1 + \Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)\,((\frac{r^2}{2}\,g - 4) + \Theta(\frac{2}{\sqrt{g}} - r)\,(4 - r^2\,g))\,, \tag{4.116}$$

$$\sin\theta = \Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)\left((-1)\,(\frac{r^4 g^2}{4} - 3\,r^2 g + 8 + \Theta(\frac{2}{\sqrt{g}} - r)(2\,r^2\,g - 8))\right)^{\frac{1}{2}}. \tag{4.117}$$

The r and z derivatives are

$$\partial_r \cos\theta = r\,g\,\Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)(1 - 2\,\Theta(\frac{2}{\sqrt{g}} - r))\,, \tag{4.118}$$

$$\partial_z \cos\theta = \frac{r^2 g'}{2}\,\Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)(1 - 2\,\Theta(\frac{2}{\sqrt{g}} - r)) \tag{4.119}$$

$$\partial_r \sin\theta = \Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)(-1)(r^3\,g^2 - 6\,r\,g + \Theta(\frac{2}{\sqrt{g}} - r)\,4\,r\,g)\sqrt{g} - r) \times \left((-1)\,(\frac{r^4 g^2}{4} - 3\,r^2 g + 8 + \Theta(\frac{2}{\sqrt{g}} - r)(2\,r^2\,g - 8))\right)^{-\frac{1}{2}} \tag{4.120}$$

The electric field for the extended solution is given by

$$\vec{E} = \kappa\left[(\frac{g'}{2}\,\partial_r\gamma\,r - g\partial_z\gamma)\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} - \frac{g'\,k}{2}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + g\,k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]\Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)(1 - 2\,\Theta(\frac{2}{\sqrt{g}} - r))$$

The square of the electromagnetic fields is

$$\vec{E}^2 = \kappa^2(\frac{g'^2}{4}\,r^2\,(r^2\,(\partial_r\gamma)^2 + k^2) - g\,g'\frac{1}{2}r^3\partial_r\gamma\,\partial_z\gamma + g^2(r^2(\partial_z\gamma)^2 + k^2))\,\Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r) \tag{4.121}$$

For constant $g(z) = \bar{g}$ we obtain

$$\vec{E} = \kappa\,\bar{g}\begin{pmatrix} y\,\partial_z\gamma \\ -x\,\partial_z\gamma \\ k \end{pmatrix}\Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)(1 - 2\,\Theta(\frac{2}{\sqrt{g}} - r)) \tag{4.122}$$

and

$$\vec{E}^2 = \kappa^2\,\bar{g}^2(r^2(\partial_z\gamma)^2 + k^2)\,\Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r). \tag{4.123}$$

In case we continue the solution beyond $r = \frac{2}{\sqrt{g}}$ as done above we obtain the following expressions for energy and spin:

$$\begin{aligned}
E &= \varepsilon_0 \,\kappa^2 \int d^3x\, g^2\,(r^2(\partial_z\gamma)^2 + \frac{1}{2}k^2) \\
&= \varepsilon_0 \,\kappa^2 \int dz \int_0^{2\pi} d\varphi \int_0^{\frac{2\sqrt{2}}{\sqrt{g}}} dr\, r g^2\,(r^2(\partial_z\gamma)^2 + \frac{1}{2}k^2) \\
&= \varepsilon_0 \,\kappa^2 \int dz \int_0^{2\pi} d\varphi\, g^2\,(\frac{r^4}{4}\,(\partial_z\gamma)^2 + \frac{r^2}{4}k^2)_0^{\frac{2\sqrt{2}}{\sqrt{g}}} \\
&= \varepsilon_0 \,\kappa^2 \int dz \int_0^{2\pi} d\varphi\,(16\,(\partial_z\gamma)^2 + g\,2\;k^2) \\
&= |\gamma = \gamma(z) = \frac{\omega}{c}\,z| = \varepsilon_0\,\kappa^2\,2\pi\;(z_1 - z_0)\,(16\,\frac{\omega^2}{c^2} + 2\,g\,k^2) \\
&= |(z_1 - z_0) = \lambda = \frac{2\,\pi}{\omega}\,c| = \varepsilon_0\,\kappa^2\,2\pi\;\lambda\,(16\,\frac{\omega^2}{c^2} + 2\,g\,k^2) \\
&= 16\pi\,\alpha_f\;(\hbar\omega + \frac{1}{8}\,\hbar\,k^2\,\frac{c^2 g}{\omega}).
\end{aligned} \tag{4.124}$$

$$\begin{aligned}
\vec{S} &= \frac{\kappa\,\varepsilon_0}{c}\int dz \int d\varphi \int r\,dr\kappa\,g\frac{\omega}{c}\,r \begin{pmatrix} \sin(\gamma + k\,\varphi)\,\partial_r \sin\theta \\ \cos(\gamma + k\,\varphi)\,\partial_r \sin\theta \\ \partial_r \cos\theta \end{pmatrix} \\
&= g\frac{\kappa^2\,\varepsilon_0}{c}\frac{\omega}{c}\int dz \int dr r^2 \begin{pmatrix} 0 \\ 0 \\ 2\,\pi\,\partial_r \cos\theta \end{pmatrix} \\
&= g^2\frac{\kappa^2\,\varepsilon_0}{c}\frac{\omega}{c}\int dz \int dr\, r^3\,2\,\pi\,\Theta(\frac{2\sqrt{2}}{\sqrt{g}} - r)(1 - 2\,\Theta(\frac{2}{\sqrt{g}} - r)) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= g^2\frac{\kappa^2\,\varepsilon_0}{c}\frac{\omega}{c}\int dz\,2\,\pi((-1)\cdot\frac{r^4}{4}|_0^{\frac{2}{\sqrt{g}}} + \frac{r^4}{4}|_{\frac{2}{\sqrt{g}}}^{\frac{2\sqrt{2}}{\sqrt{g}}})) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= g^2\frac{\kappa^2\,\varepsilon_0}{c}\frac{\omega}{c}\int dz\,2\,\pi(\frac{8}{g^2}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{\kappa^2\,\varepsilon_0}{c}\frac{\omega}{c}\int dz\,16\,\pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\kappa^2\,\varepsilon_0}{c}\frac{\omega}{c}(z_1 - z_0)\,16\,\pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= 4\,\alpha_f\frac{\hbar\omega}{c}\;(z_1 - z_0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= |z_1 - z_0 = \frac{2\pi}{\omega}\,c| = 8\pi\,\alpha_f\,\hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned} \tag{4.125}$$

4.5 FERREIRA

In [Fer06] Ferreira gives a class of solutions for the problem discussed above.
In particular he introduces a complex scalar field $u : \mathbb{R}^3 \to \mathbb{C}$ and expresses $\vec{n}$ through

$$\vec{n} = \frac{1}{(1+|u|^2)} \begin{pmatrix} u + u^* \\ -i(u - u^*) \\ |u|^2 - 1 \end{pmatrix} . \tag{4.126}$$

The inverse transformation might be given by

$$u = \frac{n_1 + in_2}{1 - n_3}. \tag{4.127}$$

In terms of the field u the field strength can be expressed as

$$H_{\mu\nu} = -2i\, \frac{\partial_\mu u\, \partial_\nu u^* - \partial_\nu u\, \partial_\mu u^*}{(1+|u|^2)^2} \tag{4.128}$$

and the e.om. takes the form

$$\partial_\mu \mathcal{K}^\mu = 0\,,\ \partial_\mu \mathcal{K}^{*\,\mu} = 0 \quad \text{where} \quad \mathcal{K}_\mu := H_{\mu\nu}\partial^\nu u\,. \tag{4.129}$$

The model is conserved under the area preserving diffeomorphisms of S^2. The infinite number of associated conserved currents is given by

$$J^G_\mu = \frac{\delta G}{\delta u}\mathcal{K}_\mu + \frac{\delta G}{\delta u^*}\mathcal{K}^*_\mu \tag{4.130}$$

where G is any functional of u and u^*, but not of their derivatives. Conservation of these currents is immediately clear from the e.o.m. .
Ferreira goes on to introduce the slightly modified toroidal coordinates

$$x^0 = \frac{a}{p}\,\sin\zeta\,, \tag{4.131}$$

$$x_1 = \frac{a}{p}\frac{\cos\varphi}{\sqrt{1+y}}\ ,\ x_2 = \frac{a}{p}\frac{\sin\varphi}{\sqrt{1+y}}\ ,\ x_3 = \frac{a}{p}\sin\xi\sqrt{\frac{y}{1+y}} \tag{4.132}$$

where $p := \cos\zeta - \cos\xi\sqrt{\frac{y}{1+y}}$ and the range of the new coordinates is the following

$$y \geq 0\,,\ 0 \leq \xi, \phi \leq\ 2\pi\,,\ 0 \leq \zeta \leq\ 2\pi\,. \tag{4.133}$$

The following relationship to standard toroidal coordinates holds.
First set $y = \frac{1}{\sinh^2\eta}$ thereby

$$\frac{1}{\sqrt{1+y}} = \frac{\sinh\eta}{\cosh\eta}\,,\ \sqrt{\frac{y}{1+y}} = \frac{1}{\cosh y\eta}\ ,\ p = \frac{1}{\cosh\eta}(\cosh\eta\,\cos\zeta - \cos\xi), \tag{4.134}$$

and further

$$x^0 = \frac{a\,\sin\zeta\,\cosh\eta}{\cosh\eta\,\cos\zeta - \cos\xi}\ , \tag{4.135}$$

$$x_1 = \frac{a\,\cos\varphi\,\sinh\eta}{\cosh\eta\,\cos\zeta - \cos\xi}\ ,\ x_2 = \frac{a\,\sin\varphi\,\sinh\eta}{\cosh\eta\,\cos\zeta - \cos\xi}\ ,\ x_3 = \frac{a\sin\xi}{\cosh\eta\,\cos\zeta - \cos\xi} \tag{4.136}$$

Then if we set $\zeta = 0$ we find

$$x^0 = 0\,, \tag{4.137}$$

$$x_1 = \frac{a\,\cos\varphi\,\sinh\eta}{\cosh\eta - \cos\xi}\,,\; x_2 = \frac{a\,\sin\varphi\,\sinh\eta}{\cosh\eta - \cos\xi}\,,\; x_3 = \frac{a\sin\xi}{\cosh\eta - \cos\xi}\,, \tag{4.138}$$

that is standard toroidal coordinates.
The inverse coordinate transformations are given by

$$y = \frac{(a^2+s^2)^2 + 4a^2x_3^2}{4a^2\rho^2}\,,\; \tan\varphi = \frac{x_2}{x_1}\,,\; \tan\zeta = \frac{2ax_0}{a^2-s^2}\,,\; \tan\xi = -\frac{2ax_3}{a^2+s^2} \tag{4.139}$$

where s is the Minkowski length of (x_0, x_1, x_2, x_3) and ρ is the radius in cylinder coordinates i.e.

