NUMERICAL INTEGRATION OF THE LANDAU-LIFSHITZ-GILBERT EQUATION

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Landau-Lifshitz-Gilbert equation (LLG)

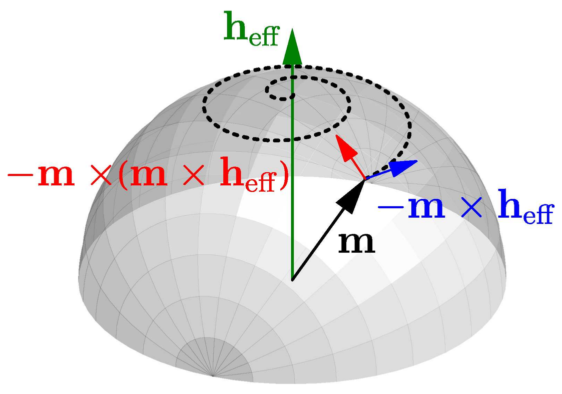
Problem formulation:

Find $\mathbf{m} : \Omega_T \to \mathbb{R}^3$ with $|\mathbf{m}| = 1$ such that

$$\mathbf{m}_t = -\frac{1}{1+\alpha^2}\mathbf{m} \times \mathbf{h}_{\text{eff}} - \frac{\alpha}{1+\alpha^2}\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}})$$

where

- $\alpha > 0 \rightsquigarrow$ Gilbert damping constant
- $\mathbf{h}_{\text{eff}} = C_{\text{exch}}\Delta\mathbf{m} + \boldsymbol{\pi}(\mathbf{m}) + \mathbf{f} \rightsquigarrow$ general effective field
- initial and boundary conditions
  - $\mathbf{m}(0) = \mathbf{m}_0$ with $\mathbf{m}_0 : \Omega \to \mathbb{R}^3$ s.t. $|\mathbf{m}_0| = 1$
  - $\frac{\partial \mathbf{m}}{\partial \mathbf{n}} = 0$ on $(0,T) \times \partial\Omega$



Total magnetic Gibbs free energy:

$$\mathcal{E}(\mathbf{m}) = \frac{C_{\text{exch}}}{2}\int_\Omega |\nabla \mathbf{m}|^2 - \int_\Omega \mathbf{f} \cdot \mathbf{m} - \frac{1}{2}\int_\Omega \boldsymbol{\pi}(\mathbf{m}) \cdot \mathbf{m}$$

Equivalent reformulations of LLG:

- Gilbert form $\rightsquigarrow \mathbf{m}_t - \alpha\mathbf{m} \times \mathbf{m}_t = \mathbf{h}_{\text{eff}} \times \mathbf{m}$
- alternative form $\rightsquigarrow \alpha\mathbf{m}_t - \mathbf{m}_t \times \mathbf{m} = \mathbf{h}_{\text{eff}} - (\mathbf{m} \cdot \mathbf{h}_{\text{eff}})\,\mathbf{m}$ and $|\mathbf{m}| = 1$

Numerical challenges:

- efficient treatment of nonlinearities
- non-convex side constraint $|\mathbf{m}| = 1$
- efficient computation of (nonlocal) field contributions

Weak formulation of LLG

Find $\mathbf{m} \in \mathbf{H}^1(\Omega_T)$, with $|\mathbf{m}| = 1$ a.e. in Ω_T, s.t.

- for all $\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega_T)$

$$\begin{aligned}(\mathbf{m}_t, \boldsymbol{\varphi})_{\Omega_T} - \alpha\,(\mathbf{m} \times \mathbf{m}_t, \boldsymbol{\varphi})_{\Omega_T} \\ = -C_{\text{exch}}\,(\nabla\mathbf{m} \times \mathbf{m}, \nabla\boldsymbol{\varphi})_{\Omega_T} + (\boldsymbol{\pi}(\mathbf{m}) \times \mathbf{m}, \boldsymbol{\varphi})_{\Omega_T} + (\mathbf{f} \times \mathbf{m}, \boldsymbol{\varphi})_{\Omega_T}\end{aligned}$$

- $\mathbf{m}(0) = \mathbf{m}_0$ in the sense of traces
- for a.e. $t \in (0,T)$

$$\mathcal{E}(\mathbf{m}(t)) + \alpha\int_{t'=0}^{t} \|\mathbf{m}_t(t')\|^2_{\mathbf{L}^2(\Omega)}\,dt \le \mathcal{E}(\mathbf{m}_0)$$

Standard tangent plane scheme

Predictor-corrector strategy: linear update + nodewise projection

- based on alternative form of LLG $\rightsquigarrow$ linear in $\mathbf{v} = \mathbf{m}_t$
- $|\mathbf{m}|^2 = 1 \Rightarrow \mathbf{m} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{v}$ belongs to tangent space
- discretization of field contributions included in analysis

Set of admissible magnetizations:

$$\mathcal{M}_h := \left\{\boldsymbol{\phi}_h \in \mathcal{S}^1(\mathcal{T}_h)^3 : |\boldsymbol{\phi}_h(\mathbf{z})| = 1 \text{ for all } \mathbf{z} \in \mathcal{N}_h\right\}$$

