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Reproducing kernel almost Pontryagin spaces ☆

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ABSTRACT

An almost Pontryagin space $\mathcal{A}$ is an inner product space which admits a direct and orthogonal decomposition of the form $\mathcal{A} = \mathcal{A}_>[\dot{+}]\mathcal{A}_\leq$ with a Hilbert space $\mathcal{A}_>$ and a finite-dimensional negative semidefinite space $\mathcal{A}_\leq$. A reproducing kernel almost Pontryagin space is an almost Pontryagin space of functions (defined on some nonempty set and taking values in some Krein space), with the property that all point evaluation functionals are continuous. We address two problems.
1° In the presence of degeneracy, it is not possible to reproduce function values as inner products with a kernel function in the usual way. We obtain a natural substitute for a kernel function, and study the relation between spaces and kernel functions in detail.
2° Given an inner product space $\mathcal{L}$ of functions, does there exist a reproducing kernel almost Pontryagin space $\mathcal{A}$ which contains $\mathcal{L}$ isometrically? We characterise those spaces for which the answer is "yes". We show that, in case of existence, there is a unique such space $\mathcal{A}$ which contains $\mathcal{L}$ isometrically and densely. Its geometry, in particular its degree of degeneracy, is an important invariant of $\mathcal{L}$. It plays a role in connection with Krein's formula describing generalised resolvents and, thus, in several concrete problems related with the extension theory of symmetric operators.

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1. Introduction

A *reproducing kernel Hilbert space* is a Hilbert space whose elements are functions, and which has the property that all point evaluation functionals are continuous. This type of spaces appears in many branches of mathematics, e.g., in functional analysis, complex analysis, statistics, etc., and plays an important role. As prominent examples, let us mention the Hardy space on the upper half-plane $H^2(\mathbb{C}^+)$ and its shift-coinvariant subspaces (model subspaces, in particular de Branges spaces), cf. [24,33,12], the Dirichlet space $\mathcal{D}$ on the unit disk, cf. [36,5], or the Bergman space, cf. [16]. A (small) selection of literature about theory and application of reproducing kernel Hilbert spaces, ranging from classical papers to recent work, is [6,42,17,34,13,21].

Within the theory of indefinite inner product spaces, *reproducing kernel Pontryagin spaces* play a similarly important role. Examples are de Branges Pontryagin spaces (which appear in the study of canonical systems with inner singularities or Schrödinger equations with strongly singular potentials), cf. [27,26,18,32], or generalised Dirichlet spaces (which appear in the theory of univalent functions), cf. [37, §7.5–6]. Again, there is a vast literature; a (again small) selection of literature being [39,15,3,14,4,10,20,1,19].

In some contexts subspaces of reproducing kernel spaces appear naturally (prominently, in the theory of de Branges spaces). It is obvious that continuity of point evaluations is inherited by subspaces. However, a subspace of a Pontryagin space need not be a Pontryagin space. Provided it is closed, the obstacle is possible presence of degeneracy, i.e., existence of nonzero elements of the subspace which are orthogonal to the whole subspace. Each Pontryagin space which is not a Hilbert space has nontrivial closed and degenerated subspaces. Even more, in many situations degenerated subspaces contain crucial information about the structure of the space (as illustrated, e.g., by the constructions in [26, §2.b]).

In the present paper we axiomatically consider the kind of spaces described above: possibly degenerated closed subspaces of reproducing kernel Pontryagin spaces. We call this type of spaces *reproducing kernel almost Pontryagin spaces*. Thereby we allow the elements of the space to be Krein space valued functions. Our aim is to settle two questions which appear in this context. The first is existence of kernel functions, and the second existence of reproducing kernel space completions. Let us explain these problems in some more detail. Thereby we use some general vocabulary from the theory of reproducing kernel spaces and indefinite inner product spaces. The reader who is not familiar with this language will find the relevant definitions in Section 2 below.

Problem I. Kernel functions. At the very basis of the theory of reproducing kernel spaces lies the fact that a reproducing kernel Hilbert- or Pontryagin space $\mathcal{A}$ can be fully described by a single function of two variables, its *reproducing kernel*. Namely (e.g., let $\mathcal{A}$ be a reproducing kernel Hilbert space of complex-valued functions defined on some set M), there exists a unique function $K : M \times M \to \mathbb{C}$ such that function values of elements of $\mathcal{A}$ are reproduced by means of the formula

$$\begin{aligned} &K(\eta,\cdot)\in\mathcal{A}, \quad \eta\in M, \\ &f(\eta)=\big[f,K(\eta,\cdot)\big]_{\mathcal{A}}, \quad f\in\mathcal{A},\eta\in M. \end{aligned} \tag{1.1}$$

Here we denote by $K(\eta,\cdot)$ the function $\zeta\mapsto K(\eta,\zeta)$, $\zeta\in M$, and by $[\cdot,\cdot]_{\mathcal{A}}$ the inner product of the space $\mathcal{A}$.

When degeneracy is permitted there appears an obvious problem:

If $\mathcal{A}$ is degenerated, then there cannot exist a kernel K with (1.1). For if $f\in\mathcal{A}$ with $[f,g]_{\mathcal{A}}=0$, $g\in\mathcal{A}$, then from (1.1) it follows that $f=0$.

We show that this problem can be resolved in a certain sense. Given a reproducing kernel almost Pontryagin space $\mathcal{A}$ (of functions defined on some nonempty set and taking values in some Krein space), there exist functions K (we call them *almost reproducing kernels*), such that reproduction of function values is established by a formula very close to (1.1). This formula is a finite-dimensional perturbation of (1.1), cf. (3.1). The existence result we show, cf. Theorem 3.2, is a generalisation and refinement of [29, Proposition 5.3] (and its proof runs along the same lines). Each almost reproducing kernel of $\mathcal{A}$ is a hermitian kernel with finite negative index. The Pontryagin space generated by it coincides with $\mathcal{A}$ as a linear space and topologically, and its inner product is a finite-dimensional perturbation of the inner product of $\mathcal{A}$.

Conversely, we show that each hermitian kernel with finite negative index generates an infinite family of reproducing kernel almost Pontryagin spaces, cf. Definition 3.7 and Proposition 3.8. The proof of this fact is geometric and uses knowledge from the Pontryagin space situation.

Contrasting the nondegenerated situation, almost reproducing kernel Pontryagin spaces on the one hand and almost reproducing kernels on the other are not anymore in a one-to-one correspondence. The results mentioned above are accompanied by a detailed description of the relation between kernel functions on the one hand and reproducing kernel spaces on the other. For a comprehensively formulated summary see p. 292.

✎ The results concerning reproduction of function values are presented in Section 3 of this paper.

Problem II. Reproducing kernel space completions. In the investigation of various topics the following question appears. Given a positive semidefinite inner product space whose elements are functions, is its Hilbert space completion a reproducing kernel Hilbert space? Depending whether the answer to this question is "yes" or "no", the objects under investigation enjoy very different properties. Interestingly, also if the Hilbert space completion is not a reproducing kernel space, there may exist reproducing kernel almost Pontryagin spaces which isometrically contain the given inner product space as a dense subspace. The reason for this phenomenon is that requiring continuity of point evaluations may force presence of degeneracy. The fact whether or not such reproducing kernel almost Pontryagin spaces exist again has consequences on the problem.

A typical example of a concrete topic where existence of reproducing kernel completions plays a key role is the Hamburger moment problem. Given a positive measure, one can build a certain positive semidefinite inner product space. Depending whether or not its Hilbert space completion is a reproducing kernel space, the measure is indeterminate or determinate. Existence of a reproducing kernel almost Pontryagin space containing this inner product space isometrically is related to finiteness of index of determinacy (a notion which was studied in [9] and following papers). A thorough discussion of this topic is elaborate and beyond the scope of this paper; it will be presented in the forthcoming manuscript [31].

The question under consideration at present is the abstract one:

Given a (not necessarily positive definite) inner product space $\mathcal{L}$ of functions, does there exist a reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically as a dense subspace (we speak of a reproducing kernel space completion*)?*

We answer this question in Theorem 4.1. The proof of this theorem relies on the theory of almost Pontryagin space completions as developed in [40]. Besides the obvious condition of finiteness of negative index of $\mathcal{L}$, there appear two other relevant conditions, see Theorem 4.1, (B) and (C). Though looking similar on first sight, their roles are clearly distinguished: (C) is responsible for well-definedness and (B) for continuity of point evaluations.

An important feature is that, in case of existence, the reproducing kernel space completion is unique. The dimension of its isotropic part is an intrinsic quantity associated with the given inner product space. In applications, its value reflects in properties of the concrete problem.

The main characterisation Theorem 4.1 is accompanied by Proposition 4.8 and Proposition 4.9 where we provide an (often generic) example, and show that one may restrict to subspaces which possess a certain density property.

✎ The results concerning reproducing kernel space completions are presented in Section 4 of this paper.

The present paper is a continuation of our investigation of the geometry of almost Pontryagin spaces undertaken in [29] and [40]. In the same time it is a preparation for future work. It lays the foundations for our forthcoming manuscripts [41] (where we deal with the operator theoretic concept of directing functionals and discuss the special case of de Branges space completions), and, building further upon this, [31] (where we present a new – and more general – approach to the index of determinacy of a measure).

Almost Pontryagin spaces can be viewed as a “mildly degenerated” version of Pontryagin spaces. Recently, a similar mildly degenerated version of Krein spaces (termed almost Krein spaces) occurred and was studied in the context of basicity properties of selfadjoint operators, cf. [7]. We do not attempt at present to investigate “reproducing kernel almost Krein spaces”. The reason being that reproducing kernel Krein spaces were thoroughly studied (see, e.g., [19]), and it turned out that the situation is much more complex than in the Pontryagin space case. Also, up to the best of our knowledge, a systematic treatment of the geometry of almost Krein spaces is not yet available. Altogether, at present, there seems little hope to obtain complete and satisfactory results about reproducing kernels in this general situation; of course, this is a potential direction of future research.

Besides the already mentioned Sections 3 and 4, which form the core of the paper, the manuscript contains a short preliminary section (Section 2) and Appendix A. In Section 2 we recall the definition of almost Pontryagin spaces and define the central notion of reproducing kernel almost Pontryagin space of Krein space valued functions on a nonempty set, cf. Definition 2.4. Moreover, we recall some known facts from the nondegenerated (Pontryagin space) situation. In Appendix A we prove some statements concerning the basic theory of almost Pontryagin spaces. These are general facts which are needed in the present paper, but are not yet available in the literature. Some are straightforward generalisations of well-known results from Pontryagin space theory. More noticeable is the perturbation result, Proposition A.9, which is a most practical tool. This result has already appeared (and was extensively used) in a special situation, cf. [27, Theorem 3.3]. Here we provide a general version and give a geometric proof of it. This is based on Lemma A.10 which contains a topological property and is interesting in its own right. Appendix A closes with some supplements to the theory of almost Pontryagin space completions.

2. Continuity of point evaluations

Standard literature on indefinite inner product spaces is [11,22,2]. We use without further notice the notion of a Pontryagin space and basic results about Pontryagin spaces as found in [22, Chapter 1] (notice that in this book the roles of positive and negative subspaces are switched compared to what is common nowadays).

Notation from linear algebra 2.1. An *inner product space* is a pair $\langle\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}\rangle$ consisting of a linear space $\mathcal{L}$ over the scalar field $\mathbb{C}$ and an inner product $[\cdot,\cdot]_{\mathcal{L}}$ on $\mathcal{L}$. If no confusion can occur, we drop explicit notation of the inner product, and speak of an inner product space $\mathcal{L}$. Inner products are denoted by rounded or square brackets and have attached a subscript specifying on which space they are defined or how they are built, e.g., $(\cdot,\cdot)_{\mathcal{K}}$ is an inner product on a space $\mathcal{K}$.

Let $\mathcal{L}$ be an inner product space. A subspace $\mathcal{N}$ of $\mathcal{L}$ is called *negative*, if $[x, x] < 0$, $x \in \mathcal{N} \setminus \{0\}$. The *negative index of* $\mathcal{L}$ is the (possibly infinite) number

$$\operatorname{ind}_- \mathcal{L} := \sup\{\dim \mathcal{N} : \ \mathcal{N} \text{ negative subspace of } \mathcal{L}\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Note that we do not distinguish between different cardinalities of infinity. This notice applies always, in particular whenever we speak of the dimension of a subspace.

We denote by $\mathcal{L}^\circ$ the *isotropic part of* $\mathcal{L}$, that is the linear subspace

$$\mathcal{L}^\circ := \mathcal{L} \cap \mathcal{L}^\perp = \{x \in \mathcal{L} : [x,y]_\mathcal{L} = 0, y \in \mathcal{L}\}.$$

The dimension $\operatorname{ind}_0 \mathcal{L} := \dim \mathcal{L}^\circ \in \mathbb{N}_0 \cup \{\infty\}$ is called the *degree of degeneracy of* $\mathcal{L}$. We call $\mathcal{L}$ *nondegenerated* if $\operatorname{ind}_0 \mathcal{L} = 0$ and *degenerated* if $\operatorname{ind}_0 \mathcal{L} > 0$. ♢

Let us now recall the definition of an almost Pontryagin space. The below given axiomatic way of defining this type of spaces was introduced in [29] where we also started a systematic investigation of the properties of such spaces. The type of spaces itself of course appeared much earlier, one may say ever since Pontryagin spaces were studied. However, with a few exception, they did not receive much attention until recently. To say some historical words, existence of discontinuous isometric and bijective linear operators was observed at an early stage in the 1960's (an accessible reference being [22, Example 6.1]), and basicity properties of selfadjoint operators were studied in [8]. More recent literature is [30] were degenerated spaces appear in the study of operator pencils, [28] were a version of Krein's resolvent formula is proved, and [35] were selfadjoint operators were studied and used to investigate the Klein–Gordon equation.

Definition 2.2. We call a triple $\langle \mathcal{A}, [\cdot,\cdot]_\mathcal{A}, \mathcal{O}\rangle$ an *almost Pontryagin space*, if $\mathcal{A}$ is a linear space, $[\cdot,\cdot]_\mathcal{A}$ is an inner product on $\mathcal{A}$, and $\mathcal{O}$ is a topology on $\mathcal{A}$, such that the following axioms hold.

(aPs1) The topology $\mathcal{O}$ is a Hilbert space topology on $\mathcal{A}$ (i.e., it is induced by some inner product which turns $\mathcal{A}$ into a Hilbert space).

(aPs2) The inner product $[\cdot,\cdot]_\mathcal{A}$ is $\mathcal{O}$-continuous (i.e., it is continuous as a map of $\mathcal{A} \times \mathcal{A}$ into $\mathbb{C}$ where $\mathcal{A} \times \mathcal{A}$ carries the product topology $\mathcal{O} \times \mathcal{O}$ and $\mathbb{C}$ the Euclidean topology).

(aPs3) There exists an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension in $\mathcal{A}$, such that $\langle \mathcal{M}, [\cdot,\cdot]_\mathcal{A}|_{\mathcal{M}\times\mathcal{M}}\rangle$ is a Hilbert space.

Let $\langle \mathcal{A}, [\cdot,\cdot]_\mathcal{A}, \mathcal{O}\rangle$ and $\langle \mathcal{B}, [\cdot,\cdot]_\mathcal{B}, \mathcal{T}\rangle$ be two almost Pontryagin spaces, then we call a map $\psi : \mathcal{A} \to \mathcal{B}$ an *isomorphism* if ψ is a linear bijection of $\mathcal{A}$ onto $\mathcal{B}$, is isometric w.r.t. $[\cdot,\cdot]_\mathcal{A}$ and $[\cdot,\cdot]_\mathcal{A}$, and homeomorphic w.r.t. $\mathcal{O}$ and $\mathcal{T}$. ♢

Unless necessary we do not notate the inner product $[\cdot,\cdot]_\mathcal{A}$ and the topology $\mathcal{O}$ explicitly, and speak of an almost Pontryagin space $\mathcal{A}$. When speaking of topological properties like convergence or Cauchy-sequences in an almost Pontryagin space, we also say *w.r.t. the norm of* $\mathcal{A}$, meaning w.r.t. some norm inducing the topology of $\mathcal{A}$. Of course, there are many such norms, but each two are equivalent. Finally, note that for each almost Pontryagin space $\mathcal{A}$ it holds that $\operatorname{ind}_- \mathcal{A} < \infty$ and $\operatorname{ind}_0 \mathcal{A} < \infty$.

To have a more concrete picture of almost Pontryagin space, recall the following facts. Thereby, the equivalence of (i) and (ii) is shown in [29, Proposition 2.5], and the equivalence of (i) and (iii) is contained in [29, Propositions 3.1, 3.2].

Theorem 2.3. *(See [29].) Let $\mathcal{A}$ be a linear space, let $[\cdot,\cdot]_{\mathcal{A}}$ be an inner product on $\mathcal{A}$, and let $\mathcal{O}$ be a Hilbert space topology on $\mathcal{A}$. Then the following statements are equivalent.*

(i) $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ *is an almost Pontryagin space.*

(ii) *We have* $\operatorname{ind}_0 \mathcal{A} < \infty$*. The space $\mathcal{A}$ can be decomposed as the direct and orthogonal sum*

$$\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{A}_-[\dot{+}]\mathcal{A}^\circ,$$

with a finite dimensional negative subspace $\mathcal{A}_-$ and an $\mathcal{O}$-closed subspace $\mathcal{A}_+$ such that $\langle \mathcal{A}_+, [\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{A}_+\times\mathcal{A}_+}\rangle$ is a Hilbert space.

(iii) *There exists a Pontryagin space which contains $\mathcal{A}$ as a closed subspace and has the property that $[\cdot,\cdot]_{\mathcal{A}}$ and $\mathcal{O}$ coincide with the Pontryagin space inner product and topology restricted to $\mathcal{A}$.*

As a corollary we see that an inner product space is a Pontryagin space, if and only if it is nondegenerated and there exists a topology $\mathcal{O}$ which turns $\mathcal{A}$ into an almost Pontryagin space, cf. [29, Corollary 2.7]. Moreover, if $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ is a positive definite almost Pontryagin space, then $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}\rangle$ is a Hilbert space and the topology induced on $\mathcal{A}$ by $[\cdot,\cdot]_{\mathcal{A}}$ equals $\mathcal{O}$.

In this context, let us point out that a nondegenerated inner product space $\mathcal{A}$ may carry at most one topology $\mathcal{O}$ so that it becomes an almost Pontryagin space. In the presence of degeneracy this uniqueness property is lost, see, e.g., [22, Example 6.1] or [29, Lemma 2.8].

The objects of our study in this paper are almost Pontryagin spaces of functions where point evaluations are continuous. To fix notation: Let M be a nonempty set and let $\mathcal{K}$ be a Krein space. For each $\eta \in M$, we denote

$$\chi_\eta : \begin{cases} \mathcal{K}^M \to \mathcal{K} \\ f \mapsto f(\eta) \end{cases}$$

and speak of the *point evaluation functional at* η. Moreover, for each $a \in \mathcal{K}$ and $\eta \in M$ we set

$$\chi_{\eta,a} : \begin{cases} \mathcal{K}^M \to \mathbb{C} \\ f \mapsto (f(\eta), a)_{\mathcal{K}} \end{cases}$$

Definition 2.4. Let M be a nonempty set and let $\mathcal{K}$ be a Krein space. We call an almost Pontryagin space $\langle\mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ a *reproducing kernel almost Pontryagin space of* $\mathcal{K}$*-valued functions on* M, if

(RKS1) The elements of $\mathcal{A}$ are $\mathcal{K}$-valued functions on M, and the linear operations of $\mathcal{A}$ are given by pointwise addition and scalar multiplication.

(RKS2) For each $\eta \in M$ the point evaluation functional $\chi_\eta|_{\mathcal{A}} : \mathcal{A} \to \mathcal{K}$ is continuous w.r.t. the topology $\mathcal{O}$ on $\mathcal{A}$ and the Krein space topology on $\mathcal{K}$.

We denote the set of all reproducing kernel almost Pontryagin spaces of $\mathcal{K}$-valued functions on M as $\mathbb{RKS}(\mathcal{K}, M)$. ◊

At this place the usage of the term "reproducing kernel almost Pontryagin space" it is not justified by anything but analogy to the nondegenerated case. We will see in Section 3 that there is indeed a good reason to use this terminology, namely existence of a substitute for the reproducing kernel in the nondegenerated situation (namely, almost reproducing kernels).

Remark 2.5. We specified in (RKS2) that continuity is understood w.r.t. the Krein space topology of $\mathcal{K}$. This may be replaced by the (a priori weaker) requirement that χ_η is continuous w.r.t. the weak topology of $\mathcal{K}$: *An almost Pontryagin space* $\langle\mathcal{A}, [\cdot,\cdot], \mathcal{O}\rangle$ *is a reproducing kernel almost Pontryagin space of* $\mathcal{K}$*-valued functions on M if and only if it satisfies* (RKS1) *and*

(RKS2′) For each $a \in \mathcal{K}$ and $\eta \in M$ we have $\chi_{\eta,a} \in \mathcal{A}'$.

