A half-normal limit distribution scheme

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1 Limit theorem

Let \( c(x) = \sum_n c_n x^n \) be a generating function with non-negative coefficients, and \( c(x,u) = \sum c_n x^n u^k \) be the bivariate generating function (BGF) where a parameter of interest has been marked, i.e. \( c(x,1) = c(x) \). We define a sequence of random variables \( X_n, n \geq 1 \) by

\[
P[X_n = k] = \frac{c_{nk}}{c_n} = \frac{[x^n u^k] c(x,u)}{[x^n] c(x,1)}.
\]

Our goal is to identify the limit distribution of \( X_n \) when \( n \) tends to infinity.

In [2, Theorems 1-3] Drmota and Soria show that the limit distribution of \( X_n \) is either Gaussian, Rayleigh or a convolution of both under certain conditions and proper rescaling. We extend these results to conditions implying a half-normal distribution.

![Figure 1: Normal:](image)

![Figure 2: Rayleigh:](image)

![Figure 3: Half-normal:](image)

The technical conditions are given by Hypothesis [H] from [2]. We define Hypothesis [H'] as [H] except that \( h(\rho,1) > 0 \) is dropped. The most important condition is an algebraic singularity of the square-root type:

\[
\frac{1}{c(x,u)} = g(x,u) + h(x,u) \sqrt{1 - \frac{x}{\rho(u)}},
\]

for \( |u - 1| < \varepsilon \) and \( |x - \rho(u)| < \varepsilon \), \( \arg(x - \rho(u)) \neq 0 \), where \( \varepsilon > 0 \) is some fixed real number, and \( g(x,u), h(x,u), \) and \( \rho(u) \) are analytic functions.

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Theorem 1. Let \( c(x,u) \) be a BGF satisfying \([H']\). If \( \rho(u) = \rho = \text{const} \) for \( |u-1| < \varepsilon \), \( g_x(\rho,1) \neq 0 \), \( h_u(\rho,1) \neq 0 \), and \( h(\rho,1) = g_u(\rho,1) = g_{uu}(\rho,1) = 0 \), then the sequence of random variables \( X_n \) has a half-normal limit distribution, i.e.

\[
\frac{X_n}{\sqrt{n}} \xrightarrow{d} \mathcal{H}(\sigma),
\]

where \( \sigma = \sqrt{2} \frac{h_u(\rho,1)}{g_{ux}(\rho,1)} \) and \( \mathcal{H}(\sigma) \) has density \( \frac{\sqrt{2}}{\sigma \sqrt{\pi}} \exp \left( -\frac{x^2}{2\sigma^2} \right) \) for \( x \geq 0 \).

Note that we can also derive the asymptotic forms of the moments, and a strong limit theorem.

2 Applications to lattice paths

The motivation of this work arose in the study of a new lattice path model: the reflection-absorption model [1]. In the case of the absorption model for the final altitude of meanders for drift 0 the above theorem shows the appearance of a half-normal distribution.

However, a variant of Theorem 1 also applies to other parameters of lattice paths:

- Returns to zero of simple aperiodic walks
  The step set of such walks is given by \( \{(1,s_1),\ldots,(1,s_k)\} \) with \( \gcd(s_2-s_1,\ldots,s_k-s_1) = 1 \). A return to zero is a point of altitude 0 after the starting point.

- Sign changes of weighted Motzkin walks
  The step set of such walks is given by \( \{(1,-1),(1,0),(1,1)\} \). It changes sign if it moves from strictly above the \( x \)-axis to strictly below, or vice versa.

![Figure 4: Motzkin walk with 7 returns to zero and 4 sign changes](image)

References
