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Homoclinic, subharmonic, and superharmonic bifurcations for a pendulum with periodically varying length

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Abstract Dynamic behavior of a weightless rod with a point mass sliding along the rod axis according to periodic law is studied. This is the simplest model of child's swing. Melnikov's analysis is carried out to find bifurcations of homoclinic, subharmonic oscillatory, and subharmonic rotational orbits. For the analysis of superharmonic rotational orbits, the averaging method is used and stability of obtained approximate solution is checked. The analytical results are compared with numerical simulation results.

Keywords Homoclinic bifurcation · Rotational orbits · Averaging method · Parametric excitation

1 Introduction

Oscillations of a pendulum with periodically varying length (PPVL) is one of the classical problems in mechanics, see [1–11]. We represent PPVL as a weightless rod with a point mass sliding along the rod axis according to periodic law. This is also a simple model of child's swing. In works [1–9] small oscillations of PPVL were studied. In [5,8,9] the instability domains of the vertical position were found both analytically and numerically. Regular rotations and chaotic regimes were also investigated, see [6,8,9].

In the literature, oscillatory and rotational approximate solutions for PPVL were obtained with quasi-linear approach, where nonlinearity was assumed to be small as well as the excitation amplitude. The only exception is [10,11], where exact stable uniform rotations were found in the case of zero damping and harmonic excitation with special amplitude and phase. But these uniform rotations occur only for very restrictive relation of parameters. On the other hand, quasi-linear approach requires taking high-order approximations for substantially nonlinear regimes, such as regular rotations with frequencies higher than the frequency of excitation, see e.g., [9]. That makes approximate expressions cumbersome and analysis difficult, while the error of approximation can still be high, because the smallness assumption of both excitation and nonlinearity might not be satisfied for any existing rotations.

In order to resolve this issue in the present paper we analyze boundaries in the parameter space for oscillatory, rotational, and more complex (rotational–oscillatory, chaotic) regimes of PPVL assuming arbitrary (not small) nonlinearity.

On the one hand, the methods used for studying other parametrically excited pendula, such as a pendulum

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with vibrating pivot [12–14], can also be applied to PPVL. In particular, we apply Melnikov's analysis in a way similar to [12], which is applicable when excitation and damping are small. We also compare the obtained boundaries with the results of numerical simulations.

On the other hand, our paper contains a methodological novelty since the rotational regimes are studied without the assumption that excitation is small. Instead of this, to apply the method of averaging, we assume that the frequency of excitation is large. In that case the unperturbed system preserves angular momentum, which is a distinctive feature of PPVL. As a result, the first-order approximation happens to be enough to find approximate solutions of different angular velocities. We also study the stability of these solutions, find their existence domains in parameter space, and compare them with the results of numerical simulations.

Numerical simulations were made in [8,9] for different amplitudes and frequencies of excitation, starting from the same initial conditions. After sufficiently high time of simulation the solutions converged to regular regimes (equilibrium, oscillations, and rotations) or remained "chaotic." Thus we have the points in parameter space, where corresponding regimes exist, for comparison with the analytically obtained approximations of the corresponding existence domains.

The paper is organized as follows. In Sect. 2, main equations of motion of a pendulum with variable length are derived and given in nondimensional form. In Sect. 3, we use Melnikov's analysis [15–18] to find bifurcations of homoclinic, oscillatory, and subharmonic rotational orbits of PPVL. Melnikov's functions are obtained and compared with the results of numerical simulation. In Sect. 4, we find bifurcations of superharmonic rotational orbits with the use of the averaging method and compare them with the numerical simulation results. In Sect. 4, we find the domains of existence of the rotational solutions under assumption that the unperturbed system preserves the angular momentum (rather than Hamiltonian as in the Melnikov's analysis) and allows for nonsmall excitation amplitude.

2 Main relations

Equation for motion of the PPVL is derived with the use of angular momentum alteration theorem and taking into account linear damping forces, see [5,6,8]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(ml^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)+\gamma l^2\frac{\mathrm{d}\theta}{\mathrm{d}t}+mgl\sin(\theta)=0, \qquad (1)$$

where m is the mass, l is the length, θ is the angle of the pendulum deviation from the vertical position, γ is the damping coefficient, and g is the acceleration due to gravity.

It is assumed that the length of the pendulum changes according to a periodic law

$$l=l_0+a\varphi(\Omega t), \qquad (2)$$

where l_0 is the mean pendulum length, a and Ω are the amplitude and frequency of the excitation, and $\varphi(\tau)$ is a smooth zero mean periodic function with period 2π.

We introduce new time $\tau = \Omega t$ and three dimensionless parameters

$$\varepsilon=\frac{a}{l_0},\quad \omega=\frac{\Omega_0}{\Omega},\quad \beta=\frac{\gamma}{m\Omega_0}, \qquad (3)$$

where $\Omega_0 = \sqrt{\frac{g}{l_0}}$ is the eigenfrequency of the pendulum with constant length $l = l_0$ and zero damping. Using these notations, Eq. (1) takes the form

$$\left((1+\varepsilon\varphi(\tau))^2\dot{\theta}\right)^{\cdot}+\beta\omega\,(1+\varepsilon\varphi(\tau))^2\dot{\theta}$$
$$+(1+\varepsilon\varphi(\tau))\,\omega^2\sin(\theta)=0, \qquad (4)$$

where the upper dot denotes differentiation with respect to new time τ.