To see this, follow the argument in the first paragraph of the proof of [3, Theorem 1.1.2]: Denote by $\mathcal{T}$ the norm topology and by $\mathcal{T}_w$ the weak topology of $\mathcal{K}$. Clearly, $\mathcal{O}$-to-$\mathcal{T}_w$-continuity of χ_η implies that the graph of χ_η is, as a subset of $\mathcal{A} \times \mathcal{K}$ closed w.r.t. $\mathcal{O} \times \mathcal{T}_w$. Since $\mathcal{T}_w$ is coarser than the norm topology, it follows that it is also closed w.r.t. $\mathcal{O} \times \mathcal{T}$. Now the Closed Graph Theorem implies that χ_η is $\mathcal{O}$-to-$\mathcal{T}$-continuous. ◊

Remark 2.6. The topology of a reproducing kernel almost Pontryagin space is uniquely determined by its inner product. This is an immediate consequence of [29, Proposition 2.9] applied with the point separating family $\{\chi_{\eta,a} : a \in \mathcal{K}, \eta \in M\}$.

Hence, the property of being a reproducing kernel almost Pontryagin space is a property of the inner product space alone. We thus may say without ambiguity that an inner product space is (or is not) a reproducing kernel almost Pontryagin space. ◊

The Pontryagin space case

Let us very briefly revisit the well-known nondegenerated case, and recall some notions and facts needed in the sequel.

First, the notion of a hermitian kernel.

Definition 2.7. Let M be a nonempty set and let $\mathcal{K}$ be a Krein space. A function $K : M \times M \to \mathcal{K}^{\mathcal{K}}$ is called a *$\mathcal{K}$-valued hermitian kernel on M*, if

$$K(\eta,\zeta) \in \mathcal{B}(\mathcal{K}), \qquad K(\eta,\zeta)^* = K(\zeta,\eta), \quad \eta,\zeta \in M.$$

Here $\mathcal{B}(\mathcal{K})$ denotes the set of all continuous linear operators on $\mathcal{K}$, and * denotes the Krein space adjoint. We denote the set of all $\mathcal{K}$-valued hermitian kernels on M by $\mathbb{K}(\mathcal{K}, M)$.

If K is a $\mathcal{K}$-valued hermitian kernel on M, we denote by $\operatorname{ind}_- K \in \mathbb{N}_0 \cup \{\infty\}$ the supremum of the numbers of negative squares of quadratic forms

$$Q(\xi_1,\ldots,\xi_m) := \sum_{i,j=1}^{m} \big(K(\eta_i,\eta_j)a_i,a_j\big)_{\mathcal{K}} \xi_i\overline{\xi_j}, \quad m \in \mathbb{N}_0,\ a_i \in \mathcal{K},\ \eta_i \in M,$$

and refer to $\operatorname{ind}_- K$ as the *negative index of K*. $\Diamond$

The fact that the topological dual space of a Pontryagin space is exhausted by the functionals $[\cdot, y]_{\mathcal{A}}$, $y \in \mathcal{A}$, leads to the following result, cf. [3, Theorem 1.1.2].

Theorem 2.8. *(See, e.g., [3].) Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{A}$ be a reproducing kernel Pontryagin space of $\mathcal{K}$-valued functions on M. Then there exists a unique function $K : M \times M \to \mathcal{K}^{\mathcal{K}}$, such that*

$$\begin{aligned} &K(\eta,\cdot)a \in \mathcal{A}, \quad a \in \mathcal{K},\ \eta \in M, \\ &\big(f(\eta),a\big)_{\mathcal{K}} = \big[f, K(\eta,\cdot)a\big]_{\mathcal{A}}, \quad f \in \mathcal{A},\ a \in \mathcal{K},\ \eta \in M. \end{aligned} \tag{2.1}$$

This function is a $\mathcal{K}$-valued hermitian kernel on M, and $\operatorname{ind}_- K = \operatorname{ind}_- \mathcal{A}$.

The unique function K whose existence is ensured by the above theorem is called the *reproducing kernel of $\mathcal{A}$*.

Also a converse result holds, cf. [3, Theorem 1.1.3]. The proof is established by taking the Pontryagin space completion of a certain inner product space generated from the given kernel.

Theorem 2.9. *(See, e.g., [3].) Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let K be a $\mathcal{K}$-valued hermitian kernel on M with* $\operatorname{ind}_- K < \infty$. *Then there exists a unique* *reproducing kernel Pontryagin space $\mathcal{A}$ of $\mathcal{K}$-valued functions on M, such that K is the reproducing kernel of $\mathcal{A}$.*

Another way of dealing with of reproducing kernel Pontryagin (or Krein-) spaces is via *Kolmogoroff decompositions*. See, e.g., the decomposition of K in [3, Theorem 1.1.2], or [14] for the Krein space case. In degenerated spaces, this approach seems problematic; at least there are serious obstacles originating in the fact that adjoint operators cannot be well-defined as soon as degeneracy is present.

3. Reproduction of function values

As we already explained in the introduction, the traditional formula (2.1) cannot hold if $\mathcal{A}$ is degenerated. Hence, we need to define in which sense we wish to understand reproduction of function values in a reproducing kernel almost Pontryagin space. Of course, the formula to be invented should reduce to the classical one if the space is nondegenerated. Experience gives the hint to use particular finite rank perturbations. Compare, e.g., with the method used in [27, Theorem 3.3] or, specifically, with the treatment of the scalar-valued case in [29, Proposition 5.3].

Definition 3.1. Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{A}$ be a reproducing kernel almost Pontryagin space of $\mathcal{K}$-valued functions on M. We call a function $K : M \times M \to \mathcal{K}^{\mathcal{K}}$ an *almost reproducing kernel of* $\mathcal{A}$, if it satisfies the following axioms.

(aRK1) K is a $\mathcal{K}$-valued hermitian kernel on M.
(aRK2) For each $a \in \mathcal{K}$ and $\eta \in M$ the function $K(\eta, \cdot)a$ belongs to $\mathcal{A}$.
(aRK3) There exists data $\mathfrak{d} = ((a_i)_{i=1}^n; (\eta_i)_{i=1}^n; (\gamma_i)_{i=1}^n) \in \mathcal{K}^n \times M^n \times \mathbb{R}^n$ where $n := \operatorname{ind}_0 \mathcal{A}$, such that

$$\big(f(\eta), a\big)_{\mathcal{K}} = \big[f, K(\eta, \cdot)a\big]_{\mathcal{A}} + \sum_{i=1}^{n} \gamma_i \cdot \chi_{\eta_i,a_i}(f)\overline{\chi_{\eta_i,a_i}\big(K(\eta, \cdot)a\big)},$$
$$f \in \mathcal{A},\ a \in \mathcal{K},\ \eta \in M. \quad \Diamond \tag{3.1}$$

Note that (aRK3) is only meaningful in presence of (aRK2). Hence, whenever we refer to (aRK3) or to (3.1), we tacitly assume that (aRK2) holds.

The requirement that $n = \operatorname{ind}_0 \mathcal{A}$ in (aRK3) is made to ensure that the perturbation is not unnecessarily large. A formula of the form (3.1) may hold also for larger values of n (with all γ_i being nonzero), however, one must allow at least $\operatorname{ind}_0 \mathcal{A}$ summands additional to $[f, K(\eta, \cdot)a]_{\mathcal{A}}$ so that (3.1) can possibly hold, cf. Lemma 3.4(i).

➪ In the following we investigate in detail the relation between reproducing kernel almost Pontryagin spaces on the one hand and almost reproducing kernels on the other. In abstract terms, we consider a relation Ξ between the set of "hermitian kernels plus data δ" on the one hand and the set of "reproducing kernel almost Pontryagin spaces" on the other. At the end of this section (see p. 292) we return to this – abstract, but comprehensive and clean – viewpoint, and indicate what the proven statements mean in terms of Ξ. We recommend the reader to visit this summary already during the presentation.

Denote by Ξ the relation between the sets $\mathbb{K}(\mathcal{K}, M) \times \bigcup_{n\in\mathbb{N}_0}(\mathcal{K}^n \times M^n \times \mathbb{R}^n)$ and $\mathbb{RKS}(\mathcal{K}, M)$ defined as

$$\Xi := \big\{\big((K;\delta);\mathcal{A}\big) : \ K \text{ is almost reproducing kernel of } \mathcal{A} \text{ with data } \delta \text{ in (3.1)}\big\}.$$

Our first result about almost reproducing kernels is that each reproducing kernel almost Pontryagin space possesses many almost reproducing kernels. The scalar-valued case was studied previously in [29, Proposition 5.3], where we showed existence of one almost reproducing kernel. There the essence of the proof was to reduce to the Hilbert space case by a cleverly chosen perturbation of the inner product. The proof of the below theorem further exploits this idea. Additional arguments are necessary, since we allow Krein space valued functions as elements of the space and include some refinements to control the variety of choices of almost reproducing kernels.

Theorem 3.2. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{A}$ be a reproducing kernel almost Pontryagin space of $\mathcal{K}$-valued functions on M. Moreover, let $n \in \mathbb{N}_0$, $(a_i)_{i=1}^n \in \mathcal{K}^n$ and $(\eta_i)_{i=1}^n \in M^n$, and assume that*

$$\mathcal{A}^\circ \cap \bigcap_{i=1}^n \ker \chi_{\eta_i,a_i} = \{0\}. \tag{3.2}$$

Then there exists a closed and nowhere dense subset Ω of $\mathbb{R}^n$ with the following property: For each tuple $(\gamma_i)_{i=1}^n \in \mathbb{R}^n \setminus \Omega$ there exists a $\mathcal{K}$-valued hermitian kernel on M such that the formula (3.1) holds with the data $\delta = ((a_i)_{i=1}^n; (\eta_i)_{i=1}^n; (\gamma_i)_{i=1}^n)$.

For the value $n := \operatorname{ind}_0 \mathcal{A}$ a choice of $(a_i)_{i=1}^n$ and $(\eta_i)_{i=1}^n$ can be made such that (3.2) holds. For each such choice, we obtain a family of almost reproducing kernels of $\mathcal{A}$.

Proof. If $n = 0$ we necessarily have $\mathcal{A}^\circ = \{0\}$, and may refer to Theorem 2.8. Note here that for $n = 0$ the index set in both, the sum in (3.1) and the intersection in (3.2) are empty. Hence, assume throughout the proof that $n > 0$.

Step 1. Appropriate choice of inner product: Choose a Hilbert space inner product $(\cdot,\cdot)_\mathcal{A}$ which induces the topology of $\mathcal{A}$, and let G be the Gram operator of $[\cdot,\cdot]_\mathcal{A}$ w.r.t. $(\cdot,\cdot)_\mathcal{A}$. Since 0 is an isolated point of the spectrum of G (or belongs to its resolvent set), we may choose $\delta \in (0, 1]$ with $\sigma(G) \cap (-\delta, 0) = \sigma(G) \cap (0, \delta) = \emptyset$.

We pass to an equivalent Hilbert space inner product which is more suitable for our needs. To this end, set (E denotes the spectral measure of G)

$$P_+ := E\big([\delta,\infty)\big), \qquad P_0 := E\big(\{0\}\big), \qquad P_- := E\big((-\infty,-\delta]\big).$$

Due to our choice of δ, we have $P_+ + P_0 + P_- = I$. Consider the bounded and selfadjoint operator

$$Q := GP_+ + P_0 - GP_-.$$

It holds that (remember that $\delta \leq 1$)

$$\begin{aligned}(Qx,x)_{\mathcal{A}} &= (GP_+x,P_+x)_{\mathcal{A}} + (P_0x,P_0x)_{\mathcal{A}} - (GP_-x,P_-x)_{\mathcal{A}} \\ &\geq \delta(P_+x,P_+x)_{\mathcal{A}} + \delta(P_0x,P_0x)_{\mathcal{A}} + \delta(P_-x,P_-x)_{\mathcal{A}} = \delta(x,x)_{\mathcal{A}}, \quad x \in \mathcal{A},\end{aligned}$$

i.e., Q is strictly positive. Hence, the inner product

$$(x,y) := (Qx,y)_{\mathcal{A}}, \quad x,y \in \mathcal{A},$$

is equivalent to $(\cdot,\cdot)_{\mathcal{A}}$.

The Gram operator H of $[\cdot,\cdot]_{\mathcal{A}}$ w.r.t. $(\cdot,\cdot)$ is given as

$$\begin{aligned}H &= Q^{-1}G = Q^{-1}(GP_+ + GP_-) \\ &= Q^{-1}\big(Q + (2GP_- - P_0)\big) = I + \underbrace{Q^{-1}(2GP_- - P_0)}_{=:R}.\end{aligned}$$

Observe that $\dim(\operatorname{ran} R) < \infty$.

Step 2. Admissible values of $\boldsymbol{\gamma}$*:* Assume that $(a_i)_{i=1}^n \in \mathcal{K}^n$ and $(\eta_i)_{i=1}^n \in M^n$ are given and satisfy (3.2).

Let $L : M \times M \to \mathcal{B}(\mathcal{K})$ be the reproducing kernel of the reproducing kernel Hilbert space $\langle \mathcal{A}, (\cdot,\cdot)\rangle$, cf. Theorem 2.8. For $\boldsymbol{\gamma} = (\gamma_i)_{i=1}^n \in \mathbb{C}^n$ we denote by $A_{\boldsymbol{\gamma}}$ the finite rank operator defined as

$$A_{\boldsymbol{\gamma}}x := \sum_{i=1}^n \gamma_i \cdot \chi_{\eta_i,a_i}(x)L(\eta_i,\cdot)a_i = \sum_{i=1}^n \gamma_i \cdot \big(x, L(\eta_i,\cdot)a_i\big) \cdot L(\eta_i,\cdot)a_i, \quad x \in \mathcal{A},$$

and set

$$B_{\boldsymbol{\gamma}} := H + A_{\boldsymbol{\gamma}} = I + (R + A_{\boldsymbol{\gamma}}), \quad \boldsymbol{\gamma} = (\gamma_i)_{i=1}^n \in \mathbb{C}^n.$$

The operator H is $(\cdot,\cdot)$-selfadjoint. Moreover, we have (here * denotes the $(\cdot,\cdot)$-adjoint)

$$A_\gamma = (A_{\bar{\gamma}})^*, \qquad B_\gamma = (B_{\bar{\gamma}})^* \quad \text{where } \bar{\gamma} = (\overline{\gamma}_i)_{i=1}^n. \tag{3.3}$$

We are aiming to show existence of values $\gamma \in \mathbb{R}^n$ for which B_γ is boundedly invertible.

Set $\mathcal{B} := \ker R \cap \bigcap_{i=1}^n \ker \chi_{\eta_i,a_i}$, and consider the orthogonal decomposition

$$\mathcal{A} = \mathcal{B} \,(\dot{+})\, \mathcal{B}^{(\perp)}$$

of $\mathcal{A}$. For $x \in \mathcal{B}$ we have $B_\gamma x = x$, in particular, $B_\gamma(\mathcal{B}) \subseteq \mathcal{B}$, $\gamma \in \mathbb{C}^n$. Using (3.3), it follows that also $B_\gamma(\mathcal{B}^{(\perp)}) \subseteq \mathcal{B}^{(\perp)}$, $\gamma \in \mathbb{C}^n$. We conclude that the operator B_γ can be written as the block operator matrix

$$B_\gamma = \begin{pmatrix} I|_{\mathcal{B}} & 0 \\ 0 & B_\gamma|_{\mathcal{B}^{(\perp)}} \end{pmatrix} : \begin{matrix} \mathcal{B} \\ (\dot{+}) \\ \mathcal{B}^{(\perp)} \end{matrix} \quad \longrightarrow \quad \begin{matrix} \mathcal{B} \\ (\dot{+}) \\ \mathcal{B}^{(\perp)} \end{matrix}$$

This shows that $0 \in \rho(B_\gamma)$ if and only if $0 \in \rho(B_\gamma|_{\mathcal{B}^{(\perp)}})$.

Since R is a finite rank operator, the space $\mathcal{B}^{(\perp)}$ is finite dimensional. Thus $0 \in \rho(B_\gamma|_{\mathcal{B}^{(\perp)}})$ if and only if $B_\gamma|_{\mathcal{B}^{(\perp)}}$ is injective. Consider a point $\gamma \in (\mathbb{C}^+)^n$ (here $\mathbb{C}^+$ denotes the open upper half-plane). If $x \in \ker B_\gamma|_{\mathcal{B}^{(\perp)}}$, then

$$(Hx, x) + (A_\gamma x, x) = (B_\gamma x, x) = 0.$$

The number (Hx, x) is real, hence it follows that $\operatorname{Im}(A_\gamma x, x) = 0$. However,

$$\operatorname{Im}(A_\gamma x, x) = \sum_{i=1}^n \operatorname{Im} \gamma_i \cdot \left|\chi_{\eta_i,a_i}(x)\right|^2,$$

and it follows that $x \in \bigcap_{i=1}^n \ker \chi_{\eta_i,a_i}$. In particular, $A_\gamma x = 0$, and hence $Hx = B_\gamma x - A_\gamma x = 0$. This says that $x \in \mathcal{A}^\circ$, and now our hypothesis (3.2) implies that $x = 0$. The analogous argument applies if $\gamma \in (\mathbb{C}^-)^n$ (where $\mathbb{C}^-$ denotes the open lower half-plane). We conclude that

$$0 \in \rho(B_\gamma|_{\mathcal{B}^{(\perp)}}), \quad \gamma \in \left(\mathbb{C}^+\right)^n \cup \left(\mathbb{C}^-\right)^n.$$

Consider the determinant

$$p(\gamma_1, \ldots, \gamma_n) := \det(B_\gamma|_{\mathcal{B}^{(\perp)}}), \quad \gamma \in \mathbb{C}^n.$$

Then p is a polynomial in the complex variables $\gamma_1, \ldots, \gamma_n$, in particular, p is analytic on all of $\mathbb{C}^n$. Since p does not vanish identically, the zero set of p cannot contain any relatively open subset of $\mathbb{R}^n$, see, e.g., [38]. Clearly, the set

$$\Omega := \{\gamma \in \mathbb{R}^n : p(\gamma_1, \ldots, \gamma_n) = 0\}$$

is closed. Thus, Ω is nowhere dense.

Step 3. Construction of kernel functions: Let $\gamma \in \mathbb{R}^n \setminus \Omega$, then $0 \in \rho(B_\gamma)$. We set

$$k(\eta, a) := B_\gamma^{-1}\big(L(\eta, \cdot)a\big) \in \mathcal{A}, \quad a \in \mathcal{K},\ \eta \in M,$$

and define $K : M \times M \to \mathcal{K}^{\mathcal{K}}$ as

$$K(\eta, \zeta)a := k(\eta, a)(\zeta), \quad a \in \mathcal{K},\ \zeta, \eta \in M.$$

Obviously, it holds that $K(\eta, \cdot)a = k(\eta, a) \in \mathcal{A}$, $a \in \mathcal{K}$, $\eta \in M$.

We have (here norms are understood w.r.t. $(\cdot,\cdot)$)

$$\begin{aligned}\big\|K(\eta, \zeta)a\big\| &\leq \|\chi_\zeta\| \cdot \big\|k(\eta, a)\big\| \leq \|\chi_\zeta\| \cdot \big\|B_\gamma^{-1}\big\| \cdot \underbrace{\big\|L(\eta, \cdot)a\big\|}_{=\|\chi_{\eta,a}\|} \\ &\leq \|\chi_\zeta\| \cdot \big\|B_\gamma^{-1}\big\| \cdot \|\chi_\eta\| \cdot \|a\|, \quad a \in \mathcal{K},\ \eta, \zeta \in M.\end{aligned}$$

This shows that K maps $M \times M$ into $B(\mathcal{K})$. Since B_γ^{-1} is selfadjoint, we have

$$\begin{aligned}\big(K(\eta, \zeta)^* a, b\big)_{\mathcal{K}} &= \big(a, K(\eta, \zeta)b\big)_{\mathcal{K}} = \big(L(\zeta, \cdot)a, k(\eta, b)\big) \\ &= \big(L(\zeta, \cdot)a, B_\gamma^{-1}L(\eta, \cdot)b\big) = \big(B_\gamma^{-1}L(\zeta, \cdot)a, L(\eta, \cdot)b\big) \\ &= \big(k(\zeta, a), L(\eta, \cdot)b\big) = \big(K(\zeta, \eta)a, b\big)_{\mathcal{K}}, \quad a, b \in \mathcal{K},\ \eta, \zeta \in M.\end{aligned}$$

This shows that $K(\eta, \zeta)^* = K(\zeta, \eta)$, $\eta, \zeta \in M$, and we see that K is a hermitian kernel.