$$s^2 = x_0^2 - \rho^2 - x_3^2\,,\; \rho^2 = x_1^2 + x_2^2\,. \tag{4.140}$$

Setting $\tau = \frac{ct}{a}$ we see

$$\begin{aligned}
&\tau^2\,p^2 = 1 - \cos^2\zeta \\
&\Rightarrow\; \tau^2\,(\cos^2\zeta + \cos^2\xi\frac{y}{1+y} - 2\cos\zeta\,\cos\xi\sqrt{\frac{y}{1+y}}) = 1 - \cos^2\zeta \\
&\Rightarrow \cos^2\zeta - \frac{\tau^2}{1+\tau^2}\,2\,\cos\xi\,\sqrt{\frac{y}{1+y}}\cos\zeta - \frac{1}{1+\tau^2}(1 - \tau^2\,\cos\xi\frac{y}{1+y}) = 0 \\
&\Rightarrow \cos\zeta_{1,2} = \frac{\tau^2}{1+\tau^2}\,\cos^2\xi\,\sqrt{\frac{y}{1+y}} \pm \sqrt{\left(\frac{\tau^2}{1+\tau^2}\right)^2\cos^2\xi\,\frac{y}{1+y} + \frac{1}{1+\tau^2}(1 - \tau^2\,\cos^2\xi\frac{y}{1+y})} \\
&= \frac{1}{1+\tau^2}(\tau^2\,\cos\xi\,\sqrt{\frac{y}{1+y}} \pm \sqrt{\tau^2\,\cos^2\xi\,\frac{y}{1+y} + (1+\tau^2)\cdot(1 - \tau^2\,\cos^2\xi\frac{y}{1+y}})) \\
&= \frac{1}{1+\tau^2}(\tau^2\,\cos\xi\,\sqrt{\frac{y}{1+y}} \pm \sqrt{1 + \tau^2 - \tau^2\,\cos^2\xi\frac{y}{1+y}}) \\
&= \frac{1}{(1+\tau^2)\sqrt{1+y}}(\tau^2\,\cos\xi\,\sqrt{y} \pm \sqrt{1 + y + \tau^2(1 + y\sin^2\xi}))\,.
\end{aligned} \tag{4.141}$$

Thereby we may express p through

$$p = \frac{1}{1+\tau^2}\frac{1}{\sqrt{1+y}}\left(\pm\sqrt{(1+y) + \tau^2(1 + y\sin^2\xi)} - \cos\xi\sqrt{y}\right)\,. \tag{4.142}$$

The coordinates then take the form

$$x_1 = \frac{a}{q}\cos\varphi\,,\; x_2 = \frac{a}{q}\sin\varphi\,,\; x_3 = \frac{a}{q}\sin\xi\sqrt{y} \tag{4.143}$$

where we have defined $q := p\cdot\sqrt{1+y}$.
The ansatz made in [Fer06] is the following

$$u = \sqrt{\frac{(1-g)}{g}}\,e^{i(m_1\xi + m_2\phi + m_3\zeta)} \tag{4.144}$$

where g only depends on y and $0 \le g \le 1$. This ansatz implies the following form for $\vec{n}$

$$\vec{n} = \begin{pmatrix} 2\sqrt{g(1-g)}\,\cos(m_1\,\xi + m_2\,\varphi + m_3\,\zeta) \\ 2\sqrt{g(1-g)}\,\sin(m_1\,\xi + m_2\,\varphi + m_3\,\zeta) \\ 1-2g \end{pmatrix}. \tag{4.145}$$

Single valuedness of this ansatz requires

$$m_1,\, m_2 \in \mathbb{Z}\,,\; m_1 + m_2 + m_3 \in 2\mathbb{Z}. \tag{4.146}$$

Inserting (4.144) into the e.o.m. one finds

$$\partial_y(\Delta\partial_y g) = 0 \;\text{ where }\; \Delta := m_1^2\,(1+y) + m_2^2\,y(1+y) - m_3^2\,y\,. \tag{4.147}$$

Analysis of the solutions leads to further restrictions on m_1, m_2, m_3 in order to yield physical sensible solutions. One finds that not both

$$(m_1+m_3)/m_2 \ge 1 \;\text{ and }\; (m_1-m_3)/m_2 \le -1 \tag{4.148}$$

can hold as well as not both

$$(m_1-m_3)/m_2 \ge 1 \;\text{ and }\; (m_1+m_3)/m_2 \le -1\,. \tag{4.149}$$

The Hopf charge

In [Fer06] the Hopf charge of the $\vec{n}$ -field configuration is calculated in the following way. First map $\mathbb{R}^3$ to S^3 through

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-g}\, e^{i(\,m_1\xi + m_3\,\zeta} \\ \sqrt{g}\, e^{-i(\,m_2\phi} \end{pmatrix}\,,\; Z^\dagger Z = 1. \tag{4.150}$$

Then the map S^3 to S^2 is given by

$$u = \frac{z_1}{z_2}\,,\; n_i = Z^\dagger \sigma_i Z. \tag{4.151}$$

The Hop index then can be written as

$$Q_H = \frac{1}{4\pi^2}\int d^3x\,\vec{A}\cdot(\nabla\wedge\vec{A}) \tag{4.152}$$

with $\vec{A} = i(Z^\dagger\vec{\nabla}Z - \vec{\nabla}Z^\dagger Z)/2$.
We can express Z from u through

$$|z_1| = \frac{|u|}{\sqrt{1+|u|^2}}\,,\; |z_2| = \frac{1}{\sqrt{1+|u|^2}}\,. \tag{4.153}$$

Since $u = \frac{z_1}{z_2}$ only the relative phase matters and we may choose $\varphi_u = \varphi_{z_1}$ and $\varphi_{z_2} = 0$. Indeed a change of the phase of Z by a phase i.e $\tilde{Z} = e^{i\,\alpha}Z$ changes the potential $\vec{A}$ only through a gauge transformation

$$\begin{aligned} \tilde{A}_i &= \frac{i}{2}(\tilde{Z}^\dagger\nabla_i\tilde{Z} - \nabla_i\tilde{Z}^\dagger\tilde{Z}) = \frac{i}{2}(Z^\dagger(i\,\nabla_i\alpha\,Z + \nabla_i Z) - (-i\,\nabla_i\alpha\,Z^\dagger + \nabla_i Z^\dagger)Z) \\ &= \frac{i}{2}(i\,\nabla_i\alpha + (Z^\dagger\nabla_i Z + \nabla_i Z^\dagger Z)) = \frac{-\nabla_i\alpha}{2} + A_i\,. \end{aligned} \tag{4.154}$$

This however does not change the Hopf charge since obviously $\nabla \wedge \vec{A}$ is not influenced and further

$$\int d^3x(\frac{-\vec{\nabla}\alpha}{2} + \vec{A}) \cdot (\nabla \wedge \vec{A}) = \int d^3x(\frac{-\nabla_i\alpha}{2}\varepsilon_{ijk}\nabla_j A_k) + \vec{A} \cdot (\nabla \wedge \vec{A}) =$$
$$\int d^3x\nabla_i(\frac{-\alpha}{2}\varepsilon_{ijk}\nabla_j A_k) + \vec{A} \cdot (\nabla \wedge \vec{A}) = \int_{\mathcal{O}\to\infty}(\frac{-\alpha}{2}\varepsilon_{ijk}\nabla_j A_k) + \int d^3x\vec{A} \cdot (\nabla \wedge \vec{A}) \quad (4.155)$$

where me assume the surface term to vanish if there are no charges, i.e., for photons only . We may further express z_1, z_2 through the components of the $\vec{n}$ field in the following way

$$|z_1| = \sqrt{\frac{1+n_3}{2}} \, , \, |z_2| = \sqrt{\frac{1-n_3}{2}} \quad (4.156)$$

with relative phase $\varphi_{\Delta z_{12}} = \varphi_u = \arctan \frac{n_2}{n_1}$. So we may choose

$$z_1 = \sqrt{\frac{1+n_3}{2}} \frac{n_1 + i\, n_2}{\sqrt{n_1^2 + n_2^2}} = \frac{n_1 + i\, n_2}{\sqrt{2(1-n_3)}} \, , \, z_2 = \sqrt{\frac{1-n_3}{2}} \, . \quad (4.157)$$

Appendix A

Standard electrodynamics

A.1 Maxwell Equations

The classical Maxwell equations are (in SI-units)

$$\nabla \cdot \mathbf{B} = 0 \quad , \quad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0, \tag{A.1}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla \times \mathbf{B} - \frac{1}{c^2}\,\partial_t \mathbf{E} = \mu_0 \mathbf{j}. \tag{A.2}$$

The two equations (A.1) are called homogenous Maxwell equations, while the pair of equations (A.2) are referred to as the inhomogeneous Maxwell equations.
The homogenous Maxwell equations can be solved with help of the potentials $\phi(t, \mathbf{x})$, $\mathbf{A}(t, \mathbf{x})$, setting

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{A.3}$$

We combine the potentials $\phi, \mathbf{A}$ to give the four potential A^μ and the sources $\rho, \mathbf{j}$ to give four current j^μ

$$A^\mu = \begin{pmatrix} \frac{\phi}{c} \\ \mathbf{A} \end{pmatrix} \quad , \quad j^\mu = \begin{pmatrix} c\rho \\ \mathbf{j} \end{pmatrix}. \tag{A.4}$$

The electromagnetic fields can be defined with help of the field strength $F_{\mu\nu}$ tensor in the following way

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{A.5}$$

Defining the dual field strength as ${}^*F_{\mu\nu}$ through

$${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}, \tag{A.6}$$

we may write the homogenous and inhomogeneous Maxwell equations in the following fashion:

$$\partial_\rho {}^*F^{\rho\sigma} = 0\,, \tag{A.7}$$
$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{A.8}$$