Discrete tangent space:

$$\mathcal{K}_{\mathbf{m}_h^i} := \left\{\boldsymbol{\psi}_h \in \mathcal{S}^1(\mathcal{T}_h)^3 : \boldsymbol{\psi}_h(\mathbf{z}) \cdot \mathbf{m}_h^i(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in \mathcal{N}_h\right\}$$

Time-marching scheme:

Let $0 \le \theta \le 1$. For $0 \le i \le N-1$ iterate:

- compute $\mathbf{v}_h^i \in \mathcal{K}_{\mathbf{m}_h^i}$ s. t.

$$\begin{aligned}\alpha\left(\mathbf{v}_h^i, \boldsymbol{\phi}_h\right)_\Omega + \left(\mathbf{m}_h^i \times \mathbf{v}_h^i, \boldsymbol{\phi}_h\right)_\Omega + C_{\text{exch}}\theta k\left(\nabla\mathbf{v}_h^i, \nabla\boldsymbol{\phi}_h\right)_\Omega \\ = -C_{\text{exch}}\left(\nabla\mathbf{m}_h^i, \nabla\boldsymbol{\phi}_h\right)_\Omega + \left(\boldsymbol{\pi}_h(\mathbf{m}_h^i), \boldsymbol{\phi}_h\right)_\Omega + \left(\mathbf{f}_h^i, \boldsymbol{\phi}_h\right)_\Omega\end{aligned}$$

  for all $\boldsymbol{\phi}_h \in \mathcal{K}_{\mathbf{m}_h^i}$;
- define $\mathbf{m}_h^{i+1} \in \mathcal{M}_h$ by $\mathbf{m}_h^{i+1}(\mathbf{z}) = \dfrac{\mathbf{m}_h^i(\mathbf{z}) + k\mathbf{v}_h^i(\mathbf{z})}{|\mathbf{m}_h^i(\mathbf{z}) + k\mathbf{v}_h^i(\mathbf{z})|}$ for all $\mathbf{z} \in \mathcal{N}_h$

Convergence result:

Unconditional convergence (up to a subsequence) towards a weak solution of LLG provided

- $1/2 < \theta \le 1$, angle condition on triangulation $\mathcal{T}_h$
- uniform boundedness of $\boldsymbol{\pi}_h$
- weak convergence properties of $\boldsymbol{\pi}_h$, $\mathbf{f}_h$, $\mathbf{m}_h^0$

Sketch of proof:

- boundedness of discrete energy
- abstract arguments provide convergent subsequences
- identify limit with weak solution of LLG

Projection-free tangent plane scheme

Nodewise projection step is removed $\Rightarrow \mathbf{m}_h^{i+1} = \mathbf{m}_h^i + k\mathbf{v}_h^i \in \mathcal{S}^1(\mathcal{T}_h)^3$

Main features:

- fully linear scheme for strongly nonlinear PDE
- unconditional convergence result is preserved
- avoid angle condition on triangulation $\mathcal{T}_h$
- unit-length constraint is violated at nodes of triangulation
- violation is controlled by time-step size independently of number of iterations

Coupling with other PDEs

Effective field often comprises nonlocal field contributions

$\Rightarrow$ (nonlinear) coupling of LLG with other PDEs

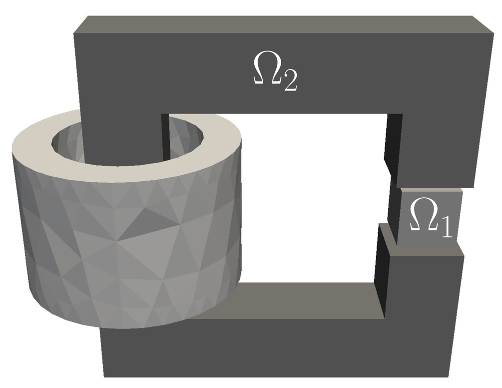
Examples:

- multiscale modeling
- spin-polarized transport in ferromagnetic multilayers

Multiscale modeling

Setting:

- multiple domains Ω_1 and Ω_2 of different scales
- solve LLG on microscopic domain Ω_1
- magnetostatic Maxwell equations and nonlinear material law on macroscopic domain Ω_2



Mathematical treatment:

- strongly monotone operator
- multiscale contribution discretized by Johnson-Nédélec FEM-BEM coupling
- satisfies all assumptions for convergence of tangent plane scheme

Spin-polarized transport

Setting:

- interaction between electric current and magnetization
- Ω multilayer, $\omega \subset \Omega$ ferromagnetic part
- LLG coupled with diffusion equation for spin accumulation field

$$\begin{aligned}&\mathbf{m}_t = -\mathbf{m} \times (\mathbf{h}_{\text{eff}} + c\mathbf{s}) + \alpha\mathbf{m} \times \mathbf{m}_t && \text{in } \omega_T \\ &\mathbf{s}_t = -\nabla \cdot (\beta\mathbf{m} \otimes \mathbf{j} - D_0\,(\nabla\mathbf{s} - \beta\beta'\mathbf{m} \otimes (\nabla\mathbf{s} \cdot \mathbf{m}))) - D_0\mathbf{s} - D_0\,(\mathbf{s} \times \mathbf{m}) && \text{in } \Omega_T\end{aligned}$$

Remarks:

- decoupled algorithm for coupled system
- only two linear systems per time-step

References

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