Validity of (3.1) follows by computation using the definitions and the fact that B_γ is selfadjoint. Namely, for each $f \in \mathcal{A}$ and $a \in \mathcal{K}, \eta \in M$ we have

$$\begin{aligned}\big[f, K(\eta, \cdot)a\big]_{\mathcal{A}} &= \big(Hf, K(\eta, \cdot)a\big) = \big(f, \underbrace{B_\gamma K(\eta, \cdot)a}_{=L(\eta, \cdot)a}\big) - \big(A_\gamma f, K(\eta, \cdot)a\big) \\ &= \big(f(\eta), a\big)_{\mathcal{K}} - \sum_{i=1}^n \gamma_i \cdot \chi_{\eta_i, a_i}(f) \underbrace{\big(L(\eta_i, \cdot)a_i, K(\eta, \cdot)a\big)_{\mathcal{K}}}_{=\overline{\chi_{\eta_i, a_i}(K(\eta, \cdot)a)}}.\end{aligned}$$

Consider now the case that $n = \operatorname{ind}_0 \mathcal{A}$. Then, since the family $\{\chi_{\eta,a} : a \in \mathcal{K}, \eta \in M\}$ is point separating, we can inductively construct elements $a_1, \ldots, a_n \in \mathcal{K}$ and $\eta_1, \ldots, \eta_n \in M$ which satisfy (3.2). Namely: As a first step choose $f \in \mathcal{A}^\circ \setminus \{0\}$ and $\eta_1 \in M, a_1 \in \mathcal{K}$ with $\chi_{\eta_1,a_1}(f) \neq 0$. Then $\dim(\mathcal{A} \cap \ker \chi_{\eta_1,a_1}) = \dim \mathcal{A}^\circ - 1$. For the inductive step assume that $m < n$ and that $\eta_1, \ldots, \eta_m \in M$ and $a_1, \ldots, a_m \in \mathcal{K}$ are given with $\dim(\mathcal{A}^\circ \cap \bigcap_{i=1}^m \ker \chi_{\eta_i,a_i}) = \dim \mathcal{A}^\circ - m$. Choose $f \in (\mathcal{A}^\circ \cap \bigcap_{i=1}^m \ker \chi_{\eta_i,a_i}) \setminus \{0\}$ and

$\eta_{m+1} \in M, a_{m+1} \in \mathcal{K}$ with $\chi_{\eta_{m+1},a_{m+1}}(f) \neq 0$. Then $\dim(\mathcal{A}^\circ \cap \bigcap_{i=1}^{m+1} \ker \chi_{\eta_i,a_i}) = \dim \mathcal{A}^\circ - (m+1)$. $\square$

Remark 3.3. Let us comment on the variety of possible choices of δ. Firstly, as we see from the inductive argument in the last part of the proof, often "most" choices of $(a_i)_{i=1}^n \in \mathcal{K}^n$, $(\eta_i)_{i=1}^n \in M^n$ will have the required property (3.2); think for instance of the case when the elements of $\mathcal{A}$ are analytic functions. Of course, this is just a vague statement.

Secondly, and more precisely, we can obtain knowledge about the exceptional set Ω. Namely, whenever $\gamma : \mathbb{C} \to \mathbb{C}^n$ is a polynomial curve which intersects $(\mathbb{C}^+)^n \cup (\mathbb{C}^-)^n$, then $\gamma(\mathbb{C}) \cap \Omega$ is finite and (here $\deg \gamma$ denotes that degree of the polynomial curve γ, meaning the maximum degree of its component functions)

$$\#\big(\gamma(\mathbb{C}) \cap \Omega\big) \leq \deg \gamma \cdot (\operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A} + n).$$

To see this, notice that $q : \xi \mapsto \det(B_{\gamma(\xi)}|_{\mathcal{B}^{(\perp)}})$ is a polynomial whose degree does not exceed $\deg \gamma \cdot \dim \mathcal{B}^{(\perp)}$. However, we can estimate $\dim \mathcal{B}^{(\perp)}$ as

$$\begin{aligned}\dim \mathcal{B}^{(\perp)} &= \operatorname{codim} \mathcal{B} \leq \operatorname{codim}(\ker R) + \operatorname{codim}\left(\bigcap_{i=1}^{n} \ker \chi_{\eta_i,a_i}\right) \\ &= \dim(\operatorname{ran} R) + n \leq \operatorname{ind}_- \mathcal{A} + \operatorname{ind}_0 \mathcal{A} + n.\end{aligned}$$

Since q does not vanish identically, the number of its zeros is finite and bounded by its degree.

Observe that, if the polynomial γ has real coefficients, then with finitely many exceptions $\gamma(\mathbb{R})$ consists of admissible values for $(\gamma_i)_{i=1}^n$. $\Diamond$

It is easy to see that validity of (3.1) implies (3.2). Moreover, it can be characterised geometrically whether or not the perturbation term in (3.1) has minimal number of summands (i.e., whether or not we speak of an almost reproducing kernel).

Lemma 3.4. *Let $\mathcal{A}$ be a reproducing kernel almost Pontryagin space of $\mathcal{K}$-valued functions on M, and let K be a $\mathcal{K}$-valued hermitian kernel on M. Let $n \in \mathbb{N}_0$ and $\delta \in \mathcal{K}^n \times M^n \times \mathbb{R}^n$, and assume that the formula (3.1) holds. Then the following statements hold.*

(i) *We have*

$$\mathcal{A}^\circ \cap \bigcap_{\substack{i=1 \\ \gamma_i \neq 0}}^{n} \ker \chi_{\eta_i,a_i} = \{0\}, \tag{3.4}$$

in particular, $n \geq \operatorname{ind}_0 \mathcal{A}$.

(ii) *Set* $\mathcal{L} := \operatorname{span}\{K(\eta,\cdot)a : a \in \mathcal{K}, \eta \in M\}$, *then*

$$\mathcal{A}^\circ \subseteq \mathcal{L}^\perp \subseteq \operatorname{span}\big\{K(\eta_i,\cdot)a_i : \gamma_i \neq 0\big\}.$$

(iii) *The following conditions are equivalent.*
  (a) *It holds that* $n = \operatorname{ind}_0 \mathcal{A}$.
  (b) *The set* $\{K(\eta_i,\cdot)a_i : i = 1,\ldots,n\}$ *is a basis of* $\mathcal{A}^\circ$.
  (c) *We have (here* δ_{ij} *denotes the Kronecker-Delta)*

$$\gamma_i \neq 0,\ i = 1,\ldots,n \quad \textit{and} \quad \big(K(\eta_j,\eta_i)a_j, a_i\big)_{\mathcal{K}} = \delta_{ij}\frac{1}{\gamma_i}, \quad i,j = 1,\ldots,n. \tag{3.5}$$

Proof. To show (3.4), assume that $f \in \mathcal{A}^\circ \cap \bigcap_{i=1}^n \{\ker \chi_{\eta_i,a_i} : \gamma_i \neq 0\}$. Then, by (3.1), it holds that $(f(\eta), a)_{\mathcal{K}} = 0$, $a \in \mathcal{K}, \eta \in M$, and hence $f = 0$. From (3.4) we obtain

$$\dim \mathcal{A}^\circ \leq \dim\Bigg(\mathcal{A}/\bigcap_{\substack{i=1\\ \gamma_i \neq 0}}^n \ker \chi_{\eta_i,a_i}\Bigg) \leq n.$$

Also item (ii) is rather straightforward from (3.1). First, notice that

$$\chi_{\eta_i,a_i}\big(K(\eta,\cdot)a\big) = \big(K(\eta,\eta_i)a, a_i\big)_{\mathcal{K}} = \big(a, K(\eta_i,\eta)a_i\big)_{\mathcal{K}}.$$

Now (3.1) yields that, for each $f \in \mathcal{L}^\perp$,

$$\big(f(\eta), a\big)_{\mathcal{K}} = \sum_{i=1}^n \gamma_i \cdot \chi_{\eta_i,a_i}(f)\big(K(\eta_i,\eta)a_i, a\big)_{\mathcal{K}}, \quad a \in \mathcal{K},\ \eta \in M.$$

From this we have

$$f = \sum_{i=1}^n \gamma_i \cdot \chi_{\eta_i,a_i}(f)K(\eta_i,\cdot)a_i = \sum_{\substack{i=1\\ \gamma_i \neq 0}}^n \gamma_i \cdot \chi_{\eta_i,a_i}(f)K(\eta_i,\cdot)a_i, \quad f \in \mathcal{L}^\perp. \tag{3.6}$$

We come to the proof of item (iii). Assume that (a) holds. Then, from the already proved item (ii), we must have

$$\mathcal{A}^\circ = \operatorname{span}\big\{K(\eta_i,\cdot)a_i : i = 1,\ldots,n, \gamma_i \neq 0\big\}.$$

This implies that the set written on the right side must contain n linearly independent elements. Thus $\gamma_i \neq 0$, $i = 1,\ldots,n$, and $\{K(\eta_i,\cdot)a_i : i = 1,\ldots,n\}$ is linearly independent. We see that (b) holds. The converse implication "(b) $\Rightarrow$ (a)" is of course trivial.

Assume next that (b) holds. Then (3.6) yields the representation

$$K(\eta_j,\cdot)a_j=\sum_{i=1}^n \gamma_i\cdot\chi_{\eta_i,a_i}\big(K(\eta_j,\cdot)a_j\big)\cdot K(\eta_i,\cdot)a_i.$$

By linear independence, thus,

$$\gamma_i\cdot\chi_{\eta_i,a_i}\big(K(\eta_j,\cdot)a_j\big)=\delta_{ij},\quad i,j=1,\dots,n.$$

This implies (3.5).

Finally, assume that (3.5) holds. Substituting in (3.1) gives

$$\big[f,K(\eta_j,\cdot)a_j\big]_{\mathcal{A}}=\underbrace{\big(f(\eta_j),a_j\big)_{\mathcal{K}}}_{=\chi_{\eta_j,a_j}(f)}-\sum_{i=1}^n\gamma_i\cdot\chi_{\eta_i,a_i}(f)\,\underbrace{\overline{\chi_{\eta_i,a_i}\big(K(\eta_j,\cdot)a_j\big)}}_{=\delta_{ij}\frac{1}{\gamma_i}}=0,\quad f\in\mathcal{A},$$

i.e., $K(\eta_j,\cdot)a_j\in\mathcal{A}^\circ$, $j=1,\dots,n$. Consider the map

$$\Lambda:\begin{cases}\mathcal{K}^M\to\mathbb{C}^n\\ f\mapsto\big(\chi_{\eta_i,a_i}(f)\big)_{i=1}^n\end{cases}\tag{3.7}$$

Then

$$\Lambda\big(\operatorname{span}\big\{K(\eta_i,\cdot)a_i:i=1,\dots,n\big\}\big)=\mathbb{C}^n,$$

and hence $\{K(\eta_i,\cdot)a_i:i=1,\dots,n\}$ must be linearly independent. Since $\dim\mathcal{A}^\circ$ cannot exceed n by (i), we see that (b) holds. □

Next, we determine those hermitian kernels which may appear as almost reproducing kernels of some reproducing kernel almost Pontryagin space. It is not a surprise that finiteness of negative index is the decisive factor. The crucial construction for the proof is to mimick the formula (3.1) on an abstract level.

Definition 3.5. Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let K be a $\mathcal{K}$-valued hermitian kernel on M. Moreover, let data $n\in\mathbb{N}_0$ and $\boldsymbol{\delta}=\big((a_i)_{i=1}^n;(\eta_i)_{i=1}^n;(\gamma_i)_{i=1}^n\big)\in\mathcal{K}^n\times M^n\times\mathbb{R}^n$ be given.

We denote by $\mathcal{F}(\mathcal{K},M)$ the space of all $\mathcal{K}$-valued functions on M with finite support, i.e.,

$$\mathcal{F}(\mathcal{K},M):=\big\{f:M\to\mathcal{K}\colon\big\{\zeta\in M:f(\zeta)\neq 0\big\}\text{ is finite}\big\},$$

and define linear functionals

$$\phi_i : \begin{cases} \mathcal{F}(\mathcal{K}, M) \to \mathbb{C}, \\ f \mapsto \sum_{\zeta \in M} (K(\zeta, \eta_i) f(\zeta), a_i)_{\mathcal{K}}, \end{cases} \quad i = 1, \ldots, n.$$

Next, we define a sesquilinear form $[\cdot,\cdot]_{K,\mathfrak{d}} : \mathcal{F}(\mathcal{K}, M) \times \mathcal{F}(\mathcal{K}, M) \to \mathbb{C}$ as

$$[f, g]_{K,\mathfrak{d}} := \sum_{\zeta,\eta \in M} \big(K(\zeta, \eta) f(\zeta), g(\eta)\big)_{\mathcal{K}} - \sum_{i=1}^{n} \gamma_i \cdot \phi_i(f) \overline{\phi_i(g)}, \quad f, g \in \mathcal{F}(\mathcal{K}, M). \quad \Diamond$$

From the fact that K is a hermitian kernel, it is obvious that $[\cdot,\cdot]_{K,\mathfrak{d}}$ is an inner product on $\mathcal{F}(\mathcal{K}, M)$.

Two inner products $[\cdot,\cdot]_{K,\mathfrak{d}_1}$ and $[\cdot,\cdot]_{K,\mathfrak{d}_2}$ may coincide. For example, this is certainly the case if $\mathfrak{d}_k = ((a_i^k)_{i=1}^{n_k}; (\eta_i^k)_{i=1}^{n_k}; (\gamma_i^k)_{i=1}^{n_k})$, $k = 1, 2$, and there exists a bijection

$$\sigma : \big\{i \in \{1, \ldots, n_1\} : \gamma_i^1 \neq 0\big\} \to \big\{i \in \{1, \ldots, n_2\} : \gamma_i^2 \neq 0\big\}$$

with

$$a_i^1 = a_{\sigma(i)}^2, \qquad \eta_i^1 = \eta_{\sigma(i)}^2, \qquad \gamma_i^1 = \gamma_{\sigma(i)}^2 \quad \text{whenever } \gamma_i^1 \neq 0.$$

A particular role is played by the inner product $[\cdot,\cdot]_{K,\varnothing}$ where $\varnothing$ is the unique element of $\mathcal{K}^0 \times M^0 \times \mathbb{R}^0$. Explicitly,

$$[f, g]_{K,\varnothing} = \sum_{\zeta,\eta \in M} \big(K(\zeta, \eta) f(\zeta), g(\eta)\big)_{\mathcal{K}}, \quad f, g \in \mathcal{F}(\mathcal{K}, M).$$

Comparing with the definition of the negative index of a hermitian kernel, we see that

$$\operatorname{ind}_- K = \operatorname{ind}_- \big\langle \mathcal{F}(\mathcal{K}, M), [\cdot,\cdot]_{K,\varnothing} \big\rangle.$$

Let us introduce one more notation. We denote by $\delta_{\eta,a} \in \mathcal{F}(\mathcal{K}, M)$ the function defined as

$$\delta_{\eta,a}(\zeta) := \begin{cases} a, & \zeta = \eta, \\ 0, & \text{otherwise}, \end{cases} \quad a \in \mathcal{K},\ \eta \in M.$$

Clearly,

$$\delta_{\eta,a+b} = \delta_{\eta,a} + \delta_{\eta,b}, \qquad \delta_{\eta,\alpha a} = \alpha \delta_{\eta,a}, \quad a, b \in \mathcal{K},\ \alpha \in \mathbb{C},\ \eta \in M,$$
$$\mathcal{F}(\mathcal{K}, M) = \operatorname{span}\{\delta_{\eta,a} : a \in \mathcal{K}, \eta \in M\}.$$

Necessity of finiteness of negative index in order to be an almost reproducing kernel now follows.

Proposition 3.6. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{A}$ be a reproducing kernel almost Pontryagin space of $\mathcal{K}$-valued functions on M. Moreover, let K be a $\mathcal{K}$-valued hermitian kernel on M, let $n \in \mathbb{N}_0$ and $\mathfrak{d} \in \mathcal{K}^n \times M^n \times \mathbb{R}^n$, and assume that the formula (3.1) holds with this data. Then*

$$\operatorname{ind}_- K \leq \operatorname{ind}_- \mathcal{A} + n.$$

Proof. Consider the linear map $\iota : \mathcal{F}(\mathcal{K}, M) \to \mathcal{A}$ which is defined as

$$\iota(f) := \sum_{\zeta \in M} K(\zeta, \cdot) f(\zeta), \quad f \in \mathcal{F}(\mathcal{K}, M). \tag{3.8}$$

From the definition of $[\cdot,\cdot]_{K,\mathfrak{d}}$ and (3.1), clearly

$$\begin{aligned}\big[K(\zeta,\cdot)b, K(\eta,\cdot)a\big]_{\mathcal{A}} &= \big(K(\zeta,\eta)b, a\big)_{\mathcal{K}} - \sum_{i=1}^{n} \gamma_i \cdot \chi_{\eta_i,a_i}\big(K(\zeta,\cdot)b\big)\overline{\chi_{\eta_i,a_i}\big(K(\eta,\cdot)a\big)} \\ &= [\delta_{\zeta,b}, \delta_{\eta,a}]_{K,\mathfrak{d}}, \quad a, b \in \mathcal{K},\ \eta, \zeta \in M.\end{aligned}$$

This implies that ι is isometric, and it follows that

$$\operatorname{ind}_- \big\langle \mathcal{F}(\mathcal{K}, M), [\cdot,\cdot]_{K,\mathfrak{d}} \big\rangle \leq \operatorname{ind}_- \mathcal{A}.$$

On the linear subspace $\mathcal{L} := \bigcap_{i=1}^{n} \ker \phi_i$ of $\mathcal{A}$ the inner products $[\cdot,\cdot]_{K,\mathfrak{d}}$ and $[\cdot,\cdot]_{K,\varnothing}$ coincide. Since $\operatorname{codim} \mathcal{L} \leq n$, it follows that

$$\big|\operatorname{ind}_- K - \operatorname{ind}_- \big\langle \mathcal{F}(\mathcal{K}, M), [\cdot,\cdot]_{K,\mathfrak{d}} \big\rangle\big| \leq n.$$

Together, we obtain $\operatorname{ind}_- K \leq \operatorname{ind}_- \mathcal{A} + n$. □

For sufficiency, we again mimick the formula (3.1); this time on a concrete level.

Definition 3.7. Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let K be a $\mathcal{K}$-valued hermitian kernel on M with $\operatorname{ind}_- K < \infty$. Moreover, let data $n \in \mathbb{N}_0$ and $\mathfrak{d} \in \mathcal{K}^n \times M^n \times \mathbb{R}^n$ be given.

Denote by $\langle \mathcal{A}_K, [\![\cdot,\cdot]\!]_K \rangle$ the unique reproducing kernel Pontryagin space with reproducing kernel K, cf. Theorem 2.9, and define a sesquilinear form $[\![\cdot,\cdot]\!]_{K,\mathfrak{d}} : \mathcal{A}_K \times \mathcal{A}_K \to \mathbb{C}$ as

$$[\![f, g]\!]_{K,\mathfrak{d}} := [\![f, g]\!]_K - \sum_{i=1}^{n} \gamma_i \cdot \chi_{\eta_i,a_i}(f)\overline{\chi_{\eta_i,a_i}(g)}, \quad f, g \in \mathcal{A}_K. \tag{3.9}$$

Obviously, $[\![\cdot,\cdot]\!]_{K,\mathfrak{d}}$ is an inner product on $\mathcal{A}_K$. We denote the inner product space $\langle \mathcal{A}_K, [\![\cdot,\cdot]\!]_{K,\mathfrak{d}} \rangle$ as $\mathcal{A}_{K,\mathfrak{d}}$.

Proposition 3.8. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let K be a $\mathcal{K}$-valued hermitian kernel on M with $\operatorname{ind}_- K < \infty$. Moreover, let data $n \in \mathbb{N}_0$ and $\mathfrak{d} \in \mathcal{K}^n \times M^n \times \mathbb{R}^n$ be given. Then the following statements hold.*

(i) *The inner product space $\mathcal{A}_{K,\mathfrak{d}}$ is a reproducing kernel almost Pontryagin space. Its almost Pontryagin space topology coincides with the Pontryagin space topology of $\mathcal{A}_{K,\varnothing} = \langle \mathcal{A}_K, [\![\cdot,\cdot]\!] \rangle_K$.*

(ii) *The formula (3.1) holds in $\mathcal{A}_{K,\mathfrak{d}}$ with K and $\mathfrak{d}$. In particular, we have $\operatorname{ind}_0 \mathcal{A}_{K,\mathfrak{d}} \leq n$.*

(iii) *We have $\operatorname{ind}_0 \mathcal{A}_{K,\mathfrak{d}} = n$ if and only if*

$$\gamma_i \neq 0, \quad i = 1, \ldots, n \quad \textit{and} \quad \big(K(\eta_j, \eta_i) a_j, a_i\big)_{\mathcal{K}} = \delta_{ij} \frac{1}{\gamma_i}, \quad i, j = 1, \ldots, n.$$

Proof. Consider the space $\mathcal{A}_K$ endowed with the inner product $[\![\cdot,\cdot]\!]_{K,\mathfrak{d}}$ and with the Pontryagin space topology $\mathcal{O}$ of $\mathcal{A}_{K,\varnothing}$. The inner product $[\![f,g]\!]_{K,\mathfrak{d}}$ is a finite rank perturbation of $[\![f,g]\!]_{K,\varnothing}$ in the sense of Lemma A.8. This yields that $\langle \mathcal{A}_K, [\![\cdot,\cdot]\!]_{K,\mathfrak{d}}, \mathcal{O} \rangle$ is an almost Pontryagin space. Since the space has not changed topologically, point evaluations are continuous.