In the case of vanishing sources ($j^\mu = 0$), the Maxwell equations are symmetric/invariant under the duality transformation

$$\mathbf{E} \to c\,\mathbf{B}\,,\; c\mathbf{B} \to -\mathbf{E}. \tag{A.9}$$

A.1.1 Maxwell equations with magnetic sources

One may maintain this symmetry in the case of non vanishing sources by introducing a magnetic charge density ρ_M, as well as a magnetic current $\mathbf{j}_M$ in the following way:

$$\nabla\cdot\mathbf{B} = \rho_M \quad , \quad \nabla\times\mathbf{E} + \partial_t\mathbf{B} = -\mathbf{j}_M, \tag{A.10}$$
$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla\times\mathbf{B} - \frac{1}{c^2}\,\partial_t\mathbf{E} = \mu_0\mathbf{j}. \tag{A.11}$$

This set of modified Maxwell equations is then invariant under the transformation

$$\begin{array}{ccc} \mathbf{E} \to c\,\mathbf{B} & \rho \to \varepsilon_0 c\rho_M & \mathbf{j} \to \varepsilon_0 c\mathbf{j}_M \\ c\mathbf{B} \to -\mathbf{E} & \varepsilon_0 c\rho_M \to -\rho & \varepsilon_0 c\mathbf{j}_M \to -\mathbf{j} \end{array} . \tag{A.12}$$

A.1.2 Conserved quantities

Specifying the expressions given in (3.22, 3.14, 3.13) we obtain the well known expressions for the energy and the spin part of the electromagnetic field

$$E = cP^0 = \frac{1}{2}\int d^3x\,(\varepsilon_0\,\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2) \tag{A.13}$$
$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{A.14}$$
$$\mathbf{L} = \frac{1}{\mu_0\,c^2}\int d^3x \sum_{i=1}^{3} E_j(\mathbf{x}\times\nabla)A_j \tag{A.15}$$
$$\mathbf{S} = \frac{1}{\mu_0\,c^2}\int d^3x\,\mathbf{E}\times\mathbf{A} \tag{A.16}$$

where in the second line we have introduced the electromagnetic vector potential $\mathbf{A}$ defined by equations (A.3,A.4).

A.2 Electromagnetic waves in vacuum and photon spin

The vacuum Maxwell equations lead to

$$\Box\mathbf{E} = 0 \quad , \quad \Box\mathbf{B} = 0 \tag{A.17}$$

which have the general solutions

$$\mathbf{E} = \mathbf{E}_0 \, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + c.c \tag{A.18}$$

$$\mathbf{B} = \mathbf{B}_0 \, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + c.c \tag{A.19}$$

with

$$\mathbf{k}\cdot\mathbf{E}_0 = \mathbf{k}\cdot\mathbf{B}_0 = 0 \quad , \quad \mathbf{k}^2 = \frac{\omega^2}{c^2} \, , \tag{A.20}$$

$$\mathbf{B}_0 = \frac{1}{k\,c}\mathbf{k}\times\mathbf{E}_0 \, . \tag{A.21}$$

We would like have an understanding in how far the expression for the spin part of the angular momentum reflects the fact the photon is a spin 1 particle. Following [Jac02] we consider a general plane wave solution given by its vector potential

$$\mathbf{A}(\mathbf{x},t) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^2} [\epsilon_\lambda(\mathbf{k}) \, a_\lambda(\mathbf{k}) \, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + c.c.] \tag{A.22}$$

where $\epsilon_{1,2}(\mathbf{k})$ are two linear independent polarization vectors lying in the plane perpendicular to $\mathbf{k}$.
The time average of the spin part of the angular momentum is then given by ([Jac02],p. 404)

$$\mathbf{S} = \frac{2}{\mu_0 \, c^2} \int \frac{d^3k}{(2\pi)^3} \, \omega \, \frac{\mathbf{k}}{k} \, [|a_+|^2 - |a_-|^2]. \tag{A.23}$$

The time average of the energy is given by

$$E = \frac{2}{\mu_0 \, c^2} \int \frac{d^3k}{(2\pi)} (a_+^2 + a_-^2) \, \omega^2 \tag{A.24}$$

Specifying these expressions to a specific wave vector $\mathbf{k}$ and projecting the spin in the direction of this wave vector we have

$$\mathbf{S}\cdot\mathbf{k}\,\frac{1}{k} = \frac{2}{\mu_0 \, c^2} \, \frac{1}{(2\pi)^3} \, \omega \, [|a_+(\mathbf{k})|^2 - |a_-(\mathbf{k})|^2]. \tag{A.25}$$

Comparing this to the energy we see that we have $\pm\hbar$ units of spin angular momentum for the modes $a_\pm$ per energy $\hbar\omega$.
A similar however slightly more complicated discussion using the decomposition of the electromagnetic field into spherical harmonics gives the same quantitative result, see ([Jac02], p. 499-503).

Appendix B

Notes on the Hopf map

The Hopf map [1] is an example of a homotopical non trivial map from S^3 to S^2, i.e., it cannot continuously be deformed to the constant map. It appears in many ways in physics, see for example [Urb03].
Following [Fab12] we construct the Hopf map in the following way. We define S^3 in the standard way as the set of points in $\mathbb{R}^4$ having unit distance from the origin i.e

$$S^3 := \{(q_0, \dots, q_3) \in \mathbb{R}^4 | \sum_{i=0}^{3} q_i^2 = 1\}. \tag{B.1}$$

We may introduce the two vectors

$$\vec{\rho}_0 := \begin{pmatrix} q_3 \\ q_0 \end{pmatrix} \quad , \quad \vec{\rho}_1 := \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{B.2}$$

and define S^3 is through $\vec{\rho}_0^2 + \vec{\rho}_1^2 = 1$. Rotating the two vectors $\vec{\rho}_0$ and $\vec{\rho}_1$ in $\mathbb{R}^2$ simultaneously by an angle ψ we move on a one dimensional closed curve on S^3. For $\psi = \pi$ we arrive at $-\vec{\rho}_0$, $-\vec{\rho}_1$ corresponding to the antipodal point from where we started. Indeed one can show that this rotation by ψ describes a great circle on S^3. For a pair of vectors $\vec{\rho}_0, \vec{\rho}_1$ with $\vec{\rho}_0^2 + \vec{\rho}_1^2 = 1$ the three quantities

$$\vec{\rho}_0 \cdot \vec{\rho}_1 \; , \; \vec{\rho}_0 \times \vec{\rho}_1 \; , \; \vec{\rho}_0^2 - \vec{\rho}_1^2 \tag{B.3}$$

are invariant under the simultaneous rotation of the two vectors and completely characterize the big circle. This also proves that any two big circles on S^3 do not intersect each other (as we will see below they are however "linked" together). Indeed any two big circles C, C' are parallel to each other in the sense that any two points on C' have the same distance from C ([Thu97], Exercise 2.7.1) .
The 3 invariants defined above may be collected into a three dimensional vector in the following way

$$\begin{aligned} n_1 &= 2(\vec{\rho}_0 \cdot \vec{\rho}_1) = 2(q_0 q_2 + q_1 q_3) \\ n_2 &= 2(\vec{\rho}_0 \times \vec{\rho}_1) = 2(q_2 q_3 - q_0 q_1) \\ n_3 &= \vec{\rho}_0^2 - \vec{\rho}_1^2 = -q_1^2 - q_2^2 + q_3^2 + q_0^2. \end{aligned} \tag{B.4}$$

[1] Found by Heinz Hopf in 1931 [Hei31] .

with $\vec{n}^2 = 1$. Since the three invariants each take values in $[-1, 1]$, $\vec{n}$ defines an S^2. The map (B.4) is just the map initially proposed by Hopf.
Alternatively we may assign an $SU(2)$ element to every point $q := (q_0, \vec{q})$ on S^3 through

$$Q := q_0 - i\vec{\sigma}\vec{q}. \tag{B.5}$$

Rotations of $\vec{e}_3 = (0, 0, 1)$ under these elements define an S^2 and the resulting map $S^3 \to SU(2) \to S^2$ is just again the Hopf map.
Another way to arrive at the Hopf map is by viewing $\mathbb{R}^4$ as $\mathbb{C}^2$. S^3 is then defined to be the set of points $(z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$. In this picture the big circles can be obtained by intersecting S^3 with a complex line (a one dimensional complex subspace) [Thu97].
Formally we get a fibre bundle $p : S^3 \to S^2$ with fiber S^1. The figure below shows what this fibration looks like under stereographic. In this figure the horizontal axis is the intersection with the complex line $z_1 = 0$ and the horizontal circle is the intersection with $z_2 = 0$. The set of

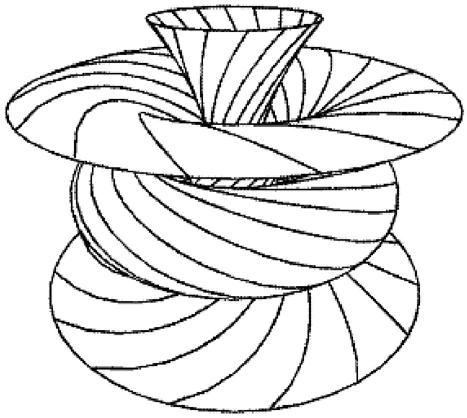

Figure B.1: Visualization of the Hopf map under stereographic projection, taken from [Thu97]

points on S^3 with $|z_2| \equiv a$ for some constant a form a torus. The big circles of S^3 are circles on this torus called Villarceau circles. They wind once around the $z_1 = 0$-circle and once around the $z_2 = 0$ circle. In fact any pair of such circles is linked once.
The Hopf map covers S^2 exactly once. This can is expressed with help of the Hopf invariant [Sch94]. For this let $f : S^3 \to S^2$ be a smooth map and ω a volume form on S^2 with $\int_{S^2} \omega = 1$. The pullback $f^*\omega$ of the volume form to S^3 is a closed form, since ω is necessarily closed on S^2 and pullbacks commute with exterior differentiation. It follows that $f^*\omega$ is exact since the second cohomology group of S^3 is trivial i.e. $H^2(S^3) = 0$. Let σ be a one-form on S^3 such that $f^*\omega = d\sigma$. Then the integral

$$H(f) = \int_{S^3} f^*\omega \wedge \sigma = \int_{S^3} d\sigma \wedge \sigma\,. \tag{B.6}$$

is the Hopf invariant. It can be proved to be an integer and not to depend on the specific choice of σ . Since $H(f)$ changes smoothly under smooth changes of f, being an integer it can not change at all under smooth deformations of f and so only depends on the homotopy class of f. In addition it is also equivalent to the linking number of two inverse images $f^-1(a)$,$f^-1(b)$, $a, b \in S^2$ see for example [RB82].