3 Melnikov's method: perturbation of a Hamiltonian system

Assuming $1+\varepsilon\varphi(\tau)>0$, Eq. (4) can be written in the following form

$$\ddot{\theta}+\left(\frac{2\varepsilon\dot{\varphi}(\tau)}{1+\varepsilon\varphi(\tau)}+\beta\omega\right)\dot{\theta}+\frac{\omega^2\sin(\theta)}{1+\varepsilon\varphi(\tau)}=0. \qquad (5)$$

Coefficients of nonlinear equation (5) explicitly depend on the periodic function $\varphi(\tau)$ and three independent dimensionless parameters: the relative excitation amplitude ε, the damping β, and the inverse relative frequency of excitation ω.

Let us assume that parameters of excitation amplitude ε and damping β are small of the same order,

$\varepsilon \sim \beta \ll 1$. Thus, we can say that dynamics of PPVL is described by the perturbed Hamiltonian system

$$\dot{\theta} = \frac{\partial H}{\partial v}, \tag{6}$$

$$\dot{v} = -\frac{\partial H}{\partial \theta} + g_1(\theta, v, \tau) + o(\varepsilon), \tag{7}$$

where the perturbation function

$$g_1(\theta, v, \tau) = (2\varepsilon \sin(\tau) - \beta\omega)\, v + \varepsilon\omega^2 \cos(\tau) \sin(\theta) \tag{8}$$

is of the first order of smallness, i.e., $g_1(\theta, v, \tau) = O(\varepsilon)$. The following function

$$H = \frac{v^2}{2} - \omega^2 \cos(\theta) \tag{9}$$

is the Hamiltonian of the unperturbed system

$$\dot{\theta} = v, \tag{10}$$

$$\dot{v} = -\Omega^2 \sin(\theta), \tag{11}$$

which is system (6)–(8) with $\varepsilon = 0$ and $\beta = 0$. The unperturbed system describes motions of the pendulum with constant length and zero damping, so the system has the first integral H=const. Unperturbed system (10)–(11) has an oscillatory solution if $H < \omega^2$ and a rotational solution if $H > \omega^2$. If $H = \omega^2$ the solution is a separatrix dividing oscillatory and rotational domains in phase space $(\theta, \dot{\theta})$.

3.1 Homoclinic bifurcations

In order to apply Melnikov's criterion [15,16] to a homoclinic orbit we find the separatrix of the unperturbed system (10)–(11) that goes through the saddle-node point $\theta = \pi$, $v = 0$. For this point, we have $H = \omega^2$ and with the use of (9) we obtain $v^2 = 2\omega^2(1 + \cos(\theta)) = 4\omega^2 \cos^2(\theta/2)$. So, the separatrix has the following form

$$v = \dot{\theta} = \pm 2\omega \cos\left(\frac{\theta}{2}\right). \tag{12}$$

This equation allows for separation of variables and the following integration

$$\ln\left(\frac{1 + \sin(\theta/2)}{\cos(\theta/2)}\right) = \pm\omega\,(\tau - \tau_0)\,, \tag{13}$$

where τ_0 is a constant of integration. Potentiation of (13) with some transformations yields

$$\begin{aligned} \cos\left(\frac{\theta}{2}\right) &= \frac{1}{\cosh\left(\omega\,(\tau - \tau_0)\right)}, \\ \sin\left(\frac{\theta}{2}\right) &= \pm \tanh\left(\omega\,(\tau - \tau_0)\right). \end{aligned} \tag{14}$$

Melnikov's distance between stable and unstable perturbed separatrices is given by the following integral

$$M^{\pm} = \int_{-\infty}^{\infty} \frac{\partial H}{\partial v}\, g_1(\theta(\tau), v(\tau), \tau)\, \mathrm{d}\tau, \tag{15}$$

where function g_1 is given in (8) and $\frac{\partial H}{\partial v} = v$. With the use of (12) and (14) integral (15) has the following expression (see Appendix 1)

$$M^{\pm} = \frac{6\pi\varepsilon \sin(\tau_0)}{\sinh\left(\frac{\pi}{2\omega}\right)} - 8\beta\omega^2. \tag{16}$$

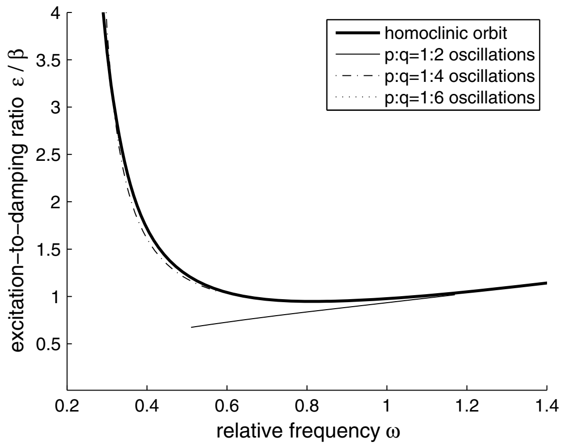
This is a sign-changing function if the following inequality is satisfied

$$\frac{\varepsilon}{\beta} > \frac{4\omega^2}{3\pi} \sinh\left(\frac{\pi}{2\omega}\right). \tag{17}$$

According to Melnikov's criterion [15], when condition (17) is satisfied homoclinic structure and complex dynamics appear near the separatrix. Condition (17) means that complex dynamics appears only when the amplitude of excitation is sufficiently large with respect to the damping coefficient. The domain of possible complex dynamics defined by inequality (17) is depicted in Fig. 1 and compared with numerical simulations in Fig. 1 for $\beta = 0.05$. The minimum of the right-hand side of (17) is reached at $\omega \approx 0.82$ and can be evaluated as 0.948. Note that condition (17) is similar to that of oscillator with quadratic nonlinearity and external periodic excitation [18]. Similar inequality for the pendulum with vertically vibrating pivot was obtained in [12], where the right-hand side function is three times greater than that in (17).