Validity of (3.1) is built in the definition. Namely, by the reproducing kernel property of K in $\mathcal{A}_K$, we have from (3.9)

$$\begin{aligned} \big(f(\eta), a\big)_{\mathcal{K}} &= [\![f, K(\eta,\cdot)a]\!]_K \\ &= [\![f, K(\eta,\cdot)a]\!]_{K,\mathfrak{d}} + \sum_{i=1}^{n} \gamma_i \cdot \chi_{\eta_i,a_i}(f) \overline{\chi_{\eta_i,a_i}\big(K(\eta,\cdot)a\big)}, \quad f \in \mathcal{A}_K. \end{aligned}$$

The statement in (iii) is immediate from Lemma 3.4(iii). $\square$

After having settled existence questions (of kernels for a given space and of spaces for a given kernel), we turn to uniqueness. First, we show uniqueness of kernels for given space and data $\mathfrak{d}$.

Proposition 3.9. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{A}$ be a reproducing kernel almost Pontryagin space of $\mathcal{K}$-valued functions on M. Moreover, set $n := \operatorname{ind}_0 \mathcal{A}$, and let $\mathfrak{d} \in \mathcal{K}^n \times M^n \times \mathbb{R}^n$.*

Then there exists at most one almost reproducing kernel K of $\mathcal{A}$ such that (3.1) holds for K with $\mathfrak{d}$.

Proof. Let K_1 and K_2 be almost reproducing kernels of $\mathcal{A}$ such that (3.1) holds for K_1 and K_2 with $\mathfrak{d} = ((a_i)_{i=1}^n; (\eta_i)_{i=1}^n; (\gamma_i)_{i=1}^n)$. Consider the map Λ defined in (3.7). Lemma 3.4(i) shows that $\Lambda|_{\mathcal{A}^\circ}$ is injective. By Lemma 3.4(iii), we have $K_l(\eta_i,\cdot)a_i \in \mathcal{A}^\circ$, $i = 1, \ldots, n$, $l = 1, 2$, and $\Lambda(K_1(\eta_i,\cdot)a_i) = \Lambda(K_2(\eta_i,\cdot)a_i)$, $i = 1, \ldots, n$. It follows that

$$K_1(\eta_i,\cdot)a_i = K_2(\eta_i,\cdot)a_i, \quad i = 1, \ldots, n.$$

From this we have

$$\begin{aligned}\chi_{\eta_i,a_i}\big(K_1(\eta,\cdot)a\big) &= \big(K_1(\eta,\eta_i)a,a_i\big)_{\mathcal{K}} = \big(a,K_1(\eta_i,\eta)a_i\big)_{\mathcal{K}} = \big(a,K_2(\eta_i,\eta)a_i\big)_{\mathcal{K}} \\ &= \big(K_2(\eta,\eta_i)a,a_i\big)_{\mathcal{K}} = \chi_{\eta_i,a_i}\big(K_2(\eta,\cdot)a\big), \quad i=1,\ldots,n,\ a\in\mathcal{K},\ \eta\in M.\end{aligned}$$

Now (3.1) yields

$$\begin{aligned}\big[f,K_1(\eta,\cdot)a\big]_{\mathcal{A}} &= \big(f(\eta),a\big)_{\mathcal{K}} - \sum_{i=1}^n \gamma_i\cdot\chi_{\eta_i,a_i}(f)\overline{\chi_{\eta_i,a_i}\big(K_1(\eta,\cdot)a\big)} \\ &= \big[f,K_2(\eta,\cdot)a\big]_{\mathcal{A}}, \quad f\in\mathcal{A},\ a\in\mathcal{K},\ \eta\in M.\end{aligned}$$

It follows that $K_1(\eta,\cdot)a - K_2(\eta,\cdot)a \in \mathcal{A}^\circ$, $a\in\mathcal{K}$, $\eta\in M$. Injectivity of $\Lambda|_{\mathcal{A}^\circ}$ implies that $K_1(\eta,\cdot)a = K_2(\eta,\cdot)a$. □

Second, we show uniqueness of the space for given kernel and data δ. In the proof, we utilise the theory of almost Pontryagin space completions as developed in [40]; for the relevant statements see Appendix A (p. 312ff).

Proposition 3.10. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let K be a $\mathcal{K}$-valued hermitian kernel on M with* $\operatorname{ind}_- K < \infty$. *Moreover, let $n\in\mathbb{N}_0$ and $\delta\in\mathcal{K}^n\times M^n\times\mathbb{R}^n$.*

Then there exists at most one reproducing kernel almost Pontryagin space $\mathcal{A}$ such that K is an almost reproducing kernel of $\mathcal{A}$ with the data δ in (3.1).

Proof. Let $\mathcal{A}$ be a reproducing kernel almost Pontryagin space, assume that (3.1) holds with K and δ and that $n = \dim\mathcal{A}^\circ$. Consider the subspace $\mathcal{L} := \operatorname{span}\{K(\eta,\cdot)a : a\in\mathcal{K},\eta\in M\}$. Then, by Lemma 3.4(ii), (iii), we have

$$\mathcal{L}^\perp = \mathcal{A}^\circ \subseteq \mathcal{L}.$$

Using Lemma A.6(iii), it follows that $\mathcal{L}$ is dense in $\mathcal{A}$.

Consider the map $\iota:\mathcal{F}(\mathcal{K},M)\to\mathcal{A}$ defined in (3.8). Then, as substituting in the definitions shows, ι is an isometry of $\langle\mathcal{F}(\mathcal{K},M),[\cdot,\cdot]_{K,\delta}\rangle$ onto $\mathcal{L}$. Hence, we may consider $\langle\iota,\mathcal{A}\rangle$ as an almost Pontryagin space completion of $\langle\mathcal{F}(\mathcal{K},M),[\cdot,\cdot]_{K,\delta}\rangle$.

Since $\{\chi_{\eta_i,a_i} : i=1,\ldots,n\}$ is point separating on $\mathcal{A}^\circ$, Proposition A.3(ii), yields

$$\mathcal{A}' = \big\{[\cdot,y]_{\mathcal{A}} : y\in\mathcal{A}\big\} + \operatorname{span}\{\chi_{\eta_i,a_i} : i=1,\ldots,n\}.$$

Remembering Lemma A.13, thus

$$\iota^*\big(\mathcal{A}'\big) = \big\langle\mathcal{F}(\mathcal{K},M),[\cdot,\cdot]_{K,\delta}\big\rangle' + \operatorname{span}\{\omega_i : i=1,\ldots,n\}, \tag{3.10}$$

where $\omega_i : \mathcal{F}(\mathcal{K}, M) \to \mathbb{C}$ are the functionals acting as

$$\omega_i(f) := \sum_{\zeta \in M} \big(K(\zeta, \eta_i) f(\zeta), a_i\big)_{\mathcal{K}}, \quad f \in \mathcal{F}(\mathcal{K}, M),\ i = 1, \ldots, n.$$

The central observation is that the right side of (3.10) of course depends on K and $\boldsymbol{\delta}$, but does not depend on $\mathcal{A}$.

Assume now that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two reproducing kernel almost Pontryagin spaces which have K as an almost reproducing kernel with the data n and $\boldsymbol{\delta}$ in (3.1).

By what we proved above, $\iota^*(\mathcal{A}_1') = \iota^*(\mathcal{A}_2')$. Theorem A.15 provides us with a linear and isometric homeomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\varphi \circ \iota = \iota$. It follows that $\varphi(f) = f$, $f \in \mathcal{L}$, i.e., for each $\eta \in M$ the restrictions of the continuous maps $\chi_\eta \circ \varphi$ and χ_η to the subspace $\mathcal{L}$ of $\mathcal{A}_1$ coincide. Since $\mathcal{L}$ is dense, these maps coincide on all of $\mathcal{A}_1$. From this, we have

$$\varphi(f) = f, \quad f \in \mathcal{A}_1,$$

and hence that $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal. □

➪ The relation Ξ (defined on p. 280) is the graph of a function which maps

$$\operatorname{dom} \Xi = \Big\{ (K; \boldsymbol{\delta}) \in \mathbb{K}(\mathcal{K}, M) \times \bigcup_{n \in \mathbb{N}_0} \big(\mathcal{K}^n \times M^n \times \mathbb{R}^n\big) : \\ \operatorname{ind}_- K < \infty,\ \gamma_i \neq 0, \big(K(\eta_j, \eta_i) a_j, a_i\big)_{\mathcal{K}} = \delta_{ij} \frac{1}{\gamma_i} \Big\}$$

surjectively onto $\mathbb{RKS}(\mathcal{K}, M)$. Here well-definedness is Proposition 3.10, the description of the domain is Proposition 3.8 for "⊇" together with Lemma 3.4(iii), and Proposition 3.6 for "⊆", and surjectivity is Theorem 3.2.

Let $\mathcal{A} \in \mathbb{RKS}(\mathcal{K}, M)$ be fixed. In order to describe the inverse image $\Xi^{-1}(\{\mathcal{A}\})$, we set $n := \operatorname{ind}_0 \mathcal{A}$ and denote by π_0, π_1, and π_2 the projections

$$\begin{aligned} \pi_0 &: \ \mathbb{K}(\mathcal{K}, M) \times \big(\mathcal{K}^n \times M^n \times \mathbb{R}^n\big) \to \mathbb{K}(\mathcal{K}, M), \\ \pi_1 &: \ \mathbb{K}(\mathcal{K}, M) \times \big(\mathcal{K}^n \times M^n \times \mathbb{R}^n\big) \to \mathcal{K}^n \times M^n, \\ \pi_2 &: \ \mathbb{K}(\mathcal{K}, M) \times \big(\mathcal{K}^n \times M^n \times \mathbb{R}^n\big) \to \mathbb{R}^n. \end{aligned}$$

Then it holds that

$$\Xi^{-1}\big(\{\mathcal{A}\}\big) \cap \pi_1^{-1}\big(\big\{\big((a_i)_{i=1}^n; (\eta_i)_{i=1}^n\big)\big\}\big) \neq \emptyset \quad \iff \quad \mathcal{A}^\circ \cap \bigcap_{i=1}^n \ker \chi_{\eta_i, a_i} = \{0\}$$

If the set on the left side is nonempty, then

$$\mathbb{R}^n \setminus \pi_2\big(\Xi^{-1}\big(\{\mathcal{A}\}\big) \cap \pi_1^{-1}\big(\big\{\big((a_i)_{i=1}^n; (\eta_i)_{i=1}^n\big)\big\}\big)\big)$$

is contained in a closed and nowhere dense subset of $\mathbb{R}^n$. For each $((a_i)_{i=1}^n; (\eta_i)_{i=1}^n; (\gamma_i)_{i=1}^n)$ we have

$$\#\big(\Xi^{-1}\big(\{\mathcal{A}\}\big) \cap (\pi_1 \times \pi_2)^{-1}\big(\big\{\big((a_i)_{i=1}^n; (\eta_i)_{i=1}^n; (\gamma_i)_{i=1}^n\big)\big\}\big)\big) \leq 1.$$

Here the implication "$\Leftarrow$" and the second statement is Theorem 3.2, the implication "$\Rightarrow$" is Lemma 3.4(i), and the last statement is Proposition 3.9.

In order to apply Proposition 3.8 to construct reproducing kernel almost Pontryagin spaces with a prescribed kernel, one has to find $\boldsymbol{\delta}$ such that (3.5) is satisfied. As a rule of thumb, this is not difficult. To illustrate this vague statement, let us consider an example.

Example 3.11. Let $a > 0$. The *Paley–Wiener space* $\mathcal{PW}_a$ is the space of all entire functions of exponential type at most a which are square integrable along the real axis. By a theorem of Paley and Wiener (see, e.g., [25, Chapter III.D]), $\mathcal{PW}_a$ is nothing but the set of Fourier transforms of square integrable functions supported on the interval $[-a, a]$. If endowed with the $L^2(\mathbb{R})$-inner product, i.e., with

$$(f, g) := \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt, \quad f, g \in \mathcal{PW}_a,$$

$\mathcal{PW}_a$ becomes a reproducing kernel Hilbert space (of complex valued functions on $\mathbb{C}$). Its reproducing kernel $K_a(\eta, \zeta)$ is given as (for $\zeta = \bar{\eta}$ this expression has to be interpreted as a derivative which is possible by analyticity)

$$K_a(\eta, \zeta) = \frac{\sin[a(\zeta - \bar{\eta})]}{\pi(\zeta - \bar{\eta})}, \quad \zeta, \eta \in \mathbb{C}.$$

This is a classical result; a proof from the reproducing kernel space perspective is given, e.g., in [12, Theorem 16].

Let us now start from the kernel function K_a, and attempt to apply Proposition 3.8. It is easy to find suitable points η_i. In fact, for arbitrary $n \in \mathbb{N}$, we simply choose pairwise different points $\eta_1, \ldots, \eta_n$ contained in an arithmetic sequence $\alpha + \frac{2\pi}{a}\mathbb{Z}$ where $\alpha \in \mathbb{R}$. Then

$$K_a(\eta_i, \eta_j) = \frac{1}{\pi}\delta_{ij}, \quad i, j = 1, \ldots, n.$$

We see that for each $n \in \mathbb{N}$ there exists a reproducing kernel almost Pontryagin space $\mathcal{A}(n)$ with $\operatorname{ind}_0 \mathcal{A}(n) = n$, such that K_a is an almost reproducing kernel of $\mathcal{A}(n)$.

Let us note that the same argument can be carried out for any (infinite dimensional) de Branges space in the sense of [29, Definition 6.4]. $\Diamond$

4. Reproducing kernel space completions

In this section we investigate the second problem posed in the introduction. The question "Given an inner product space $\mathcal{L}$ whose elements are functions, does there exist a reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically and densely?" can be answered as follows.

Theorem 4.1. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{L}$ be an inner product space of $\mathcal{K}$-valued functions on M (with linear operations acting as pointwise addition and scalar multiplication).*

There exists a reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically, if and only if $\mathcal{L}$ satisfies the following conditions (A), (B), (C).

(A) $\operatorname{ind}_- \mathcal{L} < \infty$.

(B) *There exist $N \in \mathbb{N}$, $(a_i)_{i=1}^N \in \mathcal{K}^N$, and $(\eta_i)_{i=1}^N \in M^N$, such that the following implication holds. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{L}$ with*

$$\lim_{n\to\infty}[f_n, f_n]_{\mathcal{L}} = 0, \qquad \lim_{n\to\infty}[f_n, g]_{\mathcal{L}} = 0, \quad g \in \mathcal{L},$$
$$\lim_{n\to\infty}\chi_{\eta_i,a_i}(f_n) = 0, \quad i = 1, \dots, N,$$

then $\lim_{n\to\infty}\chi_{\eta,a}(f_n) = 0$, $a \in \mathcal{K}$, $\eta \in M$.

(C) *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{L}$ with*

$$\lim_{n,m\to\infty}[f_n - f_m, f_n - f_m]_{\mathcal{L}} = 0, \qquad \lim_{n\to\infty}[f_n - f_m, g]_{\mathcal{L}} = 0, \quad g \in \mathcal{L},$$
$$\lim_{n\to\infty}\chi_{\eta,a}(f_n) = 0, \quad a \in \mathcal{K},\ \eta \in M, \tag{4.1}$$

then $\lim_{n\to\infty}[f_n, g]_{\mathcal{L}} = 0$, $g \in \mathcal{L}$.

If (A), (B) *and* (C) *hold, then there exists a unique reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically as a dense linear subspace.*

This result is an indefinite version of [6, I.4, Theorem].

Before we come to the proof, let us comment on the conditions (A), (B), (C). The role of the condition (A) is of course clear; without having finite negative index there is no chance for $\mathcal{L}$ to be contained in any almost Pontryagin space. Conditions (B) and (C) are less obvious. Informally speaking, (B) is responsible for continuity and (C) is responsible for well-definedness of point evaluation maps. This picture will become mathematically precise in the course of the proof.

The proof of Theorem 4.1 proceeds in three steps. First, we show sufficiency of (A), (B), (C), second we show the uniqueness statement, and finally necessity is established. The essential tool for the proof is the theory of almost Pontryagin space completions.

Proof of Theorem 4.1 *(Sufficiency).* Assume that $\mathcal{L}$ satisfies (A), (B), and (C). Since $\mathcal{L}$ has finite negative index, we can choose an almost Pontryagin space completion $(\iota, \mathcal{B})$ of $\mathcal{L}$ with

$$\iota^*\big(\mathcal{B}'\big) = \mathcal{L}' + \operatorname{span}\{\chi_{\eta_i,a_i}|_{\mathcal{L}} : i = 1, \ldots, N\}.$$

Moreover, we denote by $\tilde{\chi}_{\eta_i,a_i}$ the unique functional in $\mathcal{B}'$ with $\iota^*(\tilde{\chi}_{\eta_i,a_i}) = \chi_{\eta_i,a_i}|_{\mathcal{L}}$.

For each element $f \in \ker \iota$ we have

$$[f, f]_{\mathcal{L}} = [\iota f, \iota f]_{\mathcal{B}} = 0, \qquad [f, g]_{\mathcal{L}} = [\iota f, \iota g]_{\mathcal{B}} = 0, \quad g \in \mathcal{L},$$
$$\chi_{\eta_i,a_i}(f) = \tilde{\chi}_{\eta_i,a_i}(f) = 0, \quad i = 1, \ldots, N.$$

Our assumption (B) applied with the constant sequence $(f)_{n\in\mathbb{N}}$ yields that $\chi_{\eta,a}(f) = 0$, $a \in \mathcal{K}$, $\eta \in M$. This just means that $f = 0$, and we conclude that ι is injective.

Set $\lambda_{\eta,a} := \chi_{\eta,a} \circ \iota^{-1} : \operatorname{ran} \iota \to \mathbb{C}$, $a \in \mathcal{K}$, $\eta \in M$. We are going to show that $\lambda_{\eta,a}$ is bounded w.r.t. the norm of $\mathcal{B}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{ran} \iota$ with $\lim_{n\to\infty} x_n = 0$ in the norm of $\mathcal{B}$. Set $f_n := \iota^{-1} x_n$, then

$$\lim_{n\to\infty} [f_n, f_n]_{\mathcal{L}} = \lim_{n\to\infty} [x_n, x_n]_{\mathcal{B}} = 0,$$
$$\lim_{n\to\infty} [f_n, g]_{\mathcal{L}} = \lim_{n\to\infty} [x_n, \iota g]_{\mathcal{B}} = 0, \quad g \in \mathcal{L},$$
$$\lim_{n\to\infty} \chi_{\eta_i,a_i}(f_n) = \lim_{n\to\infty} \tilde{\chi}_{\eta_i,a_i}(x_n) = 0, \quad i = 1, \ldots, N.$$

It follows from (B) that

$$\lim_{n\to\infty} \lambda_{\eta,a}(x_n) = \lim_{n\to\infty} \chi_{\eta,a}(f_n) = 0, \quad a \in \mathcal{K}, \ \eta \in M.$$

This shows that indeed $\lambda_{\eta,a}$ is bounded w.r.t. the norm of $\mathcal{B}$. Let $\tilde{\chi}_{\eta,a} \in \mathcal{B}'$ be the unique extension of $\lambda_{\eta,a}$ to a continuous functional on $\mathcal{B}$. Notice that this terminology is not in conflict with what we had above: When $\tilde{\chi}_{\eta_i,a_i}$ denotes the functional introduced in the first paragraph of this proof, then $\lambda_{\eta_i,a_i} = \tilde{\chi}_{\eta_i,a_i}|_{\operatorname{ran} \iota}$. Hence, this functional is the unique extension of λ_{η_i,a_i}.

Clearly, we have $\iota^*(\tilde{\chi}_{\eta,a}) = \chi_{\eta,a}|_{\mathcal{L}}$, $a \in \mathcal{K}$, $\eta \in M$, and thus

$$\iota^*\big(\mathcal{B}'\big) = \mathcal{L}' + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \ \eta \in M\}. \tag{4.2}$$

Keep $\eta \in M$ fixed. For $x \in \mathcal{B}$ consider the linear functional ψ_x on $\mathcal{K}$ defined as

$$\psi_x(a) := \overline{\tilde{\chi}_{\eta,a}(x)}, \quad a \in \mathcal{K}.$$

We show that ψ_x is continuous. Since $\tilde{\chi}_{\eta,a} \in \mathcal{B}'$, the map $x \mapsto \psi_x$ is continuous w.r.t. the topology of $\mathcal{B}$ in its domain and the topology of pointwise convergence in its range.

Since $\operatorname{ran}\iota$ is dense in $\mathcal{B}$, thus, each functional ψ_x is the pointwise limit of a sequence of functionals of the form $\psi_{\iota f}$, $f \in \mathcal{L}$. However, for each $f \in \mathcal{L}$ we have

$$\psi_{\iota f}(a) = \overline{\tilde{\chi}_{\eta,a}(\iota f)} = \overline{\chi_{\eta,a}(\iota f)} = \big(a, f(\eta)\big)_{\mathcal{K}}, \quad a \in \mathcal{K},$$

and hence $\psi_{\iota f} \in \mathcal{K}'$. It follows, using the Principle of Uniform Boundedness, that $\psi_x \in \mathcal{K}'$, $x \in \mathcal{B}$.