Appendix C

Further calculations

1. **Example 1:**(Prof. Faber)
   Starting with

$$\zeta = z - ct \quad , \quad \eta = x\cos(k_z\zeta) + y\sin(k_z\zeta), \tag{C.1}$$

$$\phi = k_r\eta \quad , \quad \theta = \phi + k_z\zeta = k_r\eta + k_z\zeta, \tag{C.2}$$

we define the $\vec{n}$ field

$$\vec{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{C.3}$$

Its partial derivatives are

$$\partial_x\vec{n} = k_r\cos(k_z\zeta)\begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \\ -\sin\theta \end{pmatrix} \quad , \quad \partial_y\vec{n} = k_r\sin(k_z\zeta)\begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \\ -\sin\theta \end{pmatrix} \tag{C.4}$$

$$\partial_z\vec{n} = k_r k_z\xi\begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \\ -\sin\theta \end{pmatrix} + k_z\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \tag{C.5}$$

where we have introduced $\xi := (-x\sin(k_z\zeta) + y\cos(k_z\zeta))$. The components of the connection are

$$\vec{\Gamma}_x = \partial_x\vec{n}\wedge\vec{n} = k_r\cos(k_z\zeta)\begin{pmatrix} \sin\phi + \cos\theta\sin\theta\cos\phi \\ -\cos\phi + \cos\theta\sin\theta\sin\phi \\ -\sin^2\theta \end{pmatrix} \tag{C.6}$$

$$\vec{\Gamma}_y = \partial_y\vec{n}\wedge\vec{n} = k_r\sin(k_z\zeta)\begin{pmatrix} \sin\phi + \cos\theta\sin\theta\cos\phi \\ -\cos\phi + \cos\theta\sin\theta\sin\phi \\ -\sin^2\theta \end{pmatrix} \tag{C.7}$$

$$\vec{\Gamma}_z = \partial_z\vec{n}\wedge\vec{n} = k_z(1+k_r\xi)\begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix} + k_r k_z\xi\begin{pmatrix} \cos\theta\sin\theta\cos\phi \\ \cos\theta\sin\theta\sin\phi \\ -\sin^2\theta \end{pmatrix} \tag{C.8}$$

$$\vec{\Gamma}_t = -c\vec{\Gamma}_z = c\vec{\Gamma}_0. \tag{C.9}$$

From this we can derive the stress tensor

$$\vec{R}_{yz} = \vec{\Gamma}_y \wedge \vec{\Gamma}_z = -k_r k_z \sin(k_z\zeta)\sin\theta\,\vec{n}\ ,\ \vec{R}_{zx} = \vec{\Gamma}_z \wedge \vec{\Gamma}_x = k_r k_z \cos(k_z\zeta)\sin\theta\,\vec{n} \quad \text{(C.10)}$$

$$\vec{R}_{xy} = \vec{\Gamma}_x \wedge \vec{\Gamma}_y = 0\ ,\ \vec{R}_{tx} = -c\vec{R}_{zx}\ ,\ \vec{R}_{ty} = c\vec{R}_{yz}\ ,\ \vec{R}_{tz} = 0. \quad \text{(C.11)}$$

We find the electromagnetic fields

$$\vec{E} = -\frac{e_0}{4\pi\varepsilon_0}\begin{pmatrix} \vec{R}_{yz}\cdot\vec{n} \\ \vec{R}_{zx}\cdot\vec{n} \\ \vec{R}_{xy}\cdot\vec{n} \end{pmatrix} = -\frac{e_0}{4\pi\varepsilon_0}\begin{pmatrix} -k_r k_z \sin(k_z\zeta)\sin\theta \\ k_r k_z \cos(k_z\zeta)\sin\theta \\ 0 \end{pmatrix}, \quad \text{(C.12)}$$

$$\vec{B} = -\frac{1}{c^2}\frac{e_0}{4\pi\varepsilon_0}\begin{pmatrix} \vec{R}_{tx}\cdot\vec{n} \\ \vec{R}_{ty}\cdot\vec{n} \\ \vec{R}_{tz}\cdot\vec{n} \end{pmatrix} = -\frac{1}{c}\frac{e_0}{4\pi\varepsilon_0}\begin{pmatrix} -k_r k_z \cos(k_z\zeta)\sin\theta \\ -k_r k_z \sin(k_z\zeta)\sin\theta \\ 0 \end{pmatrix}. \quad \text{(C.13)}$$

For the energy density we find

$$\Theta^{00} = \frac{e_0^2}{(4\pi)^2\varepsilon_0}\,k_r^2 k_z^2 \sin^2\theta. \quad \text{(C.14)}$$

The energy is given by

$$\begin{aligned} E &= \frac{\kappa^2\varepsilon_0}{c}\int d^3x\,k_z^2 k_r^2 \sin^2\theta = \frac{\kappa^2\varepsilon_0}{c}\int d^3x\,k_z^2 k_r^2 \sin^2(k_r(x\ \cos(k_z z) + y\ \sin(k_z z)) + k_z z) \\ &= \frac{\kappa^2\varepsilon_0}{c}\,k_z^2 k_r^2 \int dydz \frac{1}{4k_r\ \cos(k_z z)}(2\theta - \sin(2\theta))|_{x_0}^{x_1} = \quad \text{(C.15)} \\ &= \frac{\kappa^2\varepsilon_0}{c}\,k_z^2 k_r^2 \int dz \frac{1}{4k_r\ \cos(k_z z)}\Bigg(2[k_r(xy\ \cos(k_z z) + \frac{y^2}{2}\sin(k_z z)) + k_z\,zy] + \\ &\quad \frac{1}{2\,k_r\ \sin(k_z z)}\cos(2\theta)\Bigg)\Bigg|_{x_0,y_0}^{x_1,y_1} \end{aligned}$$

For $\vec{\pi}^0$ we find

$$\begin{aligned} \vec{\pi}^0 &= -\frac{\alpha_f\hbar c}{4\pi}(\vec{n}\vec{R}^{0\nu})\partial_\nu\vec{n} = \frac{\alpha_f\hbar c}{4\pi}(\vec{n}\vec{R}^0_\nu)\partial_\nu\vec{n} = \frac{\alpha_f\hbar c}{4\pi}\left[(\vec{n}\vec{R}_{0x})\partial_x\vec{n} + (\vec{n}\vec{R}_{0y})\partial_y\vec{n}\right] \quad \text{(C.16)} \\ &= -\frac{\alpha_f\hbar c}{4\pi}k_r^2 k_z \sin\theta\begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}. \quad \text{(C.17)} \end{aligned}$$

We find $\vec{\pi}^0\cdot\vec{n} = 0$ and so for the spin

$$S_i = -\frac{1}{c}k_r^2 k_z\int d^3x \sin\theta\begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} = -\frac{1}{c}k_r^2 k_z\int d^3x \sin\theta\begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \\ -\sin\theta \end{pmatrix}. \quad \text{(C.18)}$$

2. **Example 2:**(Els)

The $\vec{n}$ field is given by

$$\begin{aligned} \vec{n} &= \vec{e}_z + \sin\omega\,\vec{e}_\omega\times\vec{e}_z + (1-\cos\omega)\,\vec{e}_\omega\times(\vec{e}_\omega\times\vec{e}_z) \quad \text{(C.19)} \\ &= \vec{e}_z + \sin\omega\,\vec{e}_\omega\times\vec{e}_z + (1-\cos\omega)\,((\vec{e}_\omega\cdot\vec{e}_z)\vec{e}_\omega - \vec{e}_z) \quad \text{(C.20)} \end{aligned}$$

with

$$\vec{e}_\omega = \frac{1}{1+R^2}\begin{pmatrix} 2x \\ 2y \\ R^2-1 \end{pmatrix} \quad , \quad R^2 = x^2+y^2 \quad , \quad ||\vec{e}_\omega|| = 1. \tag{C.21}$$

For the derivatives we get

$$\begin{aligned}
\partial_x \vec{n} &= \sin\omega\, \partial_x \vec{e}_\omega \times \vec{e}_z + (1-\cos\omega)(\partial_x(\vec{e}_\omega \cdot \vec{e}_z)\vec{e}_\omega + (\vec{e}_\omega \cdot \vec{e}_z)\partial_x \vec{e}_\omega)\,, && \text{(C.22)}\\
\partial_y \vec{n} &= \sin\omega\, \partial_y \vec{e}_\omega \times \vec{e}_z + (1-\cos\omega)(\partial_y(\vec{e}_\omega \cdot \vec{e}_z)\vec{e}_\omega + (\vec{e}_\omega \cdot \vec{e}_z)\partial_y \vec{e}_\omega)\,, && \text{(C.23)}\\
\partial_z \vec{n} &= \cos\omega\, \partial_z \omega\, \vec{e}_\omega \times \vec{e}_z + \sin\omega\, \partial_z\omega\, ((\vec{e}_\omega \cdot \vec{e}_z)\vec{e}_\omega - \vec{e}_z)\,, && \text{(C.24)}\\
\partial_t \vec{n} &= -c\,\partial_z \vec{n}. && \text{(C.25)}
\end{aligned}$$

General Note

With

$$\begin{aligned}
\vec{a} &= a_1\,\vec{e}_\omega + a_2\,\vec{e}_z + a_3\,\partial_x\vec{e}_\omega + a_4\,\vec{e}_\omega \times \vec{e}_z + a_5\,\partial_x\vec{e}_\omega \times \vec{e}_z\,, && \text{(C.26)}\\
\vec{b} &= b_1\,\vec{e}_\omega + b_2\,\vec{e}_z + b_3\,\partial_y\vec{e}_\omega + b_4\,\vec{e}_\omega \times \vec{e}_z + b_5\,\partial_y\vec{e}_\omega \times \vec{e}_z\,, && \text{(C.27)}\\
\vec{c} &= c_1\,\vec{e}_\omega + c_2\,\vec{e}_z + c_3\,\vec{e}_\omega \times \vec{e}_z. && \text{(C.28)}\\
& && \text{(C.29)}
\end{aligned}$$

we see

$$\begin{aligned}
\vec{a}\times\vec{c} &= [a_1c_3(\vec{e}_\omega\cdot\vec{e}_z) + a_2c_3 + a_3c_3\,\partial_x(\vec{e}_\omega\cdot\vec{e}_z) - a_4c_1(\vec{e}_\omega\cdot\vec{e}_z) - a_4c_2]\vec{e}_\omega && \text{(C.30)}\\
&+ [-a_1c_3 - a_2c_3(\vec{e}_z\cdot\vec{e}_\omega) + a_4c_1 + a_4c_2(\vec{e}_z\cdot\vec{e}_\omega) + a_5c_3(\vec{e}_\omega\times\vec{e}_z)\cdot\partial_x\vec{e}_\omega]\vec{e}_z\\
&- a_5c_1(\vec{e}_w\cdot\vec{e}_z)\partial_x\vec{e}_\omega\\
&+ (a_1c_2 - a_2c_1)\vec{e}_\omega\times\vec{e}_z\\
&+ a_3c_2\,\partial_x\vec{e}_\omega\times\vec{e}_z\\
&+ a_3c_1\,\partial_x\vec{e}_\omega\times\vec{e}_\omega.
\end{aligned}$$