3.2 Subharmonic bifurcations of oscillatory orbits

In order to apply Melnikov's criterion to oscillatory orbits we find oscillatory solutions of the unperturbed system (10)–(11). For this reason we introduce the amplitude of oscillations A so that from the first integral for $v = 0$ we have $H = -\omega^2 \cos(A)$ and with the



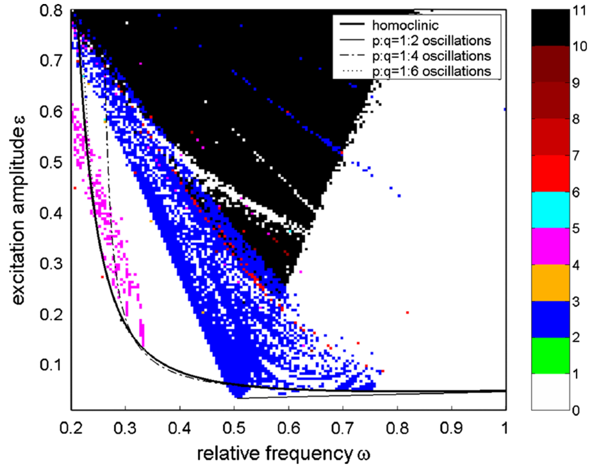


Fig. 1 Subharmonic bifurcation functions (ε/β—left, ε—right) for $q = 2$ (*solid line*), $q = 4$ (*dot-and-dash line*), $q = 6$ (*dotted line*), and $p = 1$ converge to the homoclinic bifurcation function (*bold solid line*) that approximates the domain of complex dynamics in (17). These functions approximate the domains of the corresponding resonant oscillations, depicted (*right*) with blue ($q = 2$), purple ($q = 4$), and red ($q = 6$) colors (see the color bar) on the parameter plane (ω, ε) at $\beta = 0.05$. Chaotic regimes are shown with black color

use of (9) we obtain $v^2 = 2\omega^2 (\cos(\theta) - \cos(A)) = 4\omega^2 \left(\sin^2(A/2) - \sin^2(\theta/2)\right)$. Thus, we can write the following equation

$$v = \dot{\theta} = 2\omega\, \mathrm{sign}(\dot{\theta}) \sqrt{\sin^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}, \qquad (18)$$

which allows for separation of variables. In order to integrate (18) one usually introduces (see e.g., [19] or [20]) a monotonically increasing phase ψ such that

$$\sin\left(\frac{\theta}{2}\right) = k \sin(\psi), \qquad (19)$$

where instead of amplitude A one uses $k = \sin\left(\frac{A}{2}\right) = \sqrt{\frac{H+\omega^2}{2\omega^2}}$, which is called modulus in elliptic integrals. From (19), we have equation (18) in the form $v = 2\omega k \cos\psi$. This equation along with time differentiation of (19) yields $\omega\sqrt{1 - k^2 \sin^2\psi} = \dot{\psi}$. As a result of integration starting from time τ_0, when $\psi(\tau_0) = 0$, we have $\psi = \mathrm{am}(\omega(\tau - \tau_0), k)$ and consequently

$$v = 2\omega k\, \mathrm{cn}\left(\omega(\tau - \tau_0), k\right) \qquad (20)$$

and

$$\cos\left(\tfrac{\theta}{2}\right) = \mathrm{dn}(\omega(\tau - \tau_0), k), \quad \sin\left(\tfrac{\theta}{2}\right) = k\, \mathrm{sn}(\omega(\tau - \tau_0), k), \qquad (21)$$

where $\mathrm{am}(\cdot, k)$, $\mathrm{dn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$, and $\mathrm{sn}(\cdot, k)$ are the (elliptic) Jacobi functions. The elliptic amplitude function $\mathrm{am}(\cdot, k)$ is the inverse of the incomplete elliptic integral of the first kind $\int_0^{\psi} \frac{\mathrm{d}\eta}{\sqrt{1-k^2\sin^2\eta}}$ and other elliptic functions are defined as follows: $\mathrm{sn}(\cdot, k) = \sin \mathrm{am}(\cdot, k)$, $\mathrm{cn}(\cdot, k) = \cos \mathrm{am}(\cdot, k)$, and $\mathrm{dn}(\cdot, k) = \sqrt{1 - k^2\, \mathrm{sn}^2(\cdot, k)}$. The k value follows from the resonance condition stating that period of oscillation $\frac{4K(k)}{\omega}$ and period of excitation 2π should be in rational relation

$$\frac{4K(k)}{\omega}\, p = 2\pi\, q, \qquad (22)$$

where p and q are relatively prime natural numbers and $K(k)$ is the complete elliptic integral of the first kind. Thus, for oscillation motion in resonance p:q we have the following subharmonic Melnikov's distance with the use of expressions (20) and (21):

$$M^{p/q} = \int_0^{2\pi q} \frac{\partial H}{\partial v}\, g_1(\theta(\tau), v(\tau), \tau)\, \mathrm{d}, \qquad (23)$$

where function g_1 is given in (8) and $\frac{\partial H}{\partial v} = v$. Taking the integrals in Appendix 2 for $p = 1$ and even $q = 2, 4, 6, \ldots$ we have the following Melnikov's distance

$$M^{1/q} = 4\omega^4 \left(\frac{3\varepsilon\pi \sin\tau_0}{\omega^2 \sinh(K(k')/\omega)} - 4\beta\left(E(k) - k'^2 K(k)\right) \right), \tag{24}$$

where $k'^2 = 1 - k^2$. This is a sign-changing function if the following inequality is satisfied

$$\frac{\varepsilon}{\beta} > \frac{4\omega^2}{3\pi} \left(E(k) - (k'^2) K(k) \right) \sinh\left(\frac{K(k')}{\omega} \right). \tag{25}$$

Condition (25) means that corresponding 1:q resonant oscillations appear only when the amplitude of excitation is sufficiently large with respect to the damping coefficient. The domain of possible oscillations defined by inequality (25) is depicted in Fig. 1 and compared with numerical simulations for $\beta = 0.05$. Note that condition (25) is similar to that of oscillator with quadratic nonlinearity and external periodic excitation [18]. The correspoding inequality for the pendulum with vertically vibrating pivot, see [12], has also three times greater right-hand side function than that in (25).