Since every continuous linear functional on a Krein space can be represented as an inner product, a map $\psi : \mathcal{B} \to \mathcal{K}^M$ is well-defined by requiring

$$\big((\psi x)(\eta), a\big)_{\mathcal{K}} = \tilde{\chi}_{\eta,a}(x), \quad x \in \mathcal{B},\ a \in \mathcal{K},\ \eta \in M. \tag{4.3}$$

Let us show that $\ker\psi \subseteq \mathcal{B}^\circ$. Assume that $x \in \mathcal{B}$ with $\psi x = 0$, then $\tilde{\chi}_{\eta,a}(x) = 0$, $a \in \mathcal{K}$, $\eta \in M$. Choose a sequence $(f_n)_{n\in\mathbb{N}}$ of elements $f_n \in \mathcal{L}$ with $\lim_{n\to\infty} \iota f_n = x$. Then, in particular, $(\iota f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the norm of $\mathcal{B}$, and it follows that

$$\lim_{n,m\to\infty} [f_n - f_m, f_n - f_m]_{\mathcal{L}} = \lim_{n,m\to\infty} [\iota f_n - \iota f_m, \iota f_n - \iota f_m]_{\mathcal{B}} = 0,$$
$$\lim_{n\to\infty} [f_n - f_m, g]_{\mathcal{L}} = \lim_{n\to\infty} [\iota f_n - \iota f_m, \iota g]_{\mathcal{B}} = 0, \quad g \in \mathcal{L}.$$

Moreover, since $\tilde{\chi}_{\eta,a} \in \mathcal{B}'$,

$$\lim_{n\to\infty} \chi_{\eta,a}(f_n) = \lim_{n\to\infty} \tilde{\chi}_{\eta,a}(\iota f_n) = \tilde{\chi}_{\eta,a}(x) = 0, \quad a \in \mathcal{K},\ \eta \in M.$$

Our hypothesis (C) implies that

$$[x, \iota g]_{\mathcal{B}} = \lim_{n\to\infty} [\iota f_n, \iota g]_{\mathcal{B}} = \lim_{n\to\infty} [f_n, g]_{\mathcal{L}} = 0, \quad g \in \mathcal{L}.$$

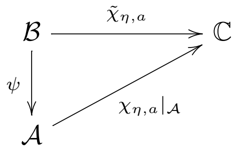
Since $\operatorname{ran}\iota$ is dense in $\mathcal{B}$, it follows that $x \in \mathcal{B}^\circ$.

Set $\mathcal{A} := \operatorname{ran}\psi$. We use Proposition A.7 to make $\mathcal{A}$ into an almost Pontryagin space. Then the map ψ becomes continuous, open, and isometric. Using the defining property (4.3) of ψ, we obtain that for each $f \in \mathcal{L}$

$$\big(f(\eta), a\big)_{\mathcal{K}} = \chi_{\eta,a}(f) = \tilde{\chi}_{\eta,a}(\iota f) = \big(\big(\psi(\iota f)\big)(\eta), a\big)_{\mathcal{K}}, \quad a \in \mathcal{K},\ \eta \in M.$$

This just says that $(\psi \circ \iota)(f) = f$, $f \in \mathcal{L}$. We see that $\mathcal{L}$ is contained in $\mathcal{A}$. Since ψ and ι are both isometric, the set-theoretic inclusion map of $\mathcal{L}$ into $\mathcal{A}$ (being equal to $\psi \circ \iota$) is also isometric.

It remains to show that $\mathcal{A}$ is a reproducing kernel almost Pontryagin space. However, again referring to (4.3), we have for each $a \in \mathcal{K}$ and $\eta \in M$



Since $\tilde{\chi}_{\eta,a}$ is continuous and $\mathcal{A}$ carries the final topology w.r.t. ψ, it follows that $\chi_{\eta,a}|_{\mathcal{A}}$ is continuous. □

Next, observe that (A), (B), (C) ensure existence of a reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically and densely. In fact, the space constructed in the above part of the proof has this property. This follows since ψ is surjective and open and $\iota(\mathcal{L})$ is dense in $\mathcal{B}$. Moreover, generally, if we have any reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically, the closure of $\mathcal{L}$ in this space is again a reproducing kernel almost Pontryagin space and contains $\mathcal{L}$ isometrically and densely.

We continue with a general observation.

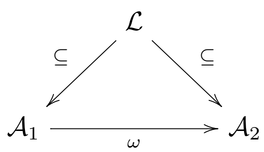
Remark 4.2. Let $\mathcal{A}$ be a reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically as a dense subspace. Then the pair $\langle \subseteq, \mathcal{A}\rangle$, where "$\subseteq$" denotes the set-theoretic inclusion map of $\mathcal{L}$ into $\mathcal{A}$, is an almost Pontryagin space completion of $\mathcal{L}$. The family $\mathcal{F} := \{\chi_{\eta,a}|_{\mathcal{A}} : a \in \mathcal{K}, \eta \in M\} \subseteq \mathcal{A}'$ is point separating, and hence in particular point separating on $\mathcal{A}^\circ$. By Proposition A.3(ii), it follows that

$$\mathcal{A}' = \bigl\{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\bigr\} + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{A}} : a \in \mathcal{K}, \eta \in M\}. \tag{4.4}$$

Clearly, we have $\subseteq^*(\chi_{\eta,a}|_{\mathcal{A}}) = \chi_{\eta,a}|_{\mathcal{L}}$. Remembering Lemma A.13, we obtain

$$\subseteq^*\bigl(\mathcal{A}'\bigr) = \mathcal{L}' + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}. \quad \Diamond \tag{4.5}$$

Proof of Theorem 4.1 *(Uniqueness statement).* Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two reproducing kernel almost Pontryagin spaces which contain $\mathcal{L}$ isometrically and densely. As a consequence of (4.5), the completions $\langle \subseteq, \mathcal{A}_1\rangle$ and $\langle \subseteq, \mathcal{A}_2\rangle$ are isomorphic, cf. Theorem A.15. Let ω be a linear and isometric homeomorphism $\omega : \mathcal{A}_1 \to \mathcal{A}_2$ with



Then $\omega(f) = f$, $f \in \mathcal{L}$, and hence

$$[\chi_{\eta,a}|_{\mathcal{A}_2} \circ \omega]|_{\mathcal{L}} = [\chi_{\eta,a}|_{\mathcal{A}_1}]|_{\mathcal{L}}, \quad a \in \mathcal{K},\ \eta \in M.$$

Since point evaluations are continuous in both spaces $\mathcal{A}_1$ and $\mathcal{A}_2$, it follows that $\chi_{\eta,a}|_{\mathcal{A}_2} \circ \omega = \chi_{\eta,a}|_{\mathcal{A}_1}$. Hence, $\omega(f) = f$ for all $f \in \mathcal{A}_1$. This just says that $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal (since ω is isometric and homeomorphic, they are equal as almost Pontryagin spaces). □

Proof of Theorem 4.1 *(Necessity).* Assume that we are given a reproducing kernel almost Pontryagin space $\mathcal{A}$ which contains $\mathcal{L}$ isometrically and, w.l.o.g., densely. Then, clearly, $\operatorname{ind}_- \mathcal{L} \le \operatorname{ind}_- \mathcal{A} < \infty$. From (4.5) we conclude that

$$\dim\bigl(\bigl[\mathcal{L}' + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}\bigr]/\mathcal{L}'\bigr) = \operatorname{ind}_0 \mathcal{A} < \infty. \tag{4.6}$$

Set $N := \operatorname{ind}_0 \mathcal{A}$, and choose $(a_i)_{i=1}^N \in \mathcal{K}^N$ and $(\eta_i)_{i=1}^N \in M^N$, such that

$$\mathcal{L}' + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\} = \mathcal{L}' + \operatorname{span}\{\chi_{\eta_i,a_i}|_{\mathcal{L}} : i = 1,\dots,N\}. \tag{4.7}$$

Let us show that the implication in (B) holds with this choice of N, $(a_i)_{i=1}^N \in \mathcal{K}^N$, and $(\eta_i)_{i=1}^N \in M^N$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence having the properties stated in the hypothesis of (B). By (4.4) and (4.7), the family $\{\chi_{\eta_i,a_i}|_{\mathcal{L}} : i = 1,\dots,N\}$ acts point separating on $\mathcal{A}^\circ$, cf. Remark A.4. Hence, we may apply Proposition A.5(i), with the subset $\mathcal{L}$ and the family $\{\chi_{\eta_i,a_i} : i = 1,\dots,n\}$, and conclude that $\lim_{n\to\infty} f_n = 0$ in the norm of $\mathcal{A}$. Continuity of $\chi_{\eta,a}|_{\mathcal{A}}$ now gives $\lim_{n\to\infty} \chi_{\eta,a}(f_n) = 0$, $a \in \mathcal{K}$, $\eta \in M$.

We show that the implication in (C) holds. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{L}$ with the properties stated in the hypothesis of (C). Then, by Proposition A.5(ii), applied with the subset $\mathcal{L}$ and the family $\{\chi_{\eta,a} : a \in \mathcal{K}, \eta \in M\}$, the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in the norm of $\mathcal{A}$. By completeness there exists $x \in \mathcal{A}$ with $\lim_{n\to\infty} f_n = x$ in the norm of $\mathcal{A}$. Since

$$\chi_{\eta,a}(x) = \lim_{n\to\infty} \chi_{\eta,a}(f_n) = 0,$$

we have $x = 0$. In particular, $\lim_{n\to\infty}[f_n, g]_{\mathcal{A}} = 0$, $g \in \mathcal{A}$, and hence (again, in particular) the conclusion in (C) holds. □

The conditions (B) and (C) of Theorem 4.1 can be reformulated in a more geometric way.

Proposition 4.3. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{L}$ be an inner product space of $\mathcal{K}$-valued functions on M with $\operatorname{ind}_- \mathcal{L} < \infty$. Consider the conditions (here $\mathcal{T}(\mathcal{L}')$ denotes the topology on $\mathcal{L}'$ introduced in Definition A.16)*

(B′) $\dim([\mathcal{L}' + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}]/\mathcal{L}') < \infty$;
(C′) $\mathcal{L}' \cap \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}$ *is* $\mathcal{T}(\mathcal{L}')$*-dense in* $\mathcal{L}'$.

Then the following implications/equivalences hold:

$$(\mathrm{B}) \Leftrightarrow (\mathrm{B}') \qquad (\mathrm{C}') \Rightarrow (\mathrm{C}) \qquad (\mathrm{B}') \wedge (\mathrm{C}) \Rightarrow (\mathrm{C}')$$

Corollary 4.4. *In Theorem 4.1 we may, separately or simultaneously, substitute* (B) *by* (B′) *and* (C) *by* (C′). □

Proof of Proposition 4.3. (B) ⇒ (B′): This has already been shown in the proof of sufficiency in Theorem 4.1. Namely, since the argument up to (4.2) does not use (C).

(B′) ⇒ (B): This has in essence been shown in the proof of necessity in Theorem 4.1. Namely, choose an almost Pontryagin space completion $\langle \iota, \mathcal{A} \rangle$ with

$$\iota^*(\mathcal{A}') = \mathcal{L}' + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}, \tag{4.8}$$

and proceed as after (4.6). Note here that (4.4) holds by the choice of $\langle \iota, \mathcal{A} \rangle$, when $\chi_{\eta,a}|_{\mathcal{A}}$ is substituted by $(\iota^*|_{\mathcal{A}'})^{-1}(\chi_{\eta,a}|_{\mathcal{L}})$.

(C′) ⇒ (C): Choose a Pontryagin space completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, and set

$$\mathcal{M} := \mathcal{L}' \cap \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}, \quad \mathcal{N} := \left(\iota^*|_{\mathcal{A}'}\right)^{-1}(\mathcal{M}).$$

Since $\iota^*|_{\mathcal{A}'}$ is a homeomorphism of $\mathcal{A}'$ with its norm topology onto $\mathcal{L}'$ with $\mathcal{T}(\mathcal{L}')$, the space $\mathcal{N}$ is dense in $\mathcal{A}'$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}$ subject to (4.1). Then $(\iota f_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in the norm of $\mathcal{A}$, cf. [22, Theorem 2.4]. Set $x := \lim_{n\to\infty}(\iota f_n)$, then

$$\left[\left(\iota^*|_{\mathcal{A}'}\right)^{-1}\phi\right](x) = \lim_{n\to\infty}\left[\left(\iota^*|_{\mathcal{A}'}\right)^{-1}\phi\right](\iota f_n) = \lim_{n\to\infty}\phi(f_n) = 0, \quad \phi \in \mathcal{M}.$$

Since $\mathcal{N}$ is dense, this implies that $x = 0$. In turn, we have $\lim_{n\to\infty}[f_n, g]_{\mathcal{L}} = \lim_{n\to\infty}[\iota f_n, \iota g]_{\mathcal{A}} = 0$, $g \in \mathcal{L}$.

(B′) ∧ (C) ⇒ (C′): Again choose an almost Pontryagin space completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$ with (4.8), and set $\tilde{\chi}_{\eta,a} := (\iota^*|_{\mathcal{A}'})^{-1}(\chi_{\eta,a}|_{\mathcal{L}})$, $a \in \mathcal{K}, \eta \in M$. We claim that the annihilator of $\operatorname{span}\{\tilde{\chi}_{\eta,a} : a \in \mathcal{K}, \eta \in M\}$ w.r.t. the canonical duality between $\mathcal{A}$ and $\mathcal{A}'$ is equal to $\{0\}$. Assume that $x \in \mathcal{A}$ with $\tilde{\chi}_{\eta,a}(x) = 0$, $a \in \mathcal{K}$, $\eta \in M$, and choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{L}$ with $\lim_{n\to\infty}(\iota f_n) = x$ in the norm of $\mathcal{A}$. Then $(f_n)_{n\in\mathbb{N}}$ satisfies (4.1), and (C) implies that

$$[x, \iota g]_{\mathcal{A}} = \lim_{n\to\infty}[\iota f_n, \iota g]_{\mathcal{A}} = \lim_{n\to\infty}[f_n, g]_{\mathcal{L}} = 0, \quad g \in \mathcal{L}.$$

It follows that $x = 0$, and our claim is established. We conclude that $\operatorname{span}\{\tilde{\chi}_{\eta,a} : a \in \mathcal{K}, \eta \in M\}$ is $\sigma(\mathcal{A}', \mathcal{A})$-dense in $\mathcal{A}'$. Since $\mathcal{A}$ is reflexive, it follows that this linear span is also norm-dense in $\mathcal{A}'$.

Set $N := \operatorname{ind}_0 \mathcal{A}$, and choose $(a_i)_{i=1}^n \in \mathcal{K}^N$ and $(\eta_i)_{i=1}^n \in M^N$ such that (4.7) holds. Then (for the notation $\mathcal{A}^\wr$ see (A.6))

$$\mathcal{A}' = \mathcal{A}^\wr \dot{+} \operatorname{span}\{\tilde{\chi}_{\eta_i,a_i} : i = 1, \dots, n\}.$$

Let P be the projection of $\mathcal{A}'$ onto $\mathcal{A}^\wr$ with kernel $\operatorname{span}\{\tilde{\chi}_{\eta_i,a_i} : i = 1, \dots, n\}$. Then $\operatorname{ran} P$ is closed by Proposition A.1 and $\ker P$ is closed by finite-dimensionality. Hence, P is continuous. Being continuous and surjective, P maps dense sets of $\mathcal{A}'$ onto dense sets of $\mathcal{A}^\wr$. Clearly,

$$\begin{aligned} P\big(\operatorname{span}\{\tilde{\chi}_{\eta,a} : a \in \mathcal{K}, \eta \in M\}\big) &= \mathcal{A}^\wr \cap \operatorname{span}\{\tilde{\chi}_{\eta,a} : a \in \mathcal{K}, \eta \in M\} \\ &= \big(\iota^*|_{\mathcal{A}'}\big)^{-1}\big(\mathcal{L}' \cap \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}\big). \end{aligned}$$

Since $\iota^*|_{\mathcal{A}^\wr}$ is a homeomorphism, validity of (C′) follows. □

Let $\mathcal{L}$ be a space of $\mathcal{K}$-valued functions on M and assume that $\mathcal{L}$ has a reproducing kernel completion. Then, generically, there exist many reproducing kernel almost Pontryagin spaces which contain $\mathcal{L}$ isometrically but not necessarily densely. Let us briefly comment on this fact.

Lemma 4.5. *Let $\mathcal{L}$ be a space of $\mathcal{K}$-valued functions on M, assume that $\mathcal{L}$ has a reproducing kernel completion, and denote it by $\mathcal{A}$. Let $\mathcal{B}$ be an almost Pontryagin space which contains $\mathcal{A}$ isometrically as a closed subspace with finite codimension. Provided that $\dim(\mathcal{B}/\mathcal{A}) \leq \dim(\mathcal{K}^M/\mathcal{A})$, there exists an isomorphic copy of $\mathcal{B}$ which is a reproducing kernel almost Pontryagin space and which contains $\mathcal{A}$ isometrically as a closed subspace with finite codimension.*

Proof. Set $n := \dim(\mathcal{B}/\mathcal{A})$, and choose elements $x_1, \dots, x_n \in \mathcal{B}$ and $g_1, \dots, g_n \in \mathcal{K}^M$ which are linearly independent modulo $\mathcal{A}$ (in $\mathcal{B}$ and in $\mathcal{K}^M$, respectively). Then we can define a map $\psi : \mathcal{B} \to \mathcal{K}^M$ by linearity and

$$\psi(f) := f, \quad f \in \mathcal{A}, \qquad \psi(x_i) := g_i, \quad i = 1, \dots, n.$$

Clearly, ψ maps $\mathcal{B}$ injectively into $\mathcal{K}^M$. Set $\tilde{\mathcal{B}} := \operatorname{ran} \psi$, and define an almost Pontryagin space structure on $\tilde{\mathcal{B}}$ by means of Proposition A.7. Then ψ becomes an isomorphism. We know that for each $a \in \mathcal{K}$ and $\eta \in M$ the functional $\chi_{\eta,a}|_{\mathcal{A}}$ is continuous. Since $\mathcal{A}$ is a closed subspace of $\tilde{\mathcal{B}}$ with finite codimension, its extension $\chi_{\eta,a}|_{\tilde{\mathcal{B}}}$ is again continuous. □

Theorem 4.1 justifies the following definition.

Definition 4.6. Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{L}$ be an inner product space of $\mathcal{K}$-valued functions on M. Assume that there exists a reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically. Then we refer to the unique reproducing kernel almost Pontryagin space which contains $\mathcal{L}$ isometrically and densely as the *reproducing kernel space completion of* $\mathcal{L}$. We denote the degree of degeneracy of this space as $\Delta(\mathcal{L})$. $\diamondsuit$

The number $\Delta(\mathcal{L})$ is an important structural constant associated with $\mathcal{L}$. As we saw in the proof of necessity, cf. (4.6), we have

$$\Delta(\mathcal{L}) = \dim\bigl(\bigl[\mathcal{L}' + \operatorname{span}\{\chi_{\eta,a}|_{\mathcal{L}} : a \in \mathcal{K}, \eta \in M\}\bigr]/\mathcal{L}'\bigr) < \infty$$

An alternative way to compute $\Delta(\mathcal{L})$, proceeds in terms of the condition (B). The following fact is seen by revisiting the proof of Theorem 4.1.

Proposition 4.7. *Let $\mathcal{L}$ be a space of $\mathcal{K}$-valued functions on M and assume that $\mathcal{L}$ has a reproducing kernel space completion. Then $\Delta(\mathcal{L})$ is the minimum of all numbers N such that* (B) *holds with $N \in \mathbb{N}_0$ and some choice of $(a_i)_{i=1}^N \in \mathcal{K}^N$, $(\eta_i)_{i=1}^N \in M^N$.*

Proof. To see this, assume first that (B) holds with N and some data $(a_i)_{i=1}^N \in \mathcal{K}^N$, $(\eta_i)_{i=1}^N \in M^N$. We can use this data in the proof of sufficiency. Remembering (4.2) and the construction of $\mathcal{A}$ as an isometric image of $\mathcal{B}$, yields

$$\Delta(\mathcal{L}) = \operatorname{ind}_0 \mathcal{A} \leq \operatorname{ind}_0 \mathcal{B} = N.$$

Conversely, revisit the proof of necessity. There we saw that (B) holds for an appropriate choice of $\Delta(\mathcal{L})$ many functionals $\chi_{\eta_i,a_i}|_{\mathcal{L}}$; see (4.7) and the argument following it. $\square$

A sufficient condition for existence of a Pontryagin space completion occurs when the space $\mathcal{L}$ is connected with an L^2-space of a positive measure. For simplicity, we restrict considerations to the scalar-valued case.