3. **Example 3: A general Ansatz**
We may try the more general ansatz

$$\vec{n} = \begin{pmatrix} f(x,y,z-ct)\sin\alpha(x,y,z-ct) \\ g(x,y,z-ct)\sin\alpha(x,y,z-ct) \\ \cos\alpha(x,y,z-ct) \end{pmatrix} \quad , \quad f^2+g^2 = 1. \tag{C.31}$$

We find

$$\partial_x\vec{n} = \begin{pmatrix} \partial_x f\sin\alpha \\ \partial_x g\sin\alpha \\ 0 \end{pmatrix} + \partial_x\alpha \begin{pmatrix} f\cos\alpha \\ g\cos\alpha \\ -\sin\alpha \end{pmatrix}\,,\ \partial_y\vec{n} = \begin{pmatrix} \partial_y f\sin\alpha \\ \partial_y g\sin\alpha \\ 0 \end{pmatrix} + \partial_y\alpha \begin{pmatrix} f\cos\alpha \\ g\cos\alpha \\ -\sin\alpha \end{pmatrix} \tag{C.32}$$

$$\partial_z\vec{n} = \begin{pmatrix} \partial_z f\sin\alpha \\ \partial_z g\sin\alpha \\ 0 \end{pmatrix} + \partial_z\alpha \begin{pmatrix} f\cos\alpha \\ g\cos\alpha \\ -\sin\alpha \end{pmatrix} \tag{C.33}$$

Further

$$\partial_x\vec{n}\wedge\partial_y\vec{n} = \begin{pmatrix} 0 \\ 0 \\ \partial_x f\partial_y g - \partial_y f\partial_x g \end{pmatrix}\sin^2\alpha \tag{C.34}$$
$$+ \ \partial_x\alpha\begin{pmatrix} \partial_y g\sin^2\alpha \\ -\partial_y f\sin^2\alpha \\ (f\partial_y g - g\partial_y f)\sin\alpha\cos\alpha \end{pmatrix} + \partial_y\alpha\begin{pmatrix} -\partial_x g\sin^2\alpha \\ \partial_x f\sin^2\alpha \\ (-f\partial_x g + g\partial_x f)\sin\alpha\cos\alpha \end{pmatrix}$$

and

$$(\partial_x\vec{n}\wedge\partial_y\vec{n})\cdot\vec{n} = \tag{C.35}$$
$$= (\partial_x f\partial_y g - \partial_y f\partial_x g)\sin^2\alpha\cos\alpha + \sin\alpha\partial_x\alpha(f\partial_y g - g\partial_y f) + \sin\alpha\partial_y\alpha(g\partial_x f - f\partial_x g) \tag{C.36}$$

We examine the case where α depends on x and y through $R := (x^2+y^2)^{\frac{1}{2}}$. The second and third term of the equation above then are equal to,

$$\sin\alpha\partial_R\alpha(f(\frac{x}{R}\partial_y g - \frac{y}{R}\partial_x g) + g(\frac{y}{R}\partial_x f - \frac{x}{R}\partial_y f)). \tag{C.37}$$

4. **Example 4: (More concrete - 2dim Hedgehog)**

Let

$$\vec{n} = \begin{pmatrix} \frac{x}{R}\sin\alpha \\ \frac{y}{R}\sin\alpha \\ \cos\alpha \end{pmatrix} \quad , \quad R = (x^2+y^2)^{1/2} \quad , \quad \alpha = \alpha((z-ct), x, y). \tag{C.38}$$

Then

$$\partial_x\vec{n} = \begin{pmatrix} \frac{y^2}{R^3}\sin\alpha \\ -\frac{xy}{R^3}\sin\alpha \\ 0 \end{pmatrix} + \partial_x\alpha\begin{pmatrix} \frac{x}{R}\cos\alpha \\ \frac{y}{R}\cos\alpha \\ -\sin\alpha \end{pmatrix} , \ \partial_y\vec{n} = \begin{pmatrix} -\frac{xy}{R^3}\sin\alpha \\ \frac{x^2}{R^3}\sin\alpha \\ 0 \end{pmatrix} + \partial_y\alpha\begin{pmatrix} \frac{x}{R}\cos\alpha \\ \frac{y}{R}\cos\alpha \\ -\sin\alpha \end{pmatrix}$$
$$\partial_z\vec{n} = \partial_z\alpha\begin{pmatrix} \frac{x}{R}\cos\alpha \\ \frac{y}{R}\cos\alpha \\ -\sin\alpha \end{pmatrix} \tag{C.39}$$

and

$$\partial_x\vec{n}\wedge\partial_y\vec{n} = \partial_y\alpha\begin{pmatrix} \frac{xy}{R^3}\sin^2\alpha \\ \frac{y^2}{R^3}\sin^2\alpha \\ \frac{y}{R^2}\cos\alpha\sin\alpha \end{pmatrix} + \partial_x\alpha\begin{pmatrix} \frac{x^2}{R^3}\sin^2\alpha \\ \frac{xy}{R^3}\sin^2\alpha \\ \frac{x}{R^2}\cos\alpha\sin\alpha \end{pmatrix}$$
$$= \partial_y\alpha\ \sin\alpha\ \frac{y}{R^2}\cdot\vec{n} + \partial_x\alpha\ \sin\alpha\ \frac{x}{R^2}\cdot\vec{n}\,, \tag{C.40}$$
$$\partial_x\vec{n}\wedge\partial_z\vec{n} = \partial_z\alpha\begin{pmatrix} \frac{xy}{R^3}\sin^2\alpha \\ \frac{y^2}{R^3}\sin^2\alpha \\ \frac{y}{R^2}\cos\alpha\sin\alpha \end{pmatrix} = \partial_z\alpha\ \frac{y}{R^2}\,\vec{n}\,, \tag{C.41}$$
$$\partial_y\vec{n}\wedge\partial_z\vec{n} = \partial_z\alpha\begin{pmatrix} -\frac{x^2}{R^3}\sin^2\alpha \\ -\frac{xy}{R^3}\sin^2\alpha \\ -\frac{x}{R^2}\cos\alpha\sin\alpha \end{pmatrix} = \partial_z\alpha\ \frac{-x}{R^2}\,\vec{n}. \tag{C.42}$$

Finally we get for the electric field

$$
\begin{aligned}
E_z &= \underbrace{-\frac{e_0}{4\pi\varepsilon_0}}_{=\kappa}(\partial_x\vec{n}\wedge\partial_y\vec{n})\cdot\vec{n} = \kappa\sin\alpha\frac{1}{R^2}(y\partial_y\alpha + x\partial_x\alpha)\,, && \text{(C.43)}\\
E_x &= \kappa(\partial_y\vec{n}\wedge\partial_z\vec{n})\cdot\vec{n} = -\kappa\frac{x}{R^2}\sin\alpha\,\partial_z\alpha\,, && \text{(C.44)}\\
E_y &= \kappa(\partial_z\vec{n}\wedge\partial_x\vec{n})\cdot\vec{n} = -\kappa\frac{y}{R^2}\sin\alpha\,\partial_z\alpha\,, && \text{(C.45)}\\
& && \text{(C.46)}
\end{aligned}
$$

For $\alpha = \alpha(\frac{x}{y}, z)$ we have that $\vec{n} = \vec{n}(\frac{x}{y}, z)$ and $E_z = 0$.

Especially for $\alpha = \alpha(R, z)$ the z component of the electric field is

$$E_z = \kappa\sin\alpha\frac{\partial_R\alpha}{R}. \tag{C.47}$$

The e.o.m. demand this to be a function of z. This leads to

$$\sin\alpha\partial_R\alpha = -\partial_R(\cos\alpha) = R\,f(z) \Rightarrow -\cos\alpha = \frac{R^2}{2}\,f(z) + g(z) \tag{C.48}$$

where $g(z)$ is a so far undetermined function of z.
For$(\frac{R^2}{2}\,f(z) + g(z)) \in [-1, 1]$ this has the solution

$$\alpha = \arccos\left(-(\frac{R^2}{2}\,f(z) + g(z))\right) = \pi - \arccos\left(\frac{R^2}{2}\,f(z) + g(z)\right) \quad + \quad (2n\pi). \tag{C.49}$$

Inserting this into the original expression for the $\vec{n}$ field leads to

$$\vec{n} = \begin{pmatrix} \frac{x}{R}(1 - (\frac{R^2}{2}\,f(z) + g(z))^2)^{\frac{1}{2}} \\ \frac{y}{R}(1 - (\frac{R^2}{2}\,f(z) + g(z))^2)^{\frac{1}{2}} \\ -(\frac{R^2}{2}\,f(z) + g(z)) \end{pmatrix}. \tag{C.50}$$

In order to have a well defined expression at $R = 0$ we choose is

$$g(z) \equiv 1. \tag{C.51}$$

Thus the domain where above expressions is valid becomes $\frac{R^2}{2}\,f(z) \in [-2, 0]$ which is equivalent to

$$f(z) = 0 \quad \vee \quad (f(z) < 0\ ,\ R \in \left[0, \frac{2}{\sqrt{-f(z)}}\right]). \tag{C.52}$$

On the "boundaries" of this domain the $\vec{n}$ field takes the form:

$$f(z) = 0\ \vee\ R = 0 : \vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad , \quad R = \frac{2}{\sqrt{-f(z)}} : \vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{C.53}$$