3.3 Subharmonic bifurcations of rotational orbits

In order to apply Melnikov's criterion to a rotational orbit we find the solution of the unperturbed system (10)–(11) for $H > \omega^2$. Thus, with the use of (9) we obtain $\dot\theta = \pm\sqrt{2}\sqrt{H + \omega^2\cos(\theta)} = \pm 2\omega k \sqrt{1 - \sin^2(\theta/2)/k^2}$, where "$\pm$" represents counter- and clockwise rotations. Since the model is symmetric with respect to the vertical axis, we will consider only the counterclockwise rotation ("+" instead of "$\pm$"). This solution has the form $\theta = 2\,\mathrm{am}(\omega k\,(\tau - \tau_0)\,, 1/k)$, so we have

$$v = \dot\theta = 2\omega k\, \mathrm{dn}\left(\omega k\,(\tau - \tau_0)\,, \frac{1}{k} \right), \tag{26}$$

$$\begin{aligned} \cos\left(\tfrac{\theta}{2}\right) &= \mathrm{cn}\left(\omega k\,(\tau - \tau_0)\,, \tfrac{1}{k}\right), \\ \sin\left(\tfrac{\theta}{2}\right) &= \mathrm{sn}\left(\omega k\,(\tau - \tau_0)\,, \tfrac{1}{k}\right). \end{aligned} \tag{27}$$

The value of k follows from the resonance condition stating that period of rotation $\frac{2K(1/k)}{\omega k}$ and period of excitation 2π should be in rational relation

$$\frac{2K\left(\frac{1}{k}\right)}{\omega k} r = 2\pi\, q, \tag{28}$$

where r and q are relatively prime natural numbers. Thus, for rotational motion in resonance r:q we have the following subharmonic Melnikov's distance

$$M^{q/r} = \int_0^{2\pi q} \frac{\partial H}{\partial v}\, g_1(\theta(\tau), v(\tau), \tau)\, \mathrm{d}\tau, \tag{29}$$

where function g_1 is given in (8) and $\frac{\partial H}{\partial v} = v$. Taking the integrals in Appendix 3 for $r = 1$ and $q = 1, 2, 3, \ldots$ we have the following Melnikov's distance

$$M^{q/1} = -2\omega^2 \left(\frac{3\varepsilon\pi}{\omega^2} \frac{\sin\tau_0}{\sinh\left(\frac{K'}{\omega k}\right)} - 4\beta k E\left(\frac{1}{k}\right) \right), \tag{30}$$

where $K' = K\left(\sqrt{1 - 1/k^2}\right)$. This is a sign-changing function if the following inequality is satisfied

$$\frac{\varepsilon}{\beta} > \frac{4\omega^2 k}{3\pi} E\left(\frac{1}{k}\right) \sinh\left(\frac{K'}{\omega k}\right). \tag{31}$$

Condition (31) means that corresponding 1:q resonant rotations appear only when the amplitude of excitation is sufficiently large with respect to the damping coefficient. The domain of possible rotations defined by inequality (31) is depicted in Fig. 2 and compared with numerical simulations for $\beta = 0.05$. Although numerically found rotations 1:2 are not monotone, these are rotation–oscillation regimes. Note that condition (31) is similar to that of oscillator with quadratic nonlinearity and external periodic excitation [18]. For the pendulum with vertically vibrating pivot similar inequality was obtained in [12], with the right-hand side function being three times greater than that in (31). Thus, the Melnikov's approach leads to similar results for different models of pendula.

4 Averaging method: superharmonic bifurcations of rotational orbits

We study resonant rotations

$$r{:}q = 1{:}1,\ 2{:}1,\ 3{:}1, \ldots, \tag{32}$$

where r is the number of full rotations during q complete periods of excitation. The new assumption here is that the relative eigenfrequency $\omega \ll 1$ is small of the same order with damping parameter $\beta \sim \omega \ll 1$, e.g., because of the small gravitation g or high excitation frequency $\Omega \gg 1$, while the excitation amplitude