Proposition 4.8. *Let $\mathcal{L}$ be an inner product space of complex-valued functions on a set M. Assume that there exists a positive measure μ defined on some σ-algebra on M, points $\eta_1, \ldots, \eta_n \in M$ and numbers $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, such that each element of $\mathcal{L}$ is square integrable w.r.t. μ and*

$$[f,g]_{\mathcal{L}} = \int\limits_M f(\lambda)\overline{g(\lambda)}\, d\mu(\lambda) + \sum_{i=1}^n \gamma_i f(\eta_i)\overline{g(\eta_i)}, \quad f,g \in \mathcal{L}. \tag{4.9}$$

Then the conditions (A) *and* (C) *hold. If, in addition, we find $\lambda_1, \ldots, \lambda_m \in M$ and $C_\eta > 0$, $\eta \in M$, with*

$$\bigl|f(\eta)\bigr|^2 \leq C_\eta \Biggl(\int\limits_M \bigl|f(\lambda)\bigr|^2\, d\mu(\lambda) + \sum_{j=1}^m \bigl|f(\lambda_j)\bigr|^2\Biggr), \quad f \in \mathcal{L},\ \eta \in M, \tag{4.10}$$

then also the condition (B) *holds.*

Proof. From the representation (4.9), we see that the inner product $[\cdot,\cdot]_{\mathcal{A}}$ is positive semidefinite on the subspace $\bigcap_{i=1}^{n}\ker(\chi_{\eta_i}|_{\mathcal{A}})$. This subspace has finite codimension, and hence $\operatorname{ind}_-\mathcal{L}<\infty$. Assume now that a sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{L}$ is given which satisfies

$$\lim_{n,m\to\infty}[f_n-f_m,f_n-f_m]_{\mathcal{L}}=0,\qquad \lim_{n\to\infty}\chi_\eta(f_n)=0,\quad \eta\in M.$$

Then (4.9) implies that

$$\lim_{n,m\to\infty}\int_M |f_n-f_m|^2\,d\mu=0.$$

Hence, there exists $f\in L^2(\mu)$ such that $\lim_{n\to\infty}f_n=f$ in the norm of $L^2(\mu)$. Convergence in $L^2(\mu)$ implies pointwise convergence μ-a.e. of some subsequence. The pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}}$, however, is equal to 0. It follows that $f=0$ μ-a.e. Using again the representation (4.9) of the inner product on $\mathcal{L}$, we see that $\lim_{n\to\infty}[f_n,g]_{\mathcal{L}}=0$ for all $g\in\mathcal{L}$.

Assume in addition that (4.10) holds. Consider the space

$$\mathcal{B}:=L^2(\mu)\times\mathbb{C}^n\times\mathbb{C}^m$$

endowed with the product topology. Clearly, $\mathcal{B}$ contains the Hilbert space $L^2(\mu)$ as a closed subspace with finite codimension. Note that the product topology is induced by the Hilbert space inner product

$$\begin{aligned}&\big(\big(h;(\alpha_i)_{i=1}^n;(\beta_i)_{j=1}^m\big),\big(h';\big(\alpha_i'\big)_{i=1}^n;\big(\beta_j'\big)_{j=1}^m\big)\big)_{\mathcal{B}}\\ &:=\int_M h(\lambda)\overline{h'(\lambda)}\,d\mu+\sum_{i=1}^n\alpha_i\overline{\alpha_i'}+\sum_{j=1}^m\beta_j\overline{\beta_j'}.\end{aligned}$$

We endow $\mathcal{B}$ with the inner product

$$\big[\big(h;(\alpha_i)_{i=1}^n;(\beta_i)_{j=1}^m\big),\big(h';\big(\alpha_i'\big)_{i=1}^n;\big(\beta_j'\big)_{j=1}^m\big)\big]_{\mathcal{B}}:=\int_M h(\lambda)\overline{h'(\lambda)}\,d\mu+\sum_{i=1}^n\gamma_i\alpha_i\overline{\alpha_i'}.$$

Clearly, this inner product is continuous w.r.t. the topology of $\mathcal{B}$, and a Hilbert space inner product on $L^2(\mu)$. We conclude that $\mathcal{B}$ is an almost Pontryagin space.

Define a map $\iota:\mathcal{L}\to\mathcal{B}$ as

$$\iota(f):=\big(f;\big(f(\eta_i)\big)_{i=1}^n;\big(f(\lambda_j)\big)_{j=1}^m\big),\quad f\in\mathcal{L}.$$

By our definition of $[\cdot,\cdot]_{\mathcal{B}}$, this map is isometric. Hence, $\langle\iota,\operatorname{Clos}(\operatorname{ran}\iota)\rangle$ is an almost Pontryagin space completion of $\mathcal{L}$. From (4.10) we see that ι is injective. The composition

$(\chi_\eta|_{\mathcal{L}}) \circ \iota^{-1}$ is a linear functional on $\operatorname{ran}\iota$, and by (4.10) it is bounded w.r.t. the norm induced by $(\cdot,\cdot)_{\mathcal{B}}$. Hence, it has an extension $\tilde{\chi}_\eta \in [\operatorname{Clos}(\operatorname{ran}\iota)]'$. Clearly, $\iota^*(\tilde{\chi}_\eta) = \chi_\eta|_{\mathcal{L}}$. We obtain

$$\dim\big(\big[\mathcal{L}' + \operatorname{span}\{\chi_\eta|_{\mathcal{L}} : \eta \in M\}\big]/\mathcal{L}'\big) \le \dim\big(\iota^*\big(\big[\operatorname{Clos}(\operatorname{ran}\iota)\big]'\big)/\mathcal{L}'\big) < \infty.$$

From the argument in the proof of Theorem 4.1, necessity, we obtain that (B) holds. □

We close this section with showing a practical result which increases applicability of Theorem 4.1. Namely, sometimes one has much better control on some subspace of an inner product space $\mathcal{L}$ than on $\mathcal{L}$ itself. The following statement says that, concerning completions, it is possible to restrict to subspaces which are in some sense dense. Of course this fact is not a surprise, however, its proof is not obvious.

Proposition 4.9. *Let M be a nonempty set, let $\mathcal{K}$ be a Krein space, and let $\mathcal{L}$ and $\mathcal{L}_0$ be inner product spaces of $\mathcal{K}$-valued functions on M such that $\mathcal{L}$ contains $\mathcal{L}_0$ isometrically. Assume that the following density condition holds: For each $f \in \mathcal{L}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in \mathcal{L}_0$, with*

$$\begin{aligned} &\lim_{n\to\infty} [f_n, f_n]_{\mathcal{L}} = \lim_{n\to\infty} [f_n, f]_{\mathcal{L}} = [f, f]_{\mathcal{L}}, \\ &\lim_{n\to\infty} \chi_{\eta,a}(f_n) = \chi_{\eta,a}(f), \quad a \in \mathcal{K},\ \eta \in M. \end{aligned} \tag{4.11}$$

Then $\mathcal{L}$ has a reproducing kernel space completion if and only if $\mathcal{L}_0$ has one. If $\mathcal{L}$ and $\mathcal{L}_0$ have reproducing kernel space completions, then they are equal.

Proof. One implication is obvious. Namely, each reproducing kernel almost Pontryagin space which isometrically contains $\mathcal{L}$ also isometrically contains $\mathcal{L}_0$. Hence existence of a reproducing kernel space completion of $\mathcal{L}$ implies existence of one for $\mathcal{L}_0$.

For the proof of the converse assume that $\mathcal{A}$ is the reproducing kernel space completion of $\mathcal{L}_0$. We are thus in the situation

$$\begin{array}{ccc} \mathcal{L} & & \mathcal{A} \\ & \rotatebox{-45}{$\supseteq$} \quad \rotatebox{45}{$\subseteq$} & \\ & \mathcal{L}_0 & \end{array}$$

By Proposition A.9, applied with the family $\{\chi_{\eta,a} : a \in \mathcal{K}, \eta \in M\}$, there exist $\eta_1, \dots, \eta_n \in M$ and $a_1, \dots, a_n \in \mathcal{K}$, such that the inner product (by possibly changing the values of a_i, we may assume that the constant γ obtained from Proposition A.9 is equal to 1)

$$(f, g)_{\mathcal{A}} = [f, g]_{\mathcal{A}} + \sum_{i=1}^{n} \chi_{\eta_i,a_i}(f)\overline{\chi_{\eta_i,a_i}(g)}, \quad f, g \in \mathcal{A},$$

is a Hilbert space scalar product on $\mathcal{A}$ (and induces the topology of $\mathcal{A}$). We define an inner product on $\mathcal{L}$ by the same formula, i.e.,

$$[\![f,g]\!]_{\mathcal{L}} := [f,g]_{\mathcal{L}} + \sum_{i=1}^{n} \chi_{\eta_i,a_i}(f)\overline{\chi_{\eta_i,a_i}(g)}, \quad f,g \in \mathcal{L}.$$

Since $\mathcal{A}$ contains $\mathcal{L}_0$ isometrically, we have

$$(f,g)_{\mathcal{A}} = [\![f,g]\!]_{\mathcal{L}}, \quad f,g \in \mathcal{L}_0.$$

If $f \in \mathcal{L}$, choose a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in \mathcal{L}_0$, with (4.11). Then

$$\begin{aligned} [\![f,f]\!]_{\mathcal{L}} &= [f,f]_{\mathcal{L}} + \sum_{i=1}^{n} \left|\chi_{\eta_i,a_i}(f)\right|^2 \\ &= \lim_{n\to\infty} \left([f_n,f_n]_{\mathcal{L}_0} + \sum_{i=1}^{n} \left|\chi_{\eta_i,a_i}(f_n)\right|^2 \right) = \lim_{n\to\infty} (f_n,f_n)_{\mathcal{A}} \geq 0. \end{aligned}$$

Due to (4.11), we have $\lim_{n\to\infty}[f_n - f, f_n - f]_{\mathcal{L}} = 0$, and in turn $\lim_{n\to\infty}[\![f_n - f, f_n - f]\!]_{\mathcal{L}} = 0$. Since $[\![\cdot,\cdot]\!]_{\mathcal{L}}$ is positive semidefinite, the triangle inequality applies, and we obtain

$$\begin{aligned} (f_n - f_m, f_n - f_m)_{\mathcal{A}}^{\frac{1}{2}} &= [\![f_n - f_m, f_n - f_m]\!]_{\mathcal{L}}^{\frac{1}{2}} \\ &\leq [\![f_n - f, f_n - f]\!]_{\mathcal{L}}^{\frac{1}{2}} + [\![f - f_m, f - f_m]\!]_{\mathcal{L}}^{\frac{1}{2}} \to 0, \quad n,m \to \infty. \end{aligned}$$

Let $g \in \mathcal{A}$ be such that $\lim_{n\to\infty} f_n = g$ in the norm of $\mathcal{A}$. Since point evaluations are continuous on $\mathcal{A}$, we obtain

$$\chi_{\eta,a}(g) = \lim_{n\to\infty} \chi_{\eta,a}(f_n) = \chi_{\eta,a}(f), \quad a \in \mathcal{K},\ \eta \in M,$$

i.e., $g = f$. This already shows that $\mathcal{L} \subseteq \mathcal{A}$. By continuity of the inner product, we moreover have

$$[f,f]_{\mathcal{A}} = \lim_{n\to\infty} [f_n,f_n]_{\mathcal{L}_0} = [f,f]_{\mathcal{L}}.$$

Using the polar identity, thus, $\mathcal{L}$ is contained isometrically in $\mathcal{A}$. Since $\mathcal{L} \supseteq \mathcal{L}_0$ and $\mathcal{L}_0$ is dense in $\mathcal{A}$, also $\mathcal{L}$ is dense in $\mathcal{A}$. Hence, $\mathcal{A}$ is the reproducing kernel completion of $\mathcal{L}$. □

Appendix A. Some supplements to the theory of almost Pontryagin spaces

In this appendix we provide some general results about almost Pontryagin spaces which are used in the present paper but are not yet available in the literature. The first couple of them (Propositions A.1–A.7) are simple and in essence straightforward generalisations of well-known Pontryagin space results. Proposition A.9 and Lemma A.10 are more involved. They contain a practical perturbation method and provide a geometric proof for it. In the last part of this appendix we recall the notion of almost Pontryagin space completions, those results about such completions which are used in the present context, and provide some supplements on this topic. In order to keep the presentation clean and mainly self-contained, we try to minimise the number of results imported from the literature.

A.1. Geometry of almost Pontryagin spaces and their duals

We frequently use the weak-star topology on the topological dual $\mathcal{A}'$ of $\mathcal{A}$, and the weak topology on $\mathcal{A}$ itself. If necessary, the topology to which a notion (like closedness, continuity, etc.) refers, is made explicit by prepending "w-" or "w^*-", respectively. Recall that we speak of "convergence in the norm of $\mathcal{A}$", meaning convergence w.r.t. some norm inducing the topology of $\mathcal{A}$.

The first object which we investigate is the topological dual of an almost Pontryagin space. It is well-known that the topological dual space $\mathcal{A}'$ of a Pontryagin space $\mathcal{A}$ is exhausted by the functionals $[\cdot, y]_{\mathcal{A}}$, $y \in \mathcal{A}$, see, e.g., [22, Lemma 5.1]. In the presence of degeneracy, this is not anymore the case; this family of functionals is not even point separating. However, the dual of an almost Pontryagin space is not much larger than that. The argument which shows this already appeared in the proof of [40, Lemma 6.5] (and is in fact a general "Banach-space argument").

Proposition A.1. *Let $\mathcal{A}$ be an almost Pontryagin space. Then the linear subspace $\{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\}$ of $\mathcal{A}'$ is w^*-closed, and*

$$\dim\bigl(\mathcal{A}'/\bigl\{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\bigr\}\bigr) = \operatorname{ind}_0 \mathcal{A}.$$

Proof. Consider the Pontryagin space $\mathcal{A}/\mathcal{A}^\circ$ and let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ be the canonical projection, cf. [29, Proposition 3.5]. Moreover, let $\pi' : (\mathcal{A}/\mathcal{A}^\circ)' \to \mathcal{A}'$ be the adjoint of π. Since π is surjective, in particular, the range of π is closed. By the Closed Range Theorem, thus $\operatorname{ran} \pi'$ is a w^*-closed subspace of $\mathcal{A}'$. Since π is isometric we have

$$\pi'\bigl([\cdot, \pi y]_{\mathcal{A}/\mathcal{A}^\circ}\bigr) = [\cdot, y]_{\mathcal{A}}, \quad y \in \mathcal{A}.$$

Now the fact that π is surjective yields

$$\pi'\bigl((\mathcal{A}/\mathcal{A}^\circ)'\bigr) = \bigl\{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\bigr\}, \tag{A.1}$$

and hence $\{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\}$ is a w^*-closed subspace of $\mathcal{A}'$. To compute the codimension of $\{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\}$, we use closedness and the fact that $\ker \pi = \mathcal{A}^\circ$ is finite dimensional. From this it follows that (here $(\ker \pi)^\perp$ denotes the annihilator of $\ker \pi$ in $\mathcal{A}'$)

$$\mathcal{A}'/\bigl\{[\cdot, y]_{\mathcal{A}} : y \in \mathcal{A}\bigr\} = \mathcal{A}'/\operatorname{ran} \pi' = \mathcal{A}'/(\ker \pi)^\perp \cong (\ker \pi)' \cong \ker \pi = \mathcal{A}^\circ. \qquad \square$$

Informally speaking, the proof of the above proposition relies on the fact that the almost Pontryagin space $\mathcal{A}$ differs from the Pontryagin space $\mathcal{A}/\mathcal{A}^\circ$ only by "something finite-dimensional". Let us exploit this idea further.

To formulate the below results, we introduce one notation.

Definition A.2. Let $\mathcal{A}$ be an almost Pontryagin space, and let $\mathcal{F}$ be a family of continuous linear functionals on $\mathcal{A}$. Then we say that $\mathcal{F}$ is *point separating on* $\mathcal{A}^\circ$, if

$$\mathcal{A}^\circ \cap \bigcap_{\varphi \in \mathcal{F}} \ker \varphi = \{0\}. \quad \Diamond$$

Proposition A.3. *Let $\mathcal{A}$ be an almost Pontryagin space, and let $\mathcal{F} \subseteq \mathcal{A}'$ be point separating on $\mathcal{A}^\circ$. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ the canonical projection. Then the following statements hold.*

(i) *The topology of $\mathcal{A}$ is the initial topology with respect to the family of maps*

$$\{\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ\} \cup \{\varphi : \mathcal{A} \to \mathbb{C}, \varphi \in \mathcal{F}\}. \tag{A.2}$$

Here $\mathcal{A}/\mathcal{A}^\circ$ is understood to be endowed with its Pontryagin space topology (and $\mathbb{C}$ with the Euclidean topology).

(ii) *The topological dual $\mathcal{A}'$ of $\mathcal{A}$ is given as*

$$\mathcal{A}' = \big\{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\big\} + \operatorname{span} \mathcal{F}. \tag{A.3}$$

Proof. Denote by $\mathcal{O}$ the topology of the almost Pontryagin space $\mathcal{A}$, and by $\mathcal{T}$ the initial topology induced on $\mathcal{A}$ by the family (A.2). Note that, since $\mathcal{F}$ is point separating on $\mathcal{A}^\circ$, and $\ker \pi = \mathcal{A}^\circ$, the topology $\mathcal{T}$ is Hausdorff. Moreover, since π as well as each $\varphi \in \mathcal{F}$ is continuous w.r.t. $\mathcal{O}$, we certainly have $\mathcal{T} \subseteq \mathcal{O}$.

Since $\dim \mathcal{A}^\circ < \infty$, we may choose a $\mathcal{T}$-closed subspace $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \dot{+} \mathcal{A}^\circ$. Then $\langle \mathcal{A}, \mathcal{T}\rangle$ is homeomorphic to $\langle \mathcal{B}, \mathcal{T}|_\mathcal{B}\rangle \times \langle \mathcal{A}^\circ, \mathcal{T}|_{\mathcal{A}^\circ}\rangle$. Clearly, $\mathcal{B}$ is also $\mathcal{O}$-closed, and hence $\langle \mathcal{A}, \mathcal{O}\rangle$ is homeomorphic to $\langle \mathcal{B}, \mathcal{O}|_\mathcal{B}\rangle \times \langle \mathcal{A}^\circ, \mathcal{O}|_{\mathcal{A}^\circ}\rangle$.

By the Open Mapping Theorem $\pi|_\mathcal{B}$ is a homeomorphism of $\langle \mathcal{B}, \mathcal{O}|_\mathcal{B}\rangle$ onto $\mathcal{A}/\mathcal{A}^\circ$. Hence, $\mathcal{O}|_\mathcal{B}$ is the initial topology on $\mathcal{B}$ w.r.t. the one-element family $\{\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ\}$. This implies that $\mathcal{O}|_\mathcal{B}$ is coarser than $\mathcal{T}|_\mathcal{B}$. Together with the fact that $\mathcal{T} \subseteq \mathcal{O}$, thus, $\mathcal{O}|_\mathcal{B} = \mathcal{T}|_\mathcal{B}$. Since $\dim \mathcal{A}^\circ < \infty$, and both of $\mathcal{O}|_{\mathcal{A}^\circ}$ and $\mathcal{T}|_{\mathcal{A}^\circ}$ are Hausdorff, we also have $\mathcal{O}|_{\mathcal{A}^\circ} = \mathcal{T}|_{\mathcal{A}^\circ}$. In total, $\mathcal{O} = \mathcal{T}$.

We come to the proof of (ii). Denote the linear space on the right side of (A.3) by $\mathcal{G}$. Since by Proposition A.1

$$\dim\big(\mathcal{G}/\big\{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\big\}\big) \le \dim\big(\mathcal{A}'/\big\{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\big\}\big) < \infty,$$

and $\{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\}$ is w^*-closed, also $\mathcal{G}$ is w^*-closed.

Assume on the contrary that the equality (A.3) does not hold, so that we have "$\supsetneq$" in (A.3). The Hahn–Banach Theorem provides us with a w^*-continuous functional on $\mathcal{A}'$ which annihilates $\mathcal{G}$ but does not vanish identically. Since every w^*-continuous functional is point evaluation at a point of $\mathcal{A}$, we obtain an element $x \in \mathcal{A} \setminus \{0\}$ with $\psi(x) = 0$, $\psi \in \mathcal{G}$. From this it follows first that $x \in \mathcal{A}^\circ$ and then by the hypothesis on $\mathcal{F}$ that $x = 0$. We have reached a contradiction. $\square$

Remark A.4. Note that, if $\mathcal{F} \subseteq \mathcal{A}'$ satisfies (A.3), then $\mathcal{F}$ must be point separating on $\mathcal{A}^\circ$. This follows since $\mathcal{A}'$ is point separating, and $\mathcal{A}^\circ$ is annihilated by $\{[\cdot, y]_\mathcal{A} : y \in \mathcal{A}\}$. $\lozenge$

Next, we turn to convergence and Cauchy-property of sequences. From (A.3) we readily see that a sequence $(x_n)_{n\in\mathbb{N}}$ in an almost Pontryagin space $\mathcal{A}$ converges weakly to an element $x \in \mathcal{A}$, if and only if

$$\lim_{n\to\infty} [x_n, y]_\mathcal{A} = [x, y]_\mathcal{A}, \quad y \in \mathcal{A}, \qquad \lim_{n\to\infty} \varphi(x_n) \to \varphi(x), \quad \varphi \in \mathcal{F},$$

provided $\mathcal{F} \subseteq \mathcal{A}'$ is point separating on $\mathcal{A}^\circ$.

The next result shows that also convergence in the norm of $\mathcal{A}$ (or being a Cauchy-sequence w.r.t. the norm of $\mathcal{A}$) can be characterised in a similar fashion. For the nondegenerated case this is a standard fact, see, e.g., [22, Theorem 2.4].