Further inserting this solution in our expressions for the electric field we find

$$\begin{aligned} E_z &= \underbrace{-\frac{e_0}{4\pi\varepsilon_0}}_{=\kappa} f(z), &\text{(C.54)}\\ E_x &= -\kappa\frac{x}{2}f'(z)\,, &\text{(C.55)}\\ E_y &= -\kappa\frac{y}{2}f'(z)\,, &\text{(C.56)}\\ & &\text{(C.57)}\end{aligned}$$

For $\vec{\pi}^0$ we find

$$\begin{aligned} \vec{\pi}^0 &= E_y\partial_x\vec{n} - E_x\partial_y\vec{n} &\text{(C.58)}\\ &= \kappa\,\partial_z\alpha\sin^2\alpha\frac{1}{R^3}\begin{pmatrix} y\\ -x\\ 0\end{pmatrix}. &\text{(C.59)}\end{aligned}$$

5. **Example 5: In spherical coordinates**

We write the $\vec{n}$ field explicitly in spherical coordinates

$$\vec{n} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}\,,\ \theta = \theta(x,y,z-ct)\,,\ \phi = \phi(x,y,z-ct). \tag{C.60}$$

So

$$\begin{aligned} \partial_x\vec{n} &= \partial_x\theta\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix} + \partial_x\phi\begin{pmatrix}-\sin\theta\sin\phi\\ \sin\theta\cos\phi\\ 0\end{pmatrix}\,, &\text{(C.61)}\\ \partial_y\vec{n} &= \partial_y\theta\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix} + \partial_y\phi\begin{pmatrix}-\sin\theta\sin\phi\\ \sin\theta\cos\phi\\ 0\end{pmatrix}\,, &\text{(C.62)}\\ \partial_z\vec{n} &= \partial_z\theta\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix} + \partial_z\phi\begin{pmatrix}-\sin\theta\sin\phi\\ \sin\theta\cos\phi\\ 0\end{pmatrix}. &\text{(C.63)}\end{aligned}$$

This yields

$$\partial_x\vec{n}\wedge\partial_y\vec{n} = \sin\theta(\partial_x\theta\partial_y\phi - \partial_y\theta\partial_x\phi)\vec{n}. \tag{C.64}$$

In general we have

$$\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n} = \sin\theta(\partial_\zeta\theta\partial_\eta\phi - \partial_\eta\theta\partial_\zeta\phi)\vec{n}. \tag{C.65}$$

If θ only depends on $R = (x^2+y^2)^{\frac{1}{2}}$ then we obtain

$$\partial_x\vec{n}\wedge\partial_y\vec{n} = \frac{\partial_R\theta}{R}(x\partial_y\phi - y\partial_x\phi). \tag{C.66}$$

6. **Example 6:** $\vec{n} = \vec{n}(\eta(x,y,z,\zeta),\zeta)$

We start with an Ansatz in spherical coordinates

$$\vec{n} = \begin{pmatrix} \sin(\pi\eta\zeta)\ \sin(\pi\eta) \\ \sin(\pi\eta\zeta)\ \cos(\pi\eta) \\ \cos(\pi\eta\zeta) \end{pmatrix} \quad \text{for } 0 \le \zeta \le 1\,,\ \vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for } \zeta \le 0\,, \vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{for } \zeta \ge 1 \tag{C.67}$$

Let $R^2 = x^2 + y^2$ and define η to be the function

$$\eta(x,y,\zeta) = \begin{cases} (1-\cos((R+\zeta)\pi))/2\,, & \text{for } (1-\cos((R+\zeta)\pi))/2 \le 1 \\ 1\,, & \text{else} \end{cases} . \tag{C.68}$$

The electric field for in for $\vec{n} = \vec{n}(\eta,\zeta)$ has the general shape [FBK07]

$$\vec{E} = \kappa\,\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})\partial_z\zeta \begin{pmatrix} -\partial_y\eta \\ \partial_x\eta \\ 0 \end{pmatrix}. \tag{C.69}$$

We find

$$\partial_\zeta\vec{n} = \partial_\zeta(\pi\eta\zeta) \begin{pmatrix} \cos(\pi\eta\zeta)\sin(\pi\eta) \\ \cos(\pi\eta\zeta)\cos(\pi\eta) \\ -\sin(\pi\eta\zeta) \end{pmatrix}, \tag{C.70}$$

$$\partial_\eta\vec{n} = \pi\zeta \begin{pmatrix} \cos(\pi\eta\zeta)\sin(\pi\eta) \\ \cos(\pi\eta\zeta)\cos(\pi\eta) \\ -\sin(\pi\eta\zeta) \end{pmatrix} + \pi \begin{pmatrix} \sin(\pi\eta\zeta)\cos(\pi\eta) \\ -\sin(\pi\eta\zeta)\sin(\pi\eta) \\ 0 \end{pmatrix} \tag{C.71}$$

and so

$$(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})\cdot\vec{n} = (\partial_\zeta(\pi\eta\zeta)\,\pi\,(-\sin(\pi\eta\zeta))\vec{n})\cdot\vec{n} = (\partial_\zeta\cos(\pi\eta\zeta)\,\pi\,\vec{n})\cdot\vec{n}\,, \tag{C.72}$$

$$\partial_x\eta = \sin((R+\zeta)\pi)\cdot\frac{x}{R}\cdot\frac{\pi}{2}. \tag{C.73}$$

The electric field then takes the form

$$\vec{E} = \kappa\pi^2\,\partial_z\zeta\,(\partial_\zeta\cos(\pi\eta\zeta))\ \sin((R+\zeta)\pi))\frac{1}{2R} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{C.74}$$

If we change the definition of η to

$$\eta(x,y,\zeta) = \begin{cases} (1-\cos((R^2+\zeta)\pi))/2, & \text{for } (-\cos((R^2+\zeta)\pi))/2 \le 1 \\ 1, & \text{else} \end{cases} \tag{C.75}$$

we get

$$\vec{E} = \kappa\pi^2\,\partial_z\zeta\,(\partial_\zeta\cos(\pi\eta\zeta))\,(\sin((R^2+\zeta)\pi))/2 \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \tag{C.76}$$

Another alternative could be

$$\eta(x,y,\zeta) = \begin{cases} 1-e^{-\frac{1}{(1-R^2-\zeta)^2}}\cdot e\,, & \text{for } 1-e^{-\frac{1}{(1-R^2-\zeta)^2}}\cdot e \le 1 \\ 1\,, & \text{else} \end{cases} . \tag{C.77}$$

7. **Example 6: $\vec{n} = \vec{n}(\eta(x, y, \zeta), \zeta)$ and spherical coordiantes**

We define

$$\eta = (x^2 + y^2)^{\frac{1}{2}} \ , \ \zeta = z. \tag{C.78}$$

We use the standard ansatz for spherical coordinates

$$\vec{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{C.79}$$

where we choose

$$\theta = \arctan\left(k_1(\eta^2 + \zeta^2)^{\frac{1}{2}}\right) \ , \ \phi = \arctan\left(k_2(\eta^2\zeta)\right) . \tag{C.80}$$

That explicit inserting these expressions gives

$$\vec{n} = \begin{pmatrix} \frac{k_1(\eta^2+\zeta^2)^{\frac{1}{2}}}{\sqrt{1+k_1^2(\eta^2+\zeta^2)}} \frac{1}{\sqrt{1+k_2^2\eta^4\zeta^2}} \\ \frac{k_1(\eta^2+\zeta^2)^{\frac{1}{2}}}{\sqrt{1+k_1^2(\eta^2+\zeta^2)}} \frac{k_2\eta^2\zeta}{\sqrt{1+k_2^2\eta^4\zeta^2}} \\ \frac{1}{\sqrt{1+k_1^2(\eta^2+\zeta^2)}} \end{pmatrix} \tag{C.81}$$

We list the different quantities/derivatives we need

$$\partial_x\eta = \frac{x}{\eta} \ , \ \partial_y\eta \ = \ \frac{y}{\eta} \ , \ \partial_z\zeta = 1 \ , \tag{C.82}$$

$$\partial_\zeta\theta = \frac{k_1}{1 + k_1^2(\eta^2 + \zeta^2)} \frac{\zeta}{(\eta^2 + \zeta^2)^{\frac{1}{2}}} \quad , \quad \partial_\eta\theta = \frac{k_1}{1 + k_1^2(\eta^2 + \zeta^2)} \frac{\eta}{(\eta^2 + \zeta^2)^{\frac{1}{2}}} \ , \tag{C.83}$$

$$\partial_\zeta\phi = \frac{k_2}{1 + k_2^2\eta^4\zeta^2}\eta^2 \quad , \quad \partial_\eta\phi = \frac{k_2}{1 + k_2^2\eta^4\zeta^2} 2\eta\zeta \ . \tag{C.84}$$

For the further calculations we set $k_1 = k_2 = 1$ and get

$$\partial_\zeta\theta\partial_\eta\phi - \partial_\eta\theta\partial_\zeta\phi = \frac{\eta}{(1 + \eta^2 + \zeta^2)(\eta^2 + \zeta^2)^{\frac{1}{2}}(1 + \eta^4\zeta^2)} (2\zeta^2 - \eta^2) \ , \tag{C.85}$$

$$\partial_\eta\vec{n} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \cdot \partial_\eta\theta + \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix} \cdot \partial_\eta\phi \ . \tag{C.86}$$

and further

$$\partial_\zeta\vec{n} \wedge \partial_\eta\vec{n} \ = \ (\partial_\zeta\theta\partial_\eta\phi - \partial_\eta\theta\partial_\zeta\phi) \ \sin\theta \, \vec{n} \tag{C.87}$$

$$= \ \frac{\eta \, (2\zeta^2 - \eta^2)}{(1 + \eta^2 + \zeta^2)(\eta^2 + \zeta^2)^{\frac{1}{2}}(1 + \eta^4\zeta^2)} \ \sin\theta \, \vec{n} \tag{C.88}$$

$$= \ \frac{\eta \, (2\zeta^2 - \eta^2)}{(1 + \eta^2 + \zeta^2)^{\frac{3}{2}}(1 + \eta^4\zeta^2)} \, \vec{n}. \tag{C.89}$$