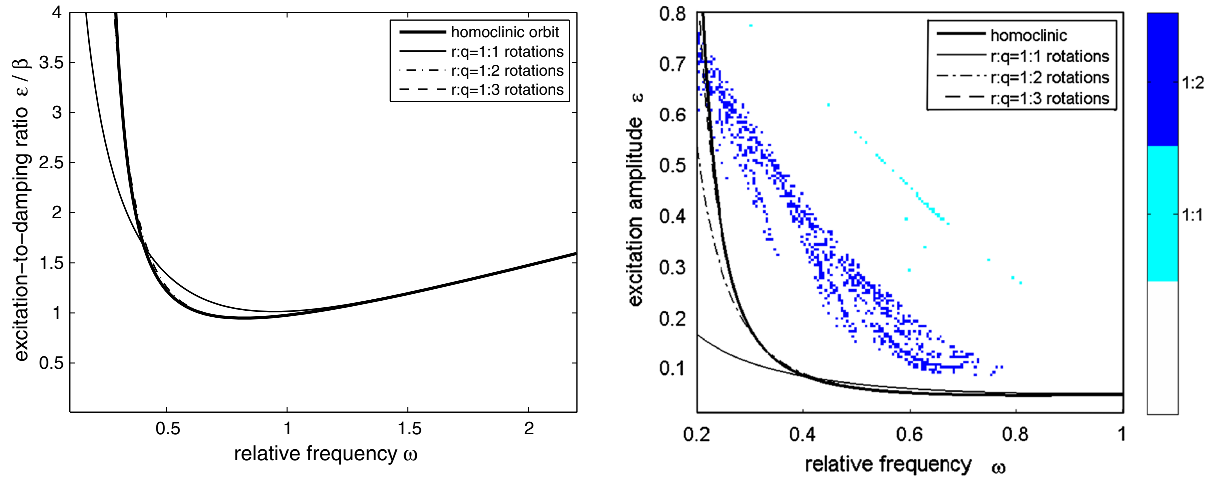


Fig. 2 Subharmonic bifurcation functions (ε/β—left, ε—right), which are the boundaries in (31) for $q = 1$ (*solid line*), $q = 2$ (*dot-and-dash line*), $q = 3$ (*dashed line*), and $r = 1$ converge to the homoclinic bifurcation function that approximates the domain of complex dynamics in (17). These functions approximate the domains of the corresponding resonant rotations, depicted with blue ($q = 1$) and azure ($q = 2$) colors (see the color bar) on the parameter plane (ω, ε) at $\beta = 0.05$

ε is not small. Then, unperturbed system (Eq. (4) with $\omega = 0$) has the specific angular momentum or sector velocity $s = (1 + \varepsilon\varphi(\tau))^2\dot{\theta}$ as its first integral. The unperturbed system is not Hamiltonian, so we cannot apply Melnikov's analysis to obtain domains of existence of the corresponding rotations. For this purpose we use the method of averaging [2,21]. Equation (4) can be written in the form of the system

$$\dot{\theta} = \frac{s}{(1 + \varepsilon\varphi(\tau))^2}, \tag{33}$$

$$\dot{s} = \omega^2 f(\theta, s, \tau), \tag{34}$$

where the perturbation function is the following

$$f(\theta, s, \tau) \equiv -\frac{\beta}{\omega} s - (1 + \varepsilon\varphi(\tau))\sin(\theta), \tag{35}$$

with the ratio $\beta/\omega = O(1)$ that can be considered as a new parameter. The unperturbed system has the following solution

$$\theta_0 = s_0\,\Phi(\tau) + \vartheta_0, \tag{36}$$

where ϑ_0 is the constant phase shift, s_0 is the constant sector velocity, and $\Phi(\tau)$ denotes the following integral

$$\Phi(\tau) = \int_0^\tau \frac{\mathrm{d}\eta}{(1 + \varepsilon\varphi(\eta))^2}. \tag{37}$$

We choose constants such as s_0 and ϑ_0, so that they approximate the perturbed solution, i.e., $\theta = \theta_0 + o(1)$. In order to do so we take a resonance condition from (32) along with the following averaged equation of (34)

$$\dot{\bar{s}} = \omega^2 F(\bar{s}), \tag{38}$$

where the first-order approximation function F is derived via the substitution in f variable θ by the expression $\bar{s}\,\Phi(\tau) + \vartheta$ and taking time average of f as if the corresponding averaged variable $\bar{s}$ and ϑ are constant, see [21]:

$$\begin{aligned} F(\bar{s}) &= \frac{1}{2\pi qr}\int_0^{2\pi qr} f(\bar{s}\,\Phi(\tau) + \vartheta, \bar{s}, \tau)\,\mathrm{d}\tau \\ &= -\frac{\beta}{\omega}\bar{s} - \int_0^{2\pi qr} \frac{1 + \varepsilon\varphi(\tau)}{2\pi qr}\sin(\bar{s}\,\Phi(\tau) + \vartheta)\,\mathrm{d}\tau \\ &= -\frac{\beta}{\omega}\bar{s} - A(\bar{s})\cos(\vartheta) - B(\bar{s})\sin(\vartheta), \end{aligned}$$

where A and B denote the following integrals

$$A(\bar{s}) = \int_0^{2\pi rq} \frac{1 + \varepsilon\varphi(\tau)}{2\pi rq}\sin(\bar{s}\,\Phi(\tau))\,\mathrm{d}\tau, \tag{39}$$

$$B(\bar{s}) = \int_0^{2\pi rq} \frac{1 + \varepsilon\varphi(\tau)}{2\pi rq}\cos(\bar{s}\,\Phi(\tau))\,\mathrm{d}\tau. \tag{40}$$

The period of averaging is chosen as $2\pi rq$ to contain integer numbers of motion and excitation periods. Notice that the averaged equation of (33) $\dot{\bar{\theta}} = \bar{s}/(1+\varepsilon\varphi(\tau))^2$ does not influence the dynamics of $\bar{s}$ in (38) and for constant (steady state) $\bar{s}$ it has the solution $\bar{\theta} = \bar{s}\,\Phi(\tau) + \vartheta$.

The steady-state value of $\bar{s}$ follows from resonance condition (32) written with the use of solution (36) and stating that period of rotation and period of excitation should be in rational relation

$$2\pi\, r = s_0\, q\, \Phi(2\pi), \tag{41}$$

where r and q are relatively prime natural numbers from (32). We have from (41) the approximate steady-state value, $\bar{s} = s_0$, where

$$s_0 = \frac{r}{q}\,\frac{2\pi}{\Phi(2\pi)}. \tag{42}$$

We find the values of ϑ from the averaged equation (38) when we set $\dot{\bar{s}} = 0$ and substitute $\bar{s}$ by its steady-state value s_0 expressed in (42), so that $F(s_0) = 0$:

$$A(s_0)\cos(\vartheta_0) + B(s_0)\sin(\vartheta_0) = -\frac{\beta}{\omega}\, s_0. \tag{43}$$