Proposition A.5. *Let $\mathcal{A}$ be an almost Pontryagin space, let $L \subseteq \mathcal{A}$ be a subset with* $\mathrm{Clos}_\mathcal{A}[\mathrm{span}\, L] = \mathcal{A}$*, and let $\mathcal{F} \subseteq \mathcal{A}'$ be point separating on $\mathcal{A}^\circ$. Moreover, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ and $x \in \mathcal{A}$.*

(i) *It holds that $\lim_{n\to\infty} x_n = x$ in the norm of $\mathcal{A}$, if and only if*

$$\lim_{n\to\infty} [x_n, y]_\mathcal{A} = [x, y]_\mathcal{A}, \quad y \in L, \qquad \lim_{n\to\infty} [x_n, x_n]_\mathcal{A} = [x, x]_\mathcal{A},$$
$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x), \quad \varphi \in \mathcal{F}.$$

(ii) *The sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in the norm of $\mathcal{A}$, if and only if*

$$\lim_{n,m\to\infty} [x_n - x_m, y]_\mathcal{A} = 0, \quad y \in L, \qquad \lim_{n,m\to\infty} [x_n - x_m, x_n - x_m]_\mathcal{A} = 0,$$
$$\lim_{n\to\infty} \varphi(x_n - x_m) = 0, \quad \varphi \in \mathcal{F}.$$

Proof. Necessity is obvious. To show sufficiency, assume that the conditions stated in (i) are satisfied. The crucial point is that, by Proposition A.3(i), the topology of $\mathcal{A}$ is the initial topology w.r.t. the family $\{\pi\} \cup \mathcal{F}$. For item (i), it thus suffices to show that

$$\lim_{n\to\infty} \pi(x_n) = \pi(x) \quad \text{and} \quad \lim_{n\to\infty} \varphi(x_n) = \varphi(x), \quad \varphi \in \mathcal{F}.$$

The second condition is written explicitly in the hypothesis. To see the first, since π is isometric, we have

$$\lim_{n\to\infty}[\pi x_n, \pi y]_{\mathcal{A}/\mathcal{A}^\circ} = [\pi x, \pi y]_{\mathcal{A}/\mathcal{A}^\circ}, \quad y \in M,$$
$$\lim_{n\to\infty}[\pi x_n, \pi x_n]_{\mathcal{A}/\mathcal{A}^\circ} = [\pi x, \pi x]_{\mathcal{A}/\mathcal{A}^\circ}.$$

An application of [22, Theorem 2.4(i)] gives $\lim_{n\to\infty}\pi(x_n) = \pi(x)$ in the norm of $\mathcal{A}/\mathcal{A}^\circ$.

The proof of item (ii) is established in the same way, referring to [22, Theorem 2.4(ii)]. □

We continue with some geometric facts. Again, the corresponding results in the Pontryagin space case are well-known, see, e.g., [22, Theorem 3.2] for (i), and [22, p. 25, Corollary 2] for (ii), and [22, Theorem 3.1] for (iii).

Lemma A.6. *Let $\langle\mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ be an almost Pontryagin space, and let $\mathcal{B}$ be an $\mathcal{O}$-closed subspace of $\mathcal{A}$.*

(i) *If $\mathcal{B}$ is nondegenerated, then $\mathcal{B}$ is orthocomplemented (i.e., $\mathcal{B} + \mathcal{B}^\perp = \mathcal{A}$).*
(ii) *If $\mathcal{B}$ is positive definite, then $\langle\mathcal{B}, [\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{B}\times\mathcal{B}}\rangle$ is a Hilbert space and the topology induced on $\mathcal{B}$ by $[\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{B}\times\mathcal{B}}$ is equal to $\mathcal{O}|_{\mathcal{B}}$.*
(iii) *If $\mathcal{B}^\perp = \mathcal{A}^\circ \subseteq \mathcal{B}$, then $\mathcal{B} = \mathcal{A}$.*

Proof. Assume that $\mathcal{B}$ is nondegenerated. We reduce to the Pontryagin space case. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ the canonical projection, and remember that π maps closed subspaces to closed subspaces, cf. [29, Proposition 3.5]. Then $\pi(\mathcal{B})$ is a closed and nondegenerated subspace of $\mathcal{A}/\mathcal{A}^\circ$, and hence

$$\pi(\mathcal{B})[\dot{+}]\pi(\mathcal{B})^\perp = \mathcal{A}/\mathcal{A}^\circ.$$

It follows that

$$\pi^{-1}\big(\pi(\mathcal{B})\big)[+]\pi^{-1}\big(\pi(\mathcal{B})^\perp\big) = \mathcal{A}.$$

However, $\pi^{-1}(\pi(\mathcal{B})) = \mathcal{B} + \mathcal{A}^\circ$ and $\pi^{-1}(\pi(\mathcal{B})^\perp) = \mathcal{B}^\perp + \mathcal{A}^\circ$. Since $\mathcal{A}^\circ \subseteq \mathcal{B}^\perp$, it follows that $\mathcal{B}[+]\mathcal{B}^\perp = \mathcal{A}$.

Assume that $\mathcal{B}$ is positive definite. Since $\mathcal{B}$ is closed, $\langle\mathcal{B}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}|_{\mathcal{B}}\rangle$ is an almost Pontryagin space. The present assertion now follows from the notice after Theorem 2.3 (uniqueness of topology).

Finally, if $\mathcal{B}^\perp = \mathcal{A}^\circ$ then we have $\pi(\mathcal{B})^\perp = \{0\}$ in the space $\mathcal{A}/\mathcal{A}^\circ$. Since $\pi(\mathcal{B})$ is closed, this implies that $\pi(\mathcal{B}) = \mathcal{A}/\mathcal{A}^\circ$, cf. [22, Theorem 3.1]. From $\mathcal{A}^\circ \subseteq \mathcal{B}$, it now follows that $\mathcal{B} = \mathcal{A}$. □

We have discussed a couple of constructions which can be carried out with almost Pontryagin spaces in [29, Propositions 3.1, 3.5]. Let us provide a reformulation of the second mentioned result which is sometimes smoother to apply.

Proposition A.7. *Let $\langle\mathcal{A},[\cdot,\cdot]_{\mathcal{A}},\mathcal{O}\rangle$ be an almost Pontryagin space, let $\mathcal{L}$ be a linear space, and let $\psi:\mathcal{A}\to\mathcal{L}$ be a linear map with $\ker\psi\subseteq\mathcal{A}^\circ$. Set $\mathcal{B}:=\operatorname{ran}\psi$ and let $\mathcal{T}$ be the final topology w.r.t. ψ. Then an inner product $[\cdot,\cdot]_{\mathcal{B}}$ is well-defined by*

$$\big[\psi(x),\psi(y)\big]_{\mathcal{B}}:=[x,y]_{\mathcal{A}},\quad x,y\in\mathcal{A},$$

the triple $\langle\mathcal{B},[\cdot,\cdot]_{\mathcal{B}},\mathcal{T}\rangle$ is an almost Pontryagin space, and the map ψ is a linear, isometric, continuous, and open surjection of $\mathcal{A}$ onto $\mathcal{B}$.

Proof. Apply [29, Proposition 3.5] with "$\mathcal{R}:=\ker\psi$", and notice that $\psi:\mathcal{A}\to\mathcal{B}$ factorises into a bijection after the canonical projection. □

A.2. A perturbation method

Another simple way of constructing new almost Pontryagin spaces from a given one is by finite rank perturbations of the inner product. This method has been applied extensively in our study of de Branges spaces, specifically see [27, Theorem 3.3]. The following lemma provides a general formulation.

Lemma A.8. *Let $\langle\mathcal{A},[\cdot,\cdot]_{\mathcal{A}},\mathcal{O}\rangle$ be an almost Pontryagin space, and let $\varphi_1,\dots,\varphi_n\in\mathcal{A}'$ and $\gamma_1,\dots,\gamma_n\in\mathbb{R}$. Define an inner product $[\![\cdot,\cdot]\!]_{\mathcal{A}}$ on $\mathcal{A}$ as*

$$[\![x,y]\!]_{\mathcal{A}}:=[x,y]_{\mathcal{A}}+\sum_{i=1}^n\gamma_i\varphi_i(x)\overline{\varphi_i(y)},\quad x,y\in\mathcal{A}.$$

Then $\langle\mathcal{A},[\![\cdot,\cdot]\!]_{\mathcal{A}},\mathcal{O}\rangle$ is an almost Pontryagin space.

Proof. Since $\varphi_i\in\mathcal{A}'$, the inner product $[\![\cdot,\cdot]\!]_{\mathcal{A}}$ is continuous w.r.t. $\mathcal{O}$. Choose an $\mathcal{O}$-closed linear subspace $\mathcal{M}$ of $\mathcal{A}$ with finite codimension in $\mathcal{A}$, such that $\langle\mathcal{M},[\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{M}\times\mathcal{M}}\rangle$ is a Hilbert space. Then the subspace

$$\mathcal{N}:=\mathcal{M}\cap\bigcap_{i=1}^n\ker\varphi_i$$

is an $\mathcal{O}$-closed subspace of $\mathcal{A}$ and in turn a closed subspace of the Hilbert space $\langle\mathcal{M},[\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{M}\times\mathcal{M}}\rangle$. Clearly, the inner products $[\cdot,\cdot]_{\mathcal{A}}$ and $[\![\cdot,\cdot]\!]_{\mathcal{A}}$ coincide on $\mathcal{N}$, and hence $\langle\mathcal{N},[\![\cdot,\cdot]\!]_{\mathcal{A}}|_{\mathcal{N}\times\mathcal{N}}\rangle$ is a Hilbert space. Clearly, the codimension of $\mathcal{N}$ in $\mathcal{A}$ is finite; it cannot exceed the codimension of $\mathcal{M}$ by more than n. □

The next proposition contains a strong converse version.

Proposition A.9. *Let $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ be an almost Pontryagin space, and let $\mathcal{F} \subseteq \mathcal{A}'$ be point separating. Then there exist $\varphi_1, \dots, \varphi_n \in \mathcal{F}$ such that for all sufficiently large values of $\gamma \in \mathbb{R}$ the space $\mathcal{A}$ becomes a Hilbert space with the inner product*

$$(x,y)_{\mathcal{A}} := [x,y]_{\mathcal{A}} + \gamma \sum_{i=1}^{n} \varphi_i(x)\overline{\varphi_i(y)}, \quad x,y \in \mathcal{A}.$$

The Hilbert space topology induced by $(\cdot,\cdot)_{\mathcal{A}}$ on $\mathcal{A}$ is equal to $\mathcal{O}$.

We give a geometric proof based on a compactness argument. A different proof (proceeding via Gram operators) could be extracted from the proof of [27, Theorem 3.3]. However, to our taste, the presently proposed intrinsic approach is more appealing. It is based on the following interesting topological property.

Lemma A.10. *Let $\langle \mathcal{A}, [\cdot,\cdot]_{\mathcal{A}}, \mathcal{O}\rangle$ be an almost Pontryagin space. Then each set*

$$A_{\leq r} := \bigl\{x \in \mathcal{A} : [x,x]_{\mathcal{A}} \leq r\bigr\}, \quad r \in \mathbb{R},$$

is w-closed.

Notice that the set $A_{\leq r}$ considered in this lemma is generically not convex; for geometric intuition think of $\mathbb{R}^3$ with $[(x_1;x_2;x_3),(y_1;y_2;y_3)] := x_1y_1 - x_2y_2 - x_3y_3$.

Proof of Lemma A.10. Choose a direct and orthogonal decomposition $\mathcal{A} = \mathcal{A}_+[\dot{+}]\mathcal{B}$ where $\mathcal{A}_+$ is closed and positive definite, and $\mathcal{B}$ is negative semidefinite (and hence finite dimensional). Existence of such a decomposition is obvious from Theorem 2.3(ii). Moreover, denote by P the projection of $\mathcal{A}$ onto $\mathcal{A}_+$ with kernel $\mathcal{B}$, and set

$$A_r^+ := \bigl\{x \in \mathcal{A}_+ : [x,x]_{\mathcal{A}} \leq r\bigr\}, \quad r \in \mathbb{R}.$$

Since the inner product $[\cdot,\cdot]_{\mathcal{A}}$ is positive definite on $\mathcal{A}_+$, the triangle inequality holds, and therefore A_r^+ is convex. Since A_r^+ is $\mathcal{O}$-closed, it is thus also w-closed. The operator $P : \mathcal{A} \to \mathcal{A}_+$ is $\mathcal{O}$-to-$\mathcal{O}|_{\mathcal{A}_+}$-continuous, and hence also w-to-w-continuous. It follows that for each $r \in \mathbb{R}$ the set

$$\bigl\{x \in \mathcal{A} : [Px,Px]_{\mathcal{A}} \leq r\bigr\} = P^{-1}\bigl(A_r^+\bigr)$$

is w-closed. In other words, the function $\nu_1 : \mathcal{A} \to \mathbb{R}$ acting as $x \mapsto [Px,Px]_{\mathcal{A}}$ is w-lower semicontinuous. The operator $I - P : \mathcal{A} \to \mathcal{B}$ is again w-to-w-continuous. Since $\dim \mathcal{B} < \infty$, it is also w-to-$\mathcal{O}|_{\mathcal{B}}$-continuous. Thus the function $\nu_2 : \mathcal{A} \to \mathbb{R}$ acting as $x \mapsto [(I-P)x,(I-P)x]_{\mathcal{A}}$ is w-continuous. It follows that $\nu_1 + \nu_2$ is w-lower semicontinuous. However, $(\nu_1 + \nu_2)(x) = [x,x]_{\mathcal{A}}$, $x \in \mathcal{A}$, and hence for each $r \in \mathbb{R}$ the set

$$A_{\leq r} = \{x \in \mathcal{A} : (\nu_1 + \nu_2)(x) \leq r\}$$

is w-closed. □

Proof of Proposition A.9. Let $\mathcal{F}_{\text{fin}}$ denote the set of all finite subsets of $\mathcal{F}$. We claim that

$$\exists M \in \mathcal{F}_{\text{fin}} : \quad [x, x]_{\mathcal{A}} > 0, \quad x \in \left[\bigcap_{\varphi \in M} \ker \varphi\right] \setminus \{0\}. \tag{A.4}$$

Denote by $\|.\|$ any norm which induces the topology of $\mathcal{A}$. Using the Banach–Alaoglu Theorem, Lemma A.10, and the fact that $\mathcal{A}$ is reflexive (since it carries a Hilbert space topology), we obtain that each set

$$C(M) := \left[\bigcap_{\varphi \in M} \ker \varphi\right] \cap \{x \in \mathcal{A} : \|x\| = 1, [x, x]_{\mathcal{A}} \leq 0\}, \quad M \in \mathcal{F}_{\text{fin}},$$

is w-compact. Assume now that (A.4) is false. This just means that $C(M) \neq \emptyset$, $M \in \mathcal{F}_{\text{fin}}$. Clearly, $C(M_1) \cap C(M_2) = C(M_1 \cup M_2)$, and hence the family $\{C(M) : M \in \mathcal{F}_{\text{fin}}\}$ has the finite intersection property. It follows that $\bigcap_{M \in \mathcal{F}_{\text{fin}}} C(M) \neq \emptyset$. However, if x belongs to this intersection, then $\|x\| = 1$ and $\varphi(x) = 0$, $\varphi \in \mathcal{F}$. Since $\mathcal{F}$ is point separating, we have reached a contradiction, and this establishes our claim (A.4).

Choose $\varphi_1, \ldots, \varphi_n \in \mathcal{F}$ such that

$$\mathcal{B} := \bigcap_{i=1}^{n} \ker \varphi_i$$

is positive definite w.r.t. $[\cdot,\cdot]_{\mathcal{A}}$. Clearly, $\mathcal{B}$ is closed in $\mathcal{A}$, and Lemma A.6 shows that $\mathcal{B}$ is orthocomplemented and that $[\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{B} \times \mathcal{B}}$ induces the topology $\mathcal{O}|_{\mathcal{B}}$ on $\mathcal{B}$.

Consider the seminorm on $\mathcal{A}$ defined as

$$p(x) := \left(\sum_{i=1}^{n} |\varphi_i(x)|^2\right)^{\frac{1}{2}}, \quad x \in \mathcal{A}.$$

Then $p(x) = 0$, $x \in \mathcal{B}$, and $p|_{\mathcal{B}^\perp}$ is a norm on $\mathcal{B}^\perp$. Since $\dim \mathcal{B}^\perp < \infty$, this norm induces the topology $\mathcal{O}|_{\mathcal{B}^\perp}$ (both being equal to the Euclidean topology). The function $x \mapsto [x, x]_{\mathcal{A}}$ is continuous on $\mathcal{B}^\perp$, and hence bounded on the unit ball of $p|_{\mathcal{B}^\perp}$. Thus, for all sufficiently large values of $\gamma > 0$ we have

$$\big|[x, x]_{\mathcal{A}}\big| < \gamma p(x)^2, \quad x \in \mathcal{B}^\perp \setminus \{0\}.$$

Consider the inner product on $\mathcal{A}$ defined as

$$(x,y)_{\mathcal{A}} := [x,y]_{\mathcal{A}} + \gamma \sum_{i=1}^{n} \varphi_i(x)\overline{\varphi_i(y)}, \quad x,y \in \mathcal{A}.$$

If $x \in \mathcal{A}$, we write $x = y + z$ with $y \in \mathcal{B}$ and $z \in \mathcal{B}^\perp$, and compute

$$(x,x)_{\mathcal{A}} = [y,y]_{\mathcal{A}} + \big([z,z]_{\mathcal{A}} + \gamma p(y+z)^2\big).$$

However, $p(y) = 0$, and hence $p(y+z) = p(z)$. Now it follows that $(x,x)_{\mathcal{A}} \geq 0$, with equality if and only if $y = z = 0$.

Next, we observe that

$$(x,y)_{\mathcal{A}} = [x,y]_{\mathcal{A}} + \gamma \sum_{i=1}^{n} \underbrace{\varphi_i(x)}_{=0}\overline{\varphi_i(y)} = [x,y]_{\mathcal{A}}, \quad x \in \mathcal{B},\ y \in \mathcal{A}.$$

This relation implies that $\mathcal{B}^{[\perp]} = \mathcal{B}^{(\perp)}$ and in turn that $\mathcal{A}$ decomposes as the $[\cdot,\cdot]_{\mathcal{A}}$-orthogonal and $(\cdot,\cdot)_{\mathcal{A}}$-orthogonal sum

$$\mathcal{A} = \mathcal{B}[\dot{+}]\mathcal{B}^\perp = \mathcal{B}(\dot{+})\mathcal{B}^\perp.$$

Hence, topologically,

$$\begin{aligned}
&\langle \mathcal{A}, \mathcal{O}\rangle \cong \langle \mathcal{B}, \mathcal{O}|_{\mathcal{B}}\rangle \times \big\langle \mathcal{B}^\perp, \mathcal{O}|_{\mathcal{B}^\perp}\big\rangle, \\
&\big\langle \mathcal{A}, (\cdot,\cdot)_{\mathcal{A}}\big\rangle \cong \big\langle \mathcal{B}, (\cdot,\cdot)_{\mathcal{A}}|_{\mathcal{B}\times\mathcal{B}}\big\rangle \times \big\langle \mathcal{B}^\perp, (\cdot,\cdot)_{\mathcal{A}}|_{\mathcal{B}^\perp\times\mathcal{B}^\perp}\big\rangle.
\end{aligned}$$

Since $(\cdot,\cdot)_{\mathcal{A}}|_{\mathcal{B}\times\mathcal{B}} = [\cdot,\cdot]_{\mathcal{A}}|_{\mathcal{B}\times\mathcal{B}}$, Lemma A.6 implies that $(\cdot,\cdot)_{\mathcal{A}}|_{\mathcal{B}\times\mathcal{B}}$ induces $\mathcal{O}|_{\mathcal{B}}$. Since $\dim \mathcal{B}^\perp < \infty$, certainly, $(\cdot,\cdot_{\mathcal{A}})|_{\mathcal{B}^\perp\times\mathcal{B}^\perp}$ induces $\mathcal{O}|_{\mathcal{B}^\perp}$. Together we see that $(\cdot,\cdot_{\mathcal{A}})$ induces $\mathcal{O}$ on $\mathcal{A}$. □

A.3. Almost Pontryagin space completions

Almost Pontryagin space completions play a central role in the present paper. In this part of the appendix, we recall the required notions and facts as given in [40, §6], and provide some supplements. Completions in the almost Pontryagin space context have been studied previously in [29, §4], some ideas are going back to [23].

Definition A.11. Let $\mathcal{L}$ be an inner product space. We call a pair $\langle \iota, \mathcal{A}\rangle$ an *almost Pontryagin space completion of* $\mathcal{L}$, if $\mathcal{A}$ is an almost Pontryagin space, and ι is a linear and isometric map of $\mathcal{L}$ onto a dense subspace of $\mathcal{A}$.