The electric field takes the form

$$\vec{E} = \kappa\vec{n}\cdot(\partial_\zeta\vec{n}\wedge\partial_\eta\vec{n})\partial_z\zeta\begin{pmatrix}-\partial_y\eta\\ \partial_x\eta\\ 0\end{pmatrix} \tag{C.90}$$

$$= \kappa\frac{(2\zeta^2-\eta^2)}{(1+\eta^4\zeta^2)(1+\eta^2+\zeta^2)^{\frac{3}{2}}}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix}, \tag{C.91}$$

is well defined everywhere, and zero along the z-axis and for $2\zeta^2-\eta^2=0$ that is for $z^2=\frac{1}{2}(x^2+y^2)$.
The energy is given by

$$E = \frac{\kappa^2\varepsilon_0}{c}\int d^3x\frac{\eta^2\,(2\zeta^2-\eta^2)^2}{(1+\eta^4\zeta^2)^2(1+\eta^2+\zeta^2)^3} \tag{C.92}$$

The spin takes the form

$$\vec{S} = -\frac{\kappa^2\,\varepsilon_0}{c}\int d^3x\frac{\eta\,(2\zeta^2-\eta^2)}{(1+\eta^2+\zeta^2)^{\frac{3}{2}}(1+\eta^4\zeta^2)}\partial_\eta\vec{n} \tag{C.93}$$

8. **Example 7: Inverse stereographic projection of the $z-(x^2+y^2)^{\frac{1}{2}}$ plane, of type $\vec{n}=\vec{n}(\eta(x,y,\zeta),\zeta)$**

We use an inverse stereographic projection to define $\vec{n}$ in the following way

$$\vec{n} = \frac{1}{R^2+1}\begin{pmatrix}2\eta\\ 2\zeta\\ R^2-1\end{pmatrix} \tag{C.94}$$

where we choose $\eta=k_1(x^2+y^2)^{\frac{1}{2}}$ and $\zeta=k_2z$ and define $R:=(\eta^2+\zeta^2)^{\frac{1}{2}}$. Since η only takes positive values we only cover half of S^2.
We find

$$\partial_x\eta = k_1^2\frac{x}{\eta}\;,\;\partial_y\eta = k_1^2\frac{y}{\eta}\;,\;\partial_z\zeta = 1k_2 \tag{C.95}$$

and using

$$\partial_\zeta\frac{1}{R^2+1} = \frac{-2\zeta}{(R^2+1)^2}\;,\;\partial_\zeta\frac{\zeta}{(1+R^2)} = \frac{\eta^2-\zeta^2+1}{(R^2+1)^2}\;,\;\partial_\zeta\frac{R^2-1}{R^2+1} = \frac{4\zeta}{(1+R^2)^2} \tag{C.96}$$

we find

$$\partial_\zeta\vec{n} = \frac{1}{(1+R^2)^2}\begin{pmatrix}-4\eta\zeta\\ 2(\eta^2-\zeta^2+1)\\ 4\zeta\end{pmatrix}, \tag{C.97}$$

$$\partial_\eta\vec{n} = \frac{1}{(1+R^2)^2}\begin{pmatrix}2(\zeta^2-\eta^2+1)\\ -4\eta\zeta\\ 4\eta\end{pmatrix}. \tag{C.98}$$

We get

$$\partial_\varsigma \vec{n} \wedge \partial_\eta \vec{n} = \frac{4}{(1+R^2)^2}\vec{n} \tag{C.99}$$

The electric field strength takes the form

$$\vec{E} = \kappa \frac{4}{(1+R^2)^2}\, k_2 \begin{pmatrix} -\frac{k_1^2 y}{\eta} \\ \frac{k_1^2 x}{\eta} \\ 0 \end{pmatrix} \tag{C.100}$$

and thus is not well defined along the z axis.
Calculating the energy we get (where we introduce the variables $\tilde{x} = k_1 x, \tilde{y} = k_1 y, \tilde{z} = k_2 z$, $d^3x = \frac{1}{k_1^2 k_2} d^3\tilde{x}$)

$$E = \frac{\varepsilon_0}{c}\int d^3x (\vec{E})^2 = \frac{\kappa^2 \varepsilon_0}{c}\frac{1}{k_1^2 k_2}\int d^3\tilde{x} \frac{16\, k_2^2 k_1^2}{(1+R^2)^4} \tag{C.101}$$

$$= \frac{\kappa^2\varepsilon_0}{c} 4\pi\, 16\, k_2 \underbrace{\int_0^\infty dR \frac{R^2}{(1+R^2)^4}}_{=\frac{\pi}{32}} = \frac{\kappa^2\varepsilon_0}{c} 2\pi^2\, k_2 = \alpha_f \frac{\hbar\pi}{2}\, k_2. \tag{C.102}$$

For the components of the spin we find

$$S_1 = \frac{-8\kappa^2\,\varepsilon_0}{c}\int d^3\tilde{x}\,\frac{1}{k_2 k_1^2}\frac{\tilde{z}^2 - (\tilde{x}^2+\tilde{y}^2)+1}{(1+\tilde{x}^2+\tilde{y}^2+\tilde{z}^2)^4}\, k_2 k_1^2 \tag{C.103}$$

$$= \frac{-8\kappa^2\,\varepsilon_0}{c}\int_{-\infty}^{\infty} d\tilde{z}\int_0^{2\pi} d\varphi \int_0^\infty d\rho\,\rho\,\frac{\tilde{z}^2-\rho^2+1}{(1+\rho^2+\tilde{z}^2)^4} = -\frac{2}{3}\frac{\pi^2\kappa^2\,\varepsilon_0}{c} \tag{C.104}$$

$$S_2 = \frac{16\kappa^2\,\varepsilon_0}{c}\int d^3x \frac{z\,(x^2+y^2)^{\frac{1}{2}}}{(1+x^2+y^2+z^2)^4} = 0 \tag{C.105}$$

$$S_3 = \frac{-16\kappa^2\,\varepsilon_0}{c}\int d^3\tilde{x}\,\frac{1}{k_2 k_1^2}\frac{(\tilde{x}^2+\tilde{y}^2)^{\frac{1}{2}}}{(1+\tilde{x}^2+\tilde{y}^2+\tilde{z}^2)^4}\, k_2 k_1^2$$

$$= \frac{-16\kappa^2\,\varepsilon_0}{c}\int_{-\infty}^{\infty} dz \int_0^{2\pi} d\varphi \int_0^\infty d\rho \frac{\rho^2}{(1+\rho^2+z^2)^4} \tag{C.106}$$

$$= -\frac{4}{3}\frac{\pi^2\kappa^2\varepsilon_0}{c} \tag{C.107}$$

so

$$\vec{S} = -\frac{2\pi^2\kappa^2\,\varepsilon_0}{c}\begin{pmatrix} 1/3 \\ 0 \\ 2/3 \end{pmatrix} = -\frac{2\pi^2\kappa^2\,\varepsilon_0}{c}\left(\frac{1}{3}, 0, \frac{2}{3}\right)^T. \tag{C.108}$$

We find

$$\frac{\vec{S}}{E} = -\left(\frac{1}{3}, 0, \frac{2}{3}\right)^T \frac{1}{k_2} \tag{C.109}$$

9. **Spherical coordinates**

We use spherical coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\,\cos\varphi\,\sin\vartheta \\ R\,\sin\varphi\,\sin\vartheta \\ R\,\cos\vartheta \end{pmatrix} \tag{C.110}$$

$$R = (x^2 + y^2 + z^2), \vartheta = \arccos\frac{z}{R}, \varphi = \arctan\frac{y}{x} \tag{C.111}$$

and try the special form $\vec{n} = \vec{n}(x, y, z, R - ct)$. We list the derivatives we will need

$$\partial_x R = \frac{x}{R} = \cos\varphi\,\sin\vartheta\,,\ \partial_y R = \frac{y}{R} = \sin\varphi\,\sin\vartheta\,,\ \partial_z R = \frac{z}{R} = \cos\vartheta \tag{C.112}$$

$$\partial_x\varphi = \frac{-y}{x^2+y^2} = \frac{-\sin\varphi}{R\,\sin\vartheta}\,,\ \partial_y\varphi = \frac{x}{x^2+y^2} = \frac{\cos\varphi}{R\,\sin\vartheta}\,,\ \partial_z\varphi = 0 \tag{C.113}$$

$$\partial_x\vartheta = \frac{z}{R^2}\frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{\cos\varphi\,\cos\vartheta}{R}\,,\ \partial_y\vartheta = \frac{z}{R^2}\frac{y}{(x^2+y^2)^{\frac{1}{2}}} = \frac{\sin\varphi\,\cos\vartheta}{R}\,, \tag{C.114}$$

$$\partial_z\vartheta = -\frac{(x^2+y^2)^{\frac{1}{2}}}{R^2} = -\frac{\sin\vartheta}{R} \tag{C.115}$$

From the explicit

We need the following expressions

$$\begin{aligned}
\partial_0\vec{n}\wedge\partial_x\vec{n} &= -\partial_R\vec{n}\wedge\partial_x\vec{n} = -\partial_x\varphi\,(\partial_R\vec{n}\wedge\partial_\varphi\vec{n}) - \partial_x\vartheta(\partial_R\vec{n}\wedge\partial_\vartheta\vec{n}) && \text{(C.116)}\\
\partial_x\vec{n}\wedge\partial_y\vec{n} &= (\partial_x R\partial_y\varphi - \partial_y R\partial_x\varphi)(\partial_R\vec{n}\wedge\partial_\varphi\vec{n}) && \text{(C.117)}\\
&+ (\partial_x R\partial_y\vartheta - \partial_y R\partial_x\vartheta)(\partial_R\vec{n}\wedge\partial_\vartheta\vec{n}) && \text{(C.118)}\\
&+ (\partial_x\varphi\partial_y\vartheta - \partial_x\vartheta\partial_y\varphi)(\partial_\varphi\vec{n}\wedge\partial_\vartheta\vec{n}). && \text{(C.119)}
\end{aligned}$$

Further

$$\begin{aligned}
&(\partial_x R\,\partial_y\varphi - \partial_y R\,\partial_x\varphi) = \frac{1}{R}\,,\ (\partial_x R\,\partial_y\vartheta - \partial_y R\,\partial_x\vartheta) = 0\,,\\
&(\partial_x\varphi\,\partial_y\vartheta - \partial_x\vartheta\,\partial_y\varphi) = \frac{-1}{R^2\,\tan\vartheta}\\
&(\partial_x R\,\partial_z\varphi - \partial_z R\,\partial_x\varphi) = \frac{\sin\varphi}{R\tan\vartheta}\,,\ (\partial_x R\,\partial_z\vartheta - \partial_z R\,\partial_x\vartheta) = -\frac{\cos\varphi}{R}\,,\\
&(\partial_x\varphi\,\partial_z\vartheta - \partial_x\vartheta\,\partial_z\varphi) = \frac{\sin\varphi}{R^2}\\
&(\partial_y R\,\partial_z\varphi - \partial_z R\,\partial_y\varphi) = -\frac{\cos\varphi}{R\tan\vartheta}\,,\ (\partial_y R\,\partial_z\vartheta - \partial_z R\,\partial_y\vartheta) = -\frac{\sin\varphi}{R}\,,\\
&(\partial_y\varphi\,\partial_z\vartheta - \partial_y\vartheta\,\partial_z\varphi) = -\frac{\cos\varphi}{R^2}
\end{aligned}$$