Thus, we find $\vartheta = \vartheta_0$, which takes values and form two branches ($\pm$) of the solution

$$\vartheta_0 = \vartheta^* + \pi \pm \arccos\left(\frac{s_0}{\sqrt{A^2(s_0) + B^2(s_0)}}\,\frac{\beta}{\omega}\right), \tag{44}$$

where the constant ϑ^* can be expressed as follows

$$\vartheta^* = \operatorname{sign}(B(s_0))\arccos\left(\frac{A(s_0)}{\sqrt{A^2(s_0) + B^2(s_0)}}\right) + 2\pi n,$$

with n being an integer number. Thus, the domain of existence of the corresponding regular rotations is approximated by the following condition that equation (43) has the solution expressed in (44),

$$\frac{\omega}{\beta} \geq \frac{s_0}{\sqrt{A^2(s_0) + B^2(s_0)}}. \tag{45}$$

The averaged variables $\bar{s}$ and $\bar{\theta}$ approximate the slow variable $s = \bar{s} + o(1)$ and fast phase $\theta = \bar{\theta} + o(1)$, which solve system (33)–(34). The solution for regular rotational motion in resonance $r{:}q$ can be written as follows

$$\theta = \bar{s}\,\Phi(\tau) + \vartheta + o(1) = s_0\,\Phi(\tau,\varepsilon) + \vartheta_0 + o(1), \tag{46}$$

where s_0 and ϑ_0 are defined in (42) and (44).

Here, we obtain only the first-order approximation with the use of averaged equation (38). In order to obtain higher-order approximations the general averaging scheme by Volosov can be applied to system (33)–(34), see [21].

4.1 Stability analysis

In order to study the stability of a solution $\bar{s}$ we perturb it by small value η in (38). Thus, we have the linearized equation

$$\dot{\eta} = \omega^2 F'(\bar{s})\eta, \tag{47}$$

with the derivative of function F in the form

$$F'(\bar{s}) = -\frac{\beta}{\omega} - A'(\bar{s})\cos(\vartheta) - B'(\bar{s})\sin(\vartheta),$$

where A' and B' are the derivatives of integrals A and B in (39) and (40):

$$A'(\bar{s}) = \int_0^{2\pi rq} \frac{1+\varepsilon\varphi(\tau)}{2\pi rq}\,\Phi(\tau)\cos(\bar{s}\,\Phi(\tau))\,\mathrm{d}\tau, \tag{48}$$

$$B'(\bar{s}) = -\int_0^{2\pi rq} \frac{1+\varepsilon\varphi(\tau)}{2\pi rq}\,\Phi(\tau)\sin(\bar{s}\,\Phi(\tau))\,\mathrm{d}\tau. \tag{49}$$

According to Lyapunov's theorem on stability based on a linear approximation, the instability or asymptotic stability of the solution $\bar{s}$ of equation (38) is determined by the instability or asymptotic stability of the linearized equation (47). Thus, for steady-state solution, $\bar{s} = s_0$, we have the condition of asymptotic stability $F'(s_0) < 0$:

$$A'(s_0)\cos(\vartheta_0) + B'(s_0)\sin(\vartheta_0) > -\frac{\beta}{\omega}. \tag{50}$$

This condition is checked numerically for various parameters ε and ω/β, see Fig. 3 (left). It turns out that only one branch of the solution, which corresponds to plus (+) in (44), can be stable. Moreover, there are regions (denoted with hatch lines in Fig. 3) where both existing

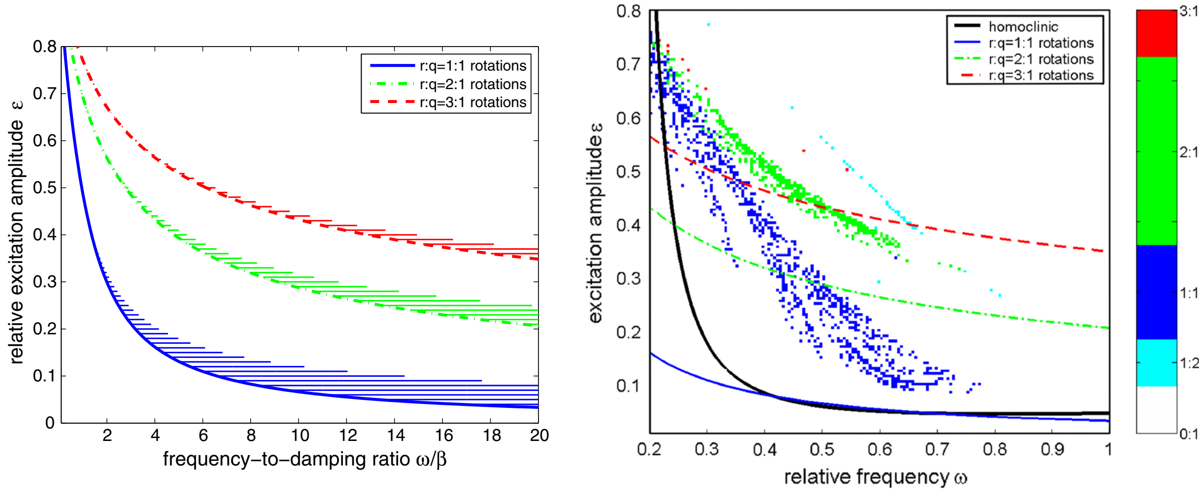


Fig. 3 Superharmonic bifurcation functions which are the boundaries in (45) for $r = 1$, $r = 2$, $r = 3$, and $q = 1$ approximate domains of the corresponding rotations, depicted (*right*) with blue ($r = 1$), green ($r = 2$), and red ($r = 3$) colors (*see the color bar*) on the parameter plane (ω, ε) at $\beta = 0.05$. Hatch lines (*left*) denote the region where corresponding solution exists but it is unstable

branches of the solution are unstable. Superharmonic bifurcations of rotational orbits happen on the upper borders of the corresponding hatched areas; although it would be natural to expect that exact (not approximate) borders of existence domain coincide with the border of stability domain.