Let $\langle \iota_i, \mathcal{A}_i\rangle$, $i = 1,2$, be two almost Pontryagin space completions of $\mathcal{L}$. We say that $\langle \iota_1, \mathcal{A}_1\rangle$ and $\langle \iota_2, \mathcal{A}_2\rangle$ are *isomorphic*, if there exists a linear and isometric homeomorphism φ of $\mathcal{A}_1$ onto $\mathcal{A}_2$ with $\varphi \circ \iota_1 = \iota_2$. ◊

It is easy to see that an inner product space $\mathcal{L}$ has an almost Pontryagin space completion if and only if $\operatorname{ind}_- \mathcal{L} < \infty$, cf. [40, Remark 6.4]. The totality of all almost Pontryagin space completions of $\mathcal{L}$ can be described via linear functionals on $\mathcal{L}$; we recall this result in Theorem A.15 below. Before that, we introduce some more terminology.

An inner product space with finite negative index carries a very particular topology which is induced by a seminorm constructed intrinsically from the inner product, cf. [11, Theorem I.11.7, Chapter IV.6]. This topology is called the *decomposition majorant* of $\mathcal{L}$. It is not necessarily a Hausdorff topology, in fact, the intersection of all neighbourhoods of zero equals $\mathcal{L}^\circ$. The seminorm used to construct this topology is not unique, however, the topology itself is. It is characterised by a minimality property of its Hausdorff quotient, cf. [11, §IV.6]

For a linear space $\mathcal{V}$, we denote by $\mathcal{V}^*$ its algebraic dual space, i.e., the linear space of all linear functionals on $\mathcal{V}$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are linear spaces and $\phi : \mathcal{V}_1 \to \mathcal{V}_2$ is a linear map, we denote by $\phi^* : \mathcal{V}_2^* \to \mathcal{V}_1^*$ its algebraic dual map, i.e., the map acting as

$$\phi^*(\psi) := \psi \circ \phi, \quad \psi \in \mathcal{V}_2^*.$$

Definition A.12. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. We denote by $\mathcal{L}'$ the linear space of all linear functionals on $\mathcal{L}$ which are bounded w.r.t. the decomposition majorant of $\mathcal{L}$. $\Diamond$

Let us point out in this place that, whenever $\langle \iota, \mathcal{A} \rangle$ is an almost Pontryagin space completion of a space $\mathcal{L}$, the map $\iota^*|_{\mathcal{A}'}$ is injective (a consequence of the fact that $\operatorname{ran} \iota$ is dense).

A description of $\mathcal{L}'$ generalising the connection pointed out in [40, Remark 6.7] is the following.

Lemma A.13. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, and let $\langle \iota, \mathcal{A} \rangle$ be an almost Pontryagin space completion of $\mathcal{L}$. Then*

$$\mathcal{L}' = \big\{ x \mapsto [\iota x, y]_{\mathcal{A}} : y \in \mathcal{A} \big\}.$$

Proof. Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}^\circ$ be the canonical projection. Then $\langle \pi \circ \iota, \mathcal{A}/\mathcal{A}^\circ \rangle$ is a Pontryagin space completion of $\mathcal{L}$. Using [40, Remark 6.7] and (A.1), we obtain

$$\mathcal{L}' = (\pi \circ \iota)^* \big((\mathcal{A}/\mathcal{A}^\circ)' \big) = \iota^* \big(\big\{ [\cdot, y]_{\mathcal{A}} : y \in \mathcal{A} \big\} \big) = \big\{ x \mapsto [\iota x, y]_{\mathcal{A}} : y \in \mathcal{A} \big\}. \qquad \Box$$

It is also not difficult to give an intrinsic description of $\mathcal{L}'$.

Lemma A.14. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$, and let $\varphi \in \mathcal{L}^*$. Then $\varphi \in \mathcal{L}'$ if and only if for each sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{L}$ which satisfies*

$$\lim_{n \to \infty} [x_n, x]_{\mathcal{L}} = 0, \quad x \in \mathcal{L}, \qquad \lim_{n \to \infty} [x_n, x_n]_{\mathcal{L}} = 0, \tag{A.5}$$

it holds that $\lim_{n \to \infty} \varphi(x_n) = 0$.

Proof. Let $\langle\iota,\mathcal{A}\rangle$ be a Pontryagin space completion of $\mathcal{L}$. For necessity, assume that there exists $y\in\mathcal{A}$ with $\varphi(x)=[\iota x,y]_{\mathcal{A}}$, $x\in\mathcal{L}$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}$ with (A.5), then the sequence $(\iota x_n)_{n\in\mathbb{N}}$ converges to 0 in the norm of $\mathcal{A}$, cf. [22, Theorem 2.4]. It follows that $\lim_{n\to\infty}\varphi(x_n)=\lim_{n\to\infty}[\iota x_n,y]_{\mathcal{A}}=0$.

Conversely, assume that the stated implication holds true. Since $\operatorname{ran}\iota$ is dense in $\mathcal{A}$ and $\mathcal{A}$ is nondegenerated, we have $\ker\iota=\mathcal{L}^\circ$. Hence, for each $x\in\ker\iota$, the hypothesis (A.5) of the stated implication is fulfilled for the constant sequence $(x)_{n\in\mathbb{N}}$, and it follows that $\varphi(x)=0$. We conclude that $\ker\iota\subseteq\ker\varphi$. Thus there exists a linear map $\psi:\operatorname{ran}\iota\to\mathbb{C}$ with $\psi\circ\iota=\varphi$. We claim that ψ is bounded w.r.t. the norm of $\mathcal{A}$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\operatorname{ran}\iota$ with $\lim_{n\to\infty}y_n=0$ in the norm of $\mathcal{A}$. Choose $x_n\in\mathcal{L}$ with $y_n=\iota x_n$, then

$$\lim_{n\to\infty}[x_n,x]_{\mathcal{L}}=\lim_{n\to\infty}[y_n,\iota x]_{\mathcal{A}}=0,\quad x\in\mathcal{L},\qquad \lim_{n\to\infty}[x_n,x_n]_{\mathcal{L}}=\lim_{n\to\infty}[y_n,y_n]_{\mathcal{A}}=0.$$

It follows that $\lim_{n\to\infty}\psi(y_n)=\lim_{n\to\infty}\varphi(x_n)=0$, and this establishes our claim. Being a bounded functional defined on a subspace of the Pontryagin space $\mathcal{A}$, ψ has a representation $[\cdot,y]_{\mathcal{A}}$ with some $y\in\mathcal{A}$. Thus $\varphi=\iota^*([\cdot,y]_{\mathcal{A}})\in\mathcal{L}'$. $\square$

We can now state a description of all almost Pontryagin space completions of a given inner product space with finite negative index.

Theorem A.15. *(See [40, Theorem 6.8].) Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_-\mathcal{L}<\infty$. The set of all isomorphy classes of almost Pontryagin space completions of $\mathcal{L}$ corresponds bijectively to the set of all linear subspaces of $\mathcal{L}^*$ which contain $\mathcal{L}'$ with finite codimension. This correspondence is established by the map*

$$\langle\iota,\mathcal{A}\rangle\mapsto\iota^*\big(\mathcal{A}'\big),$$

and we have $\dim(\iota^*(\mathcal{A}')/\mathcal{L}')=\operatorname{ind}_0\mathcal{A}$.

Next we aim at topologising the linear space $\mathcal{L}'$. It trivially carries the topology of pointwise convergence, i.e., the weak topology $\sigma(\mathcal{L}',\mathcal{L})$. However, this topology is usually too coarse. A more useful one is the following.

Definition A.16. Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_-\mathcal{L}<\infty$, and let $\langle\iota,\mathcal{A}\rangle$ be a Pontryagin space completion of $\mathcal{L}$. We denote by $\mathcal{T}(\mathcal{L}')$ the final topology on $\mathcal{L}'$ w.r.t. the map $\iota^*|_{\mathcal{A}'}:\mathcal{A}'\to\mathcal{L}'$ where $\mathcal{A}'$ is endowed with its norm topology. $\Diamond$

We need to show that $\mathcal{T}(\mathcal{L}')$ is well-defined. The following lemma says a bit more than that. Thereby, for an almost Pontryagin space $\mathcal{A}$, we set

$$\mathcal{A}^{\wr}:=\big\{[\cdot,y]_{\mathcal{A}}:y\in\mathcal{A}\big\}.\tag{A.6}$$

Note that $\mathcal{A}^{\wr}=\mathcal{A}'$ if and only if $\mathcal{A}$ is a Pontryagin space, cf. Proposition A.1.

Lemma A.17. *Let $\mathcal{L}$ be an inner product space with $\operatorname{ind}_- \mathcal{L} < \infty$. For an almost Pontryagin space completion $\langle \iota, \mathcal{A} \rangle$ of $\mathcal{L}$, let $\mathcal{T}_{\langle \iota, \mathcal{A} \rangle}$ be the final topology on $\mathcal{L}'$ w.r.t. the map $\iota^*|_{\mathcal{A}^{\wr}} : \mathcal{A}^{\wr} \to \mathcal{L}'$ where $\mathcal{A}^{\wr}$ is endowed with the norm topology inherited from $\mathcal{A}'$.*

The topology $\mathcal{T}_{\langle \iota, \mathcal{A} \rangle}$ is independent of the choice of $\langle \iota, \mathcal{A} \rangle$. In particular, $\mathcal{T}(\mathcal{L}')$ is well-defined and coincides with $\mathcal{T}_{\langle \iota, \mathcal{A} \rangle}$ for any $\langle \iota, \mathcal{A} \rangle$. Moreover, $\mathcal{T}(\mathcal{L}')$ is a Hilbert space topology.

Proof. Let $\langle \iota_1, \mathcal{A}_1 \rangle$ and $\langle \iota_2, \mathcal{A}_2 \rangle$ be almost Pontryagin space completions of $\mathcal{L}$. Denote by $\pi_i : \mathcal{A}_i \to \mathcal{A}_i / \mathcal{A}_i{}^\circ$, $i = 1, 2$, the canonical projection. Then $\langle \pi_i \circ \iota_i, \mathcal{A}_i / \mathcal{A}_i{}^\circ \rangle$, $i = 1, 2$, are Pontryagin space completions of $\mathcal{L}$, and hence are isomorphic. Let $\Phi : \mathcal{A}_1 / \mathcal{A}_1{}^\circ \to \mathcal{A}_2 / \mathcal{A}_2{}^\circ$ be a linear and isometric homeomorphism with

$$\begin{array}{ccccc} \mathcal{A}_1 & \xleftarrow{\ \iota_1\ } & \mathcal{L} & \xrightarrow{\ \iota_2\ } & \mathcal{A}_2 \\ \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_2} \\ \mathcal{A}_1/\mathcal{A}_1{}^\circ & & \xrightarrow[\ \Phi\ ]{} & & \mathcal{A}_2/\mathcal{A}_2{}^\circ \end{array}$$

Passing to adjoints, and remembering (A.1) yields

$$\begin{array}{ccccc} \mathcal{A}_1' \supseteq \operatorname{ran} \pi_1' = \mathcal{A}_1^{\wr} & \xrightarrow{\ \iota_1^*\ } & \mathcal{L}' & \xleftarrow{\ \iota_2^*\ } & \mathcal{A}_2^{\wr} = \operatorname{ran} \pi_2' \subseteq \mathcal{A}_2' \\ \uparrow{\scriptstyle \pi_1'} & & & & \uparrow{\scriptstyle \pi_2'} \\ (\mathcal{A}_1/\mathcal{A}_1{}^\circ)' & & \xleftarrow[\ \Phi'\ ]{} & & (\mathcal{A}_2/\mathcal{A}_2{}^\circ)' \end{array}$$

By the Open Mapping Theorem, π_2' is open (as a map onto its range). Hence, the map $\iota_2^* \circ \pi_2'$ is a continuous and open surjection of $(\mathcal{A}_2/\mathcal{A}_2{}^\circ)'$ endowed with its norm topology onto $\langle \mathcal{L}', \mathcal{T}_{\langle \iota_2, \mathcal{A}_2 \rangle} \rangle$. In the same way, $\iota_1^* \circ \pi_1' \circ \Phi'$ is a continuous and open surjection of $(\mathcal{A}_2/\mathcal{A}_2{}^\circ)'$ onto $\langle \mathcal{L}', \mathcal{T}_{\langle \iota_1, \mathcal{A}_1 \rangle} \rangle$. By the above diagram these two maps coincide, and it follows that $\mathcal{T}_{\langle \iota_1, \mathcal{A}_1 \rangle} = \mathcal{T}_{\langle \iota_2, \mathcal{A}_2 \rangle}$.

It remains to show that $\mathcal{T}(\mathcal{L}')$ is a Hilbert space topology. However, the norm topology of $\mathcal{A}$ and hence also the one of $\mathcal{A}'$ is such. Since $\mathcal{A}^{\wr}$ is a closed subspace of $\mathcal{A}'$ and $\iota^*|_{\mathcal{A}^{\wr}}$ is a homeomorphism of $\mathcal{A}^{\wr}$ onto $\langle \mathcal{L}', \mathcal{T}(\mathcal{L}') \rangle$, this property is inherited by $\mathcal{T}(\mathcal{L}')$. □

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References

[1] D. Alpay, A. Dijksma, H. Langer, The transformation of Issai Schur and related topics in an indefinite setting, in: System Theory, the Schur Algorithm and Multidimensional Analysis, in: Operator Theory: Advances and Applications, vol. 176, Birkhäuser, Basel, 2007, pp. 1–98.

[2] T.Ya. Azizov, I.S. Iohvidov, Linear Operators in Spaces with an Indefinite Metric, Pure and Applied Mathematics, John Wiley & Sons Ltd., Chichester, 1989, p. xii+304 (Translated from the Russian by E.R. Dawson).

[3] D. Alpay, et al., Schur Functions, Operator Colligations, and Reproducing Kernel Pontryagin Spaces, Operator Theory: Advances and Applications, vol. 96, Birkhäuser Verlag, Basel, 1997, p. xii+229.

[4] D. Alpay, The Schur Algorithm, Reproducing Kernel Spaces and System Theory, SMF/AMS Texts and Monographs, vol. 5, American Mathematical Society, Providence, RI, 2001, p. viii+150 (Translated from the 1998 French original by Stephen S. Wilson).

[5] N. Arcozzi, et al., The Dirichlet space: a survey, N.Y. J. Math. 17A (2011) 45–86.

[6] N. Aronszajn, Theory of reproducing kernels, Trans. Am. Math. Soc. 68 (1950) 337–404.

[7] T.Ya. Azizov, L.I. Soukhotcheva, Linear operators in almost Krein spaces, in: Operator Theory in Inner Product Spaces, in: Operator Theory: Advances and Applications, vol. 175, Birkhäuser, Basel, 2007, pp. 1–11.

[8] T.Ya. Azizov, Completely continuous operators that are selfadjoint with respect to a degenerate indefinite metric, Mat. Issled. 7 (4) (26) (1972) 237–240, 259.

[9] C. Berg, A.J. Duran, The index of determinacy for measures and the 1^2-norm of orthonormal polynomials, Trans. Am. Math. Soc. 347 (8) (1995) 2795–2811.

[10] V. Bolotnikov, A. Kheifets, L. Rodman, Jet functions having indefinite Caratheodory–Pick matrices, Linear Algebra Appl. 385 (2004) 215–286.

[11] J. Bognar, Indefinite Inner Product Spaces, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Band 78, Springer-Verlag, New York, 1974, p. ix+224.

[12] L. de Branges, Hilbert Spaces of Entire Functions, Prentice-Hall Inc., Englewood Cliffs, NJ, 1968, p. ix+326.

[13] A. Berlinet, C. Thomas-Agnan, Reproducing Kernel Hilbert Spaces in Probability and Statistics, With a preface by Persi Diaconis, Kluwer Academic Publishers, Boston, MA, 2004, p. xxii+355.

[14] T. Constantinescu, A. Gheondea, Representations of Hermitian kernels by means of Krein spaces, Publ. Res. Inst. Math. Sci. 33 (6) (1997) 917–951.

[15] M.A. Dritschel, J. Rovnyak, Operators on indefinite inner product spaces, in: Lectures on Operator Theory and its Applications, Waterloo, ON, 1994, in: Fields Institute Monographs, vol. 3, Amer. Math. Soc., Providence, RI, 1996, pp. 141–232.

[16] P. Duren, A. Schuster, Bergman Spaces, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, Providence, RI, 2004, p. x+318.

[17] H. Dym, J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation, CBMS Regional Conference Series in Mathematics, vol. 71, 1989, p. x+147 (Published for the Conference Board of the Mathematical Sciences, Washington, DC).

[18] C. Fulton, H. Langer, Sturm–Liouville operators with singularities and generalized Nevanlinna functions, Complex Anal. Oper. Theory 4 (2) (2010) 179–243.

[19] A. Gheondea, A survey on reproducing kernel Krein spaces, arXiv:1309.2393v2 [math.FA], 2013.

[20] M. Hein, O. Bousquet, Kernels, associated structures and generalizations, Technical Report 127, Max Planck Institute for Biological Cybernetics, 2004, http://www.kyb.tuebingen.mpg.de/fileadmin/user_upload/files/publications/pdf2816.pdf.

[21] T. Hofmann, B. Scholkopf, A.J. Smola, Kernel methods in machine learning, Ann. Stat. 36 (3) (2008) 1171–1220.

[22] I.S. Iohvidov, M.G. Krein, H. Langer, Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric, Mathematical Research, vol. 9, Akademie-Verlag, Berlin, 1982, p. 120.

[23] P. Jonas, H. Langer, B. Textorius, Models and unitary equivalence of cyclic selfadjoint operators in Pontrjagin spaces, in: Operator Theory and Complex Analysis, Sapporo, 1991, in: Operator Theory: Advances and Applications, vol. 59, Birkhäuser, Basel, 1992, pp. 252–284.

[24] P. Koosis, Introduction to H_p Spaces, 2nd ed., Cambridge Tracts in Mathematics, vol. 115, Cambridge University Press, Cambridge, 1998, p. xiv+289 (With two appendices by V.P. Havin [Viktor Petrovich Khavin]).

[25] P. Koosis, The Logarithmic Integral. I, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, Cambridge, 1998, p. xviii+606 (Corrected reprint of the 1988 original).

[26] M. Kaltenbäck, H. Woracek, Pontryagin spaces of entire functions. VI, Acta Sci. Math. (Szeged) 76 (3–4) (2010) 511–560.
[27] M. Kaltenbäck, H. Woracek, Pontryagin spaces of entire functions. I, Integral Equ. Oper. Theory 33 (1) (1999) 34–97.
[28] M. Kaltenbäck, H. Woracek, The Krein formula for generalized resolvents in degenerated inner product spaces, Monatshefte Math. 127 (2) (1999) 119–140.
[29] M. Kaltenbäck, H. Winkler, H. Woracek, Almost Pontryagin spaces, in: Recent Advances in Operator Theory and its Applications, in: Operator Theory: Advances and Applications, vol. 160, Birkhäuser, Basel, 2005, pp. 253–271.
[30] H. Langer, R. Mennicken, C. Tretter, A self-adjoint linear pencil Q-λP of ordinary differential operators, Methods Funct. Anal. Topol. 2 (1) (1996) 38–54.
[31] M. Langer, H. Woracek, A Pontryagin space approach to the index of determinacy of a measure, in preparation.
[32] M. Langer, H. Woracek, Direct and inverse spectral theorems for a class of canonical systems with two singular endpoints, in preparation.
[33] N.K. Nikolski, Operators, Functions, and Systems: An Easy Reading, vol. 1, Hardy, Hankel, and Toeplitz, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002, p. xiv+461, Translated from the French by Andreas Hartmann.
[34] J.R. Partington, Interpolation, Identification, and Sampling, London Mathematical Society Monographs. New Series, vol. 17, The Clarendon Press, Oxford University Press, New York, 1997, p. xii+267.
[35] F. Philipp, C. Trunk, G-self-adjoint operators in almost Pontryagin spaces, in: Spectral Theory in Inner Product Spaces and Applications, in: Operator Theory: Advances and Applications, vol. 188, Birkhäuser Verlag, Basel, 2009, pp. 207–235.
[36] W.T. Ross, The classical Dirichlet space, in: Recent Advances in Operator-Related Function Theory, in: Contemporary Mathematics, vol. 393, Amer. Math. Soc., Providence, RI, 2006, pp. 171–197.
[37] M. Rosenblum, J. Rovnyak, Topics in Hardy Classes and Univalent Functions, Birkhäuser Advanced Texts: Basler Lehrbücher (Birkhäuser Advanced Texts: Basel Textbooks), Birkhäuser Verlag, Basel, 1994, p. xii+250.
[38] E.D. Solomentsev, Uniqueness Properties of Analytic Functions, Encyclopedia of Mathematics, Springer.
[39] P. Sorjonen, Pontrjaginräume mit einem reproduzierenden Kern, Ann. Acad. Sci. Fenn., Ser. A 1 Math. 594 (1975) 30.
[40] H. de Snoo, H. Woracek, Sums, couplings, and completions of almost Pontryagin spaces, Linear Algebra Appl. 437 (2) (2012) 559–580.
[41] H. Woracek, Directing functionals and de Branges space completions in the almost Pontryagin space setting, in preparation.
[42] K. Yao, Applications of reproducing kernel Hilbert spaces – bandlimited signal models, Inf. Control 11 (1967) 429–444.