Using these expressions we see that the electric and magnetic field take the form

$$\vec{E} = \kappa \begin{pmatrix} (\partial_y n \wedge \partial_z \vec{n}) \cdot \vec{n} \\ -(\partial_x n \wedge \partial_z \vec{n}) \cdot \vec{n} \\ (\partial_x n \wedge \partial_y \vec{n}) \cdot \vec{n} \end{pmatrix} \tag{C.120}$$

$$= -\kappa \left(\frac{1}{R^2 \sin\vartheta}\, \vec{e}_R (\partial_\varphi \vec{n} \wedge \partial_\vartheta \vec{n}) \cdot \vec{n} + \frac{1}{R}\, \vec{e}_\varphi (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) \cdot \vec{n} - \frac{1}{R \sin\vartheta}\, \vec{e}_\vartheta (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) \cdot \vec{n} \right)$$

$$\vec{B} = -\frac{\kappa}{c} \left(\frac{1}{R\, \sin\vartheta}\, \vec{e}_\varphi (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) \cdot \vec{n} + \frac{1}{R}\, \vec{e}_\vartheta (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) \cdot \vec{n} \right) \tag{C.121}$$

Further we find

$$\partial_x(\partial_x \vec{n} \wedge \partial_0 \vec{n}) = \partial_x^2 \varphi (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \partial_x^2 \vartheta (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) + \partial_x \varphi \partial_x (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \partial_x \vartheta \partial_x (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n})$$

$$\partial_x(\partial_x \vec{n} \wedge \partial_0 \vec{n}) + \partial_y(\partial_y \vec{n} \wedge \partial_0 \vec{n}) + \partial_z(\partial_z \vec{n} \wedge \partial_0 \vec{n}) =$$
$$= \triangle\varphi (\partial_r \vec{n} \wedge \partial_\varphi \vec{n}) + \triangle\vartheta (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) + \nabla\varphi \cdot \nabla(\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \nabla\vartheta \cdot \nabla(\partial_R \wedge \partial_\vartheta \vec{n})$$
$$= \frac{\cos\vartheta}{r^2 \sin\vartheta} (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) + \frac{1}{(r\, \sin\vartheta)^2} \partial_\varphi (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \frac{1}{r^2} \partial\vartheta (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) \tag{C.122}$$

$$\partial_0(\partial_0 \vec{n} \wedge \partial_x \vec{n}) = \partial_R(\partial_x \varphi (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \partial_x \vartheta (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n})) =$$
$$= \frac{\sin\varphi}{R^2\, \sin\vartheta} (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) - \frac{\cos\varphi \cos\vartheta}{R^2} (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) + \partial_x \varphi\, \partial_R (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \partial_x \vartheta\, \partial_R (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n})$$
$$= -\frac{1}{R} \big(\partial_x \varphi (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \partial_x \vartheta (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n}) \big) + \partial_x \varphi\, \partial_R (\partial_R \vec{n} \wedge \partial_\varphi \vec{n}) + \partial_x \vartheta\, \partial_R (\partial_R \vec{n} \wedge \partial_\vartheta \vec{n} \tag{C.123}$$

NOTATION and UNITS

4-vectors of Minkowski space will mostly be written with explicit indices and we will use greek letters for these indices and use latin letters if only refering to spatial components. At times it will be convenient to write the 3-vector of spatial components of a Minkowski 4-vector without indices. In such a case we will use **boldface** to indicate the vector character of the quantity. So we may write a four vector in the following ways

$$A_\mu = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} A_0 \\ A_i \end{pmatrix} = \begin{pmatrix} A_0 \\ \mathbf{A} \end{pmatrix}. \tag{C.124}$$

Further choices of notation are

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \;,\; x^\mu = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} \;,\; \partial_\mu = \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \frac{1}{c}\partial_t \\ \nabla \end{pmatrix} \tag{C.125}$$

$$\epsilon^{\mu\nu\rho\sigma} := \begin{cases} +1 & , \text{ even permutations of } 0123 \\ -1 & , \text{ uneven permutations of } 0123 \\ 0 & , \text{ else} \end{cases} . \tag{C.126}$$

The Pauli matrices are defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;,\; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{C.127}$$

We define

$$\sigma = \vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}. \tag{C.128}$$

Some constants and relations we will use are

$$c^2 = \frac{1}{\varepsilon_0\,\mu_0} \tag{C.129}$$

$$\alpha_f = \frac{e_0^2}{4\pi\varepsilon_0\hbar c} \quad , \text{ Sommerfelds finestructure constant} \tag{C.130}$$

$$\kappa = -\frac{e_0}{4\pi\varepsilon_0}. \tag{C.131}$$

Coordinate Systems:
Coordinate systems we will use for the inner/colorspace and spatial $\mathbb{R}^3$ will follow the following notation
Spatial coordinates:
Spherical coordinates: R, ϑ, φ
Cylinder coordinates: r, φ, z

Internal coordinates:
Spherical coordinates: θ, ϕ

Formulas

Some useful formulas.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}\,, \tag{C.132}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times b)\,, \tag{C.133}$$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \tag{C.134}$$

From $\vec{n}^2 = 1$ we get

$$\vec{n}\partial_x\vec{n} = 0 \tag{C.135}$$

$$\partial_x\vec{n} \cdot \partial_x\vec{n} = -\vec{n} \cdot \partial_x^2\vec{n}\,,\ \partial_y\vec{n} \cdot \partial_x\vec{n} = -\vec{n} \cdot \partial_x\partial_y\vec{n} \tag{C.136}$$

$$\partial_y\vec{n} \cdot \partial_x^2\vec{n} + 2\,\partial_x\partial_y\vec{n} \cdot \partial_x\vec{n} = -\vec{n} \cdot \partial_x^2\partial_y\vec{n}. \tag{C.137}$$

Units

We will check the units for some of the quantities we study.
We define the basic units through **:**
$[E]$...Energy
$[l]$...length
$[t]$...time
$[v]$...velocity
$[m]$...mass
$[0]$... number
$[q]$... charge

We list the units of some quantities:

$$[\alpha_f] = [0] \tag{C.138}$$

$$[\hbar] = [h] = [E][t] \tag{C.139}$$

$$\tag{C.140}$$

From the following formulas we can obtain units for the quantities defined by them:

$$S = \int dt \int d^3x \mathcal{L} \Rightarrow [\mathcal{L}] = \frac{[E]}{[l]^3} \tag{C.141}$$

$$\vec{\Gamma}_\mu = \partial_\mu \alpha \, \vec{n} + \sin\alpha \, \cos\alpha \, \partial_\mu \vec{n} + \sin^2 \alpha \, \vec{n} \times \partial_\mu \vec{n} \Rightarrow [\vec{n}] = [0] \, , \, [\vec{\Gamma}_\mu] = \frac{1}{[l]} \tag{C.142}$$

$$\vec{\pi}^\mu = \frac{\partial \mathcal{L}}{\partial \vec{\Gamma}_\mu} \Rightarrow [\vec{\pi}^\mu] = \frac{[E]}{[\frac{1}{l}]\,[l]^3} = \frac{[E]\,[t]}{[l][l][t]} = \frac{[\text{momentum}]}{[t]\,[l]} \tag{C.143}$$

$$S_i = -\frac{1}{c} \int d^3x \vec{\pi}^0 \sin\alpha \, [\cos\alpha \, \vec{n} \times \vec{e}_i + \sin\alpha \, \vec{n} \times (\vec{n} \times \vec{e}_i)] \tag{C.144}$$

$$\Rightarrow \quad [S_i] = \frac{1}{[v]} \, [l]^3 \, \frac{[\text{momentum}]}{[l][t]} = \frac{[t]}{[l]} \, [l]^3 \, \frac{[\text{momentum}]}{[l][t]} = [\text{momentum}][l] = \tag{C.145}$$

$$[\text{angular momentum}], \tag{C.146}$$

$$\kappa = -\frac{e_0}{4\pi\varepsilon_0} = \alpha_f \frac{\hbar \, c}{e_0} \Rightarrow [\kappa] = [\frac{\hbar \, c}{e_0}] = \frac{[E][l]}{[q]} \tag{C.147}$$

$$[\mathcal{L}] = \frac{[E]}{[l]^3} \tag{C.148}$$

$$[\hbar] = [h] = [E][t] \tag{C.149}$$

$$[\vec{n}] = [0] \tag{C.150}$$

$$[\vec{\Gamma}_\nu] = \frac{1}{[l]} \tag{C.151}$$

$$[\vec{\pi}^\mu] = \frac{[E]}{[v][l]^3} = \frac{[\text{momentum}]}{[l]^3} \tag{C.152}$$

$$[S_i] = [\text{momentum}][l] = [\text{angular momentum}] = [E][t] = [\text{action}] \tag{C.153}$$

$$[\kappa] = \frac{[E][l]}{[q]} \tag{C.154}$$

Checking the dimensions of some other derived formulas:

$$\vec{S} = \frac{\kappa \, \varepsilon_0}{c} \int d^3x \, (E_x(\partial_y \vec{n}) - E_y(\partial_x \vec{n})) = \frac{\kappa^2 \, \varepsilon_0}{c} \int d^3x \, ((\partial_y \vec{n} \wedge \partial_z \vec{n}) \cdot \vec{n} \, (\partial_y \vec{n}) - (\partial_z \vec{n} \wedge \partial_x \vec{n}) \cdot \vec{n} \, (\partial_x \vec{n}))$$

$$\Rightarrow [\vec{S}] = [\frac{\kappa^2 \, \varepsilon_0}{c}][l]^3 \frac{1}{[l]^3} = [\frac{\alpha_f \, \hbar c}{4\pi \, c}] = [\hbar] = [\text{action}] = [\text{angular momentum}]. \tag{C.155}$$

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