4.2 Comparison with direct simulations

In order to find the boundaries for rotational regimes with relative angular velocities $r = 1, 2, 3$ the first approximation is enough, because excitation amplitude ε is not small in contrast to the quasi-linear approach in [9], where both ε and ω are assumed to be small and higher-order approximations of the averaging method are needed to obtain similar boundaries. Notice that it is also possible to not consider small damping β when sector velocity s is small, because they are multiplied in the first term of perturbation function (35).

The right-hand side in condition (45) can be calculated for any particular ε, r, and q, see Fig. 3, where $\varphi(\tau) = \cos(\tau)$. In order to do that, first we calculate numerically (37) as function of τ, then we take integrals (39) and (40) for s_0 obtained from (42). In right Fig. 3 we depict the points where numerical simulation converged to regular rotations. The points have different colors for relative angular velocities $r = 1, 2, 3$. All these points are above the corresponding boundaries of existence domains which have the same colors. Thus, the existence condition (45) is satisfied for all numerically obtained rotational solutions.

5 Conclusion

For the pendulum with variable length we derived analytical formulas for the boundaries of bifurcations in the space of three parameters: the relative frequency of excitation, amplitude of excitation, and damping. The boundaries for homoclinic bifurcation separating the domain of only stationary and oscillatory regimes from the domain of more complex dynamics, subharmonic oscillations, and subharmonic rotations are obtained using Melnikov's method under assumption of small damping and excitation amplitude. For the analysis of superharmonic bifurcations of rotational orbits the method of averaging is used assuming smallness of relative excitation frequency rather than that of excitation amplitude. Both methods allow to obtain in the first approximation the basic rotational orbit with angular velocity equal to the excitation frequency, 1:1. Small excitation frequency (or small gravity) allows to introduce the unperturbed system with the conservation of angular momentum, so that faster rotations are found in the first approximation by the method of averaging. In Figs. 1, 2, and 3 it is shown that the boundaries for complex dynamics, subharmonic oscillations, and superharmonic rotations are in good agreement with the results of numerical simulation.

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Appendices

Melnikov function for homoclinic orbit

$$M^{\pm} = 8\varepsilon\omega^2 \int_{-\infty}^{\infty} \frac{\sin(\tau)\ \mathrm{d}\tau}{\cosh^2(\omega\,(\tau-\tau_0))} - 4\beta\omega^3 \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{\cosh^2(\omega\,(\tau-\tau_0))} + 4\varepsilon\omega^3 \int_{-\infty}^{\infty} \frac{\cos(\tau)\sinh(\omega\,(\tau-\tau_0))}{\cosh^3(\omega\,(\tau-\tau_0))}\,\mathrm{d}\tau, \quad (51)$$

we denote $M^{\pm} = 8\varepsilon\omega^2 I_1 - 4\beta\omega^3 I_2 + 4\varepsilon\omega^3 I_3$, where

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin(\tau)\,\mathrm{d}\tau}{\cosh^2(\omega\,(\tau-\tau_0))} = \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{\sin(\tau_0+\eta/\omega)\ \mathrm{d}\eta}{\cosh^2(\eta)} = \frac{\sin(\tau_0)}{\omega}\int_{-\infty}^{\infty} \frac{\cos(\eta/\omega)\,\mathrm{d}\eta}{\cosh^2(\eta)} + \frac{\cos(\tau_0)}{\omega}\int_{-\infty}^{\infty} \frac{\sin(\eta/\omega)\,\mathrm{d}\eta}{\cosh^2(\eta)}. \quad (52)$$

The integral $\int_{-\infty}^{\infty} \frac{\sin(\eta/\omega)\,\mathrm{d}\eta}{\cosh^2(\eta)}$ is zero because its integrand is an odd function, while the other integral has even integrand and can be calculated as follows: $\int_{-\infty}^{\infty} \frac{\cos(t/\omega)}{\cosh^2 t}\,\mathrm{d}t = \frac{\pi}{\omega\sinh(\pi/2\omega)}$; hence, the first term has the expression

$$I_1 = \frac{\pi\,\sin(\tau_0)}{\omega^2\sinh(\pi/2\omega)}. \quad (53)$$

The second integral can be calculated as follows:

$$I_2 = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{\cosh^2(\omega\,(\tau-\tau_0))} = \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{\cosh^2(s)} = \frac{2}{\omega}, \quad (54)$$

while the integral I_3 can be converted to I_1 via integration by parts using the relation $\frac{\sinh(s)\,\mathrm{d}s}{\cosh^3(s)} = -\frac{1}{2}\,\mathrm{d}\frac{1}{\cosh^2(s)}$ as

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} \frac{\cos(\tau)\ \sinh(\omega\,(\tau-\tau_0))}{\cosh^3(\omega\,(\tau-\tau_0))}\,\mathrm{d}\tau, \\ &= \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{\cos(\tau_0+\eta/\omega)\ \sinh(\eta)}{\cosh^3(\eta)}\,\mathrm{d}\eta \\ &= -\frac{1}{2\omega^2}\left.\frac{\cos(\tau_0+\eta/\omega)}{\cosh^2(\eta)}\right|_{-\infty}^{\infty} - \frac{1}{2\omega^2}\int_{-\infty}^{\infty} \frac{\sin(\tau_0+\eta/\omega)}{\cosh^2(\eta)}\,\mathrm{d}\eta \\ &= -\frac{I_1}{2\omega}. \end{aligned} \quad (55)$$

Thus, $M^{\pm} = 8\varepsilon\omega^2 I_1 - 4\beta\omega^3 I_2 + 4\varepsilon\omega^3 I_3 = \frac{6\pi\varepsilon\sin(\tau_0)}{\sinh(\pi/2\omega)} - 8\beta\omega^2$.

Melnikov function for subharmonic oscillations

$$\begin{aligned} M^{p/q} &= 4\omega^2k^2 \int_0^{2\pi q} (2\varepsilon\sin(\tau) - \beta\omega)\ \mathrm{cn}^2 \times (\omega\,(\tau-\tau_0)\,,k)\ \mathrm{d}\tau \\ &\quad + 4\varepsilon\omega^3k^2 \int_0^{2\pi q} \cos(\tau)\,\mathrm{sn}(\omega\,(\tau-\tau_0)\,,k) \times \mathrm{dn}(\omega\,(\tau-\tau_0)\,,k)\ \mathrm{cn}(\omega\,(\tau-\tau_0)\,,k)\ \mathrm{d}\tau, \end{aligned} \quad (56)$$

so we denote $M^{p/q} = 8\varepsilon\omega^2k^2 I_1 - 4\beta\omega^3k^2 I_2 + 4\varepsilon\omega^3k^2 I_3$,

$$\begin{aligned} I_1 &= \int_0^{2\pi q} \sin(\tau)\,\mathrm{cn}^2(\omega\,(\tau-\tau_0)\,,k)\ \mathrm{d}\tau \\ &= -\cos(\tau)\,\mathrm{cn}^2(\omega\,(\tau-\tau_0)\,,k)\Big|_0^{2\pi q} - 2\omega I_3 \\ &= -2\omega I_3, \end{aligned} \quad (57)$$

$$I_2 = \int_0^{2\pi q} \mathrm{cn}^2(\omega\,(\tau-\tau_0)\,,k)\ \mathrm{d}\tau, \quad (58)$$

$$I_3 = \int_0^{2\pi q} \cos(\tau)\,\mathrm{cn}(\omega\,(\tau-\tau_0)\,,k)\ \mathrm{sn}(\omega\,(\tau-\tau_0)\,,k) \times \mathrm{dn}(\omega\,(\tau-\tau_0)\,,k)\ \mathrm{d}\tau, \quad (59)$$

where we use the formula $\frac{\mathrm{d}\,\mathrm{cn}(u)}{\mathrm{d}u} = -\,\mathrm{sn}(u)\,\mathrm{dn}(u)$. Thus, we have $M^{q/p} = -12\varepsilon\omega^3 k^2 I_3 - 4\beta\omega^3 k^2 I_2$.

$$I_1 = \int_0^{2\pi q} \sin(\tau)\,\mathrm{cn}^2(\omega(\tau-\tau_0),k)\,\mathrm{d}\tau = \frac{1}{\omega}\int_0^{2\pi q\omega} \sin(\tau_0+s/\omega)\,\mathrm{cn}^2(s,k)\,\mathrm{d}s = \frac{\sin(\tau_0)}{\omega}\int_0^{2\pi q\omega}\cos(s/\omega)\,\mathrm{cn}^2(s,k)\,\mathrm{d}s. \quad (60)$$

The integral I_3 vanishes except for $p=1$ and even q. In this case we have from (312.02) in [22]

$$I_2 = \frac{4\omega}{k^2}\Big(E(k) - (k'^2)K(k)\Big),$$
$$I_3 = -\frac{\pi}{k^2\omega}\frac{\sin\tau_0}{\sinh(K(k'))},$$

where $k'^2 = 1-k^2$. So we have (24).

Melnikov function for subharmonic rotations

$$M^{q/r} = 4\omega^2k^2\int_0^{2\pi q}(2\varepsilon\sin(\tau)-\beta\omega)\,\mathrm{dn}^2\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\mathrm{d}\tau + 4\varepsilon\omega^3 k\int_0^{2\pi q}\cos(\tau)\,\mathrm{sn}\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\times\mathrm{dn}\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\mathrm{cn}\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\mathrm{d}\tau, \quad (61)$$

so we denote $M^{q/r} = 8\varepsilon\omega^2k^2 I_1 - 4\beta\omega^3k^2 I_2 + 4\varepsilon\omega^3 k I_3$,

$$I_1 = \int_0^{2\pi q}\sin(\tau)\,\mathrm{dn}^2\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\mathrm{d}\tau = -\frac{2\omega}{k}I_3 - \cos(\tau)\,\mathrm{dn}^2\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\Big|_0^{2\pi q} = -\frac{2\omega}{k}I_3, \quad (62)$$

$$I_2 = \int_0^{2\pi q}\mathrm{dn}^2\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\mathrm{d}\tau, \quad (63)$$

$$I_3 = \int_0^{2\pi q}\cos(\tau)\,\mathrm{sn}\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\times\mathrm{dn}\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\mathrm{cn}\left(\omega k(\tau-\tau_0),\frac{1}{k}\right)\mathrm{d}\tau, \quad (64)$$

where we use the formula $\frac{\mathrm{d}\,\mathrm{dn}(u,\frac{1}{k})}{\mathrm{d}u} = -\frac{\mathrm{sn}(u,\frac{1}{k})\,\mathrm{cn}(u,\frac{1}{k})}{k^2}$. Thus, we have $M^{q/r} = -12\varepsilon\omega^3k^2 I_3 - 4\beta\omega^3k^2 I_2$.

The integral I_3 vanishes except for $r=1$. In this case we have

$$I_2 = \frac{2}{\omega k}E\left(\frac{1}{k}\right), \quad I_3 = -\frac{\pi}{k^2\omega}\frac{\sin\tau_0}{\sinh(K')},$$

where $K' = K\left(\sqrt{1-1/k^2}\right)$. So we have (30